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Rasheed M.A. Azzam
University of New Orleans, razzam@uno.edu

N. M. Bashara

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Loci of Invariant-Azimuth and Invariant-Ellipticity Polarization States of an Optical System

R. M. A. Azzam and N. M. Bashara

The loci of polarization states for which either the ellipticity alone or the azimuth alone remains invariant upon passing through an optical system are introduced. The cartesian equations of these two loci are derived in the complex plane in which the polarization states are represented. The equations are quartic and are conveniently expressed in terms of the elements of the Jones matrix of the optical system. As an example the loci are determined for a system composed of a $\pi/4$ rotator followed by a quarter-wave retarder.

I. Introduction

A linear nondepolarizing optical system is known to possess two eigenpolarizations¹⁻³ which are the same for both directions of propagation if reciprocity is satisfied. The primary condition for reciprocity is the absence of magnetic fields.^{2,4} The practical importance of these eigenpolarizations is evident, for example, in the study of passive and active optical resonators.² By definition, the eigenpolarizations refer to incident elliptic vibrations that propagate through and emerge from the system with both ellipticity and azimuth unaffected. It is interesting to search for the polarization states for which either ellipticity alone or azimuth alone remains unchanged after propagation through the system. For any one of the invariant-ellipticity states (IES) the polarization ellipses at input and output of the optical system have the same axial ratio and the same sense of description, although the major axis at the output is, in general, rotated with respect to that at input. On the other hand, for any one of the invariant-azimuth states (IAS) the ellipses of polarization at input and output have their major axes oriented parallel to one another, and in general, they have different ellipticities. The eigenpolarizations of the optical system represent the points of intersection of the locus of IES and that of IAS.

It is the objective of this paper to determine the loci of IES and IAS for any given optical system. The polarization is described by a single complex variable and the effect of the optical system by a bilinear

transformation.³ The Poincaré-sphere representation is not suited for this application. The circle-to-circle transformation property of an optical system can be used to introduce the loci of invariant-ellipticity states (IES) and invariant-azimuth states (IAS). Consider, for example, the locus of IES. If the emergent polarization $\bar{\xi}$ is to describe an equiellipticity circle Γ in the complex plane, the incident polarization $\bar{\chi}$ should likewise describe a circle γ . When the two circle Γ and γ intersect one another the points of intersection will represent two incident polarization states whose ellipticity is preserved after propagation through the system. This is shown in Fig. 1. The points l and r represent the polarization states at the input of the optical system which produce left (L) and right (R) circularly polarized light at its output, respectively. The circle γ represents one member of a family of circles that enclose l and r which is mapped by the optical system onto the family of equiellipticity circles Γ . There is one-to-one correspondence between the circles of the two families γ and Γ . Of particular interest are the two limiting circles γ_1 and γ_2 which touch their images Γ_1 and Γ_2 , respectively. The interior of γ_1 and γ_2 (horizontally hatched) is mapped into the interior of Γ_1 and Γ_2 (vertically hatched), respectively. Thus circles that enclose l or r and lie inside γ_1 or γ_2 do not intersect with their images. However, each circle γ outside γ_1 and γ_2 (as the one shown) intersects its image Γ (outside Γ_1 and Γ_2) in two points belonging to the locus \mathcal{E}_i of the incident IES. Therefore, for each angle of ellipticity ϵ in the range $\epsilon_1 \leq \epsilon \leq \epsilon_2$ (where ϵ_1 and ϵ_2 are determined by the equiellipticity circles Γ_1 and Γ_2 , respectively) there are two incident vibrations that emerge from the system with their ellipticity unchanged. Displacing each point on \mathcal{E}_i along the equiellipticity circle Γ that passes through it by an

The authors are with the Electrical Materials Laboratory, Department of Electrical Engineering, The University of Nebraska, Lincoln, Nebraska 68508.

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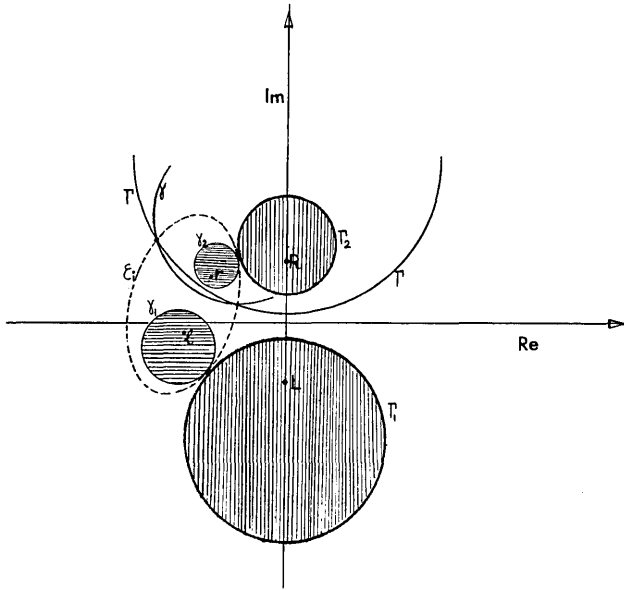


Fig. 1. Introduces the locus ξ_i of the incident invariant-ellipticity states (IES).

amount equal to the rotation of the major axis induced by the system gives the locus ξ_0 of the IES at the output of the system.

In Ref. 3 (footnote 11) it was stated without proof that "For each ellipticity within a specific range two characteristic vibrations of different azimuths pass through the system with their ellipticity unchanged." The discussion of the preceding paragraph clarifies what is meant by a specific range and shows that these "characteristic" vibrations belong to one locus ξ_i which we call the locus of the invariant-ellipticity states (IES).

The locus \mathcal{Q}_i of the incident invariant-azimuth states (IAS) can be introduced in a similar manner but is not given here for brevity.

In the following section we determine the cartesian equations of the loci ξ_i and \mathcal{Q}_i of the incident IES and IAS for any optical system in terms of the elements of its Jones matrix.

II. Cartesian Equations of the Loci of Invariant-Ellipticity and Invariant-Azimuth States

The cartesian equation of an equiellipticity circle is

$$x^2 + (y - \csc 2\epsilon)^2 = \cot^2 2\epsilon, \quad (1)$$

where $\tan \epsilon$ is the ellipticity. Put in a different form, Eq. (1) becomes

$$\begin{aligned} \sin 2\epsilon &= 2y/(1 + x^2 + y^2), \\ &= [2 \operatorname{Im}(\bar{x})]/(1 + |\bar{x}|^2), \end{aligned} \quad (2)$$

which yields the ellipticity of the vibration represented by the point \bar{x} in the complex plane. Similarly for a constant azimuth θ , the cartesian equation of the equiazimuth circle is

$$(x + \cot 2\theta)^2 + y^2 = \csc^2 2\theta, \quad (3)$$

from which

$$\begin{aligned} \tan 2\theta &= 2x/(1 - x^2 - y^2), \\ &= [2 \operatorname{Re}(\bar{x})]/(1 - |\bar{x}|^2), \end{aligned} \quad (4)$$

Equation (4) gives the azimuth of the elliptic vibration represented by the point \bar{x} in the complex plane. \bar{x} and $\bar{\xi}$, the two complex numbers representing the polarization states of the incident and outgoing waves, respectively, are related by

$$\bar{\xi} = (A\bar{x} + B)/(C\bar{x} + D), \quad (5)$$

where $A = T_{22}$, $B = T_{21}$, $C = T_{12}$, and $D = T_{11}$ are the elements of the system's Jones matrix. From Eq. (2) it is seen that if \bar{x} and $\bar{\xi}$ are to have the same ellipticity,

$$\operatorname{Im}(\bar{x})/(1 + |\bar{x}|^2) = \operatorname{Im}(\bar{\xi})/(1 + |\bar{\xi}|^2), \quad (6)$$

which determines the locus ξ_i of the IES at the input of the system. Similarly,

$$\operatorname{Re}(\bar{x})/(1 - |\bar{x}|^2) = \operatorname{Re}(\bar{\xi})/(1 - |\bar{\xi}|^2), \quad (7)$$

states that the azimuths of \bar{x} and $\bar{\xi}$ are equal [Eq. (4)] and hence defines the locus \mathcal{Q}_i of the IAS at the input of the system. Substituting

$$\bar{x} = x + jy, \quad (8)$$

$$\begin{aligned} A &= a_1 + ja_2, \quad B = b_1 + jb_2, \\ C &= c_1 + jc_2, \quad D = d_1 + jd_2, \end{aligned} \quad (9)$$

into Eq. (5) and using the result in Eqs. (6) and (7) we find that

$$(x^2 + y^2 + 1)Q_1 - yQ_2 = 0, \quad (10)$$

$$(x^2 + y^2 - 1)Q_3 - xQ_4 = 0. \quad (11)$$

Equations (10) and (11) describe the loci ξ_i and \mathcal{Q}_i of the incident IES and IAS, respectively. The Q 's in Eqs. (10) and (11) are quadratic functions of x and y that can be conveniently written in the form

$$\begin{aligned} Q_1 &= (C \times A)(x^2 + y^2) - (B \times C - D \times A)x - \\ &\quad (B \cdot C - D \cdot A)y + D \times B, \end{aligned} \quad (12)$$

$$\begin{aligned} Q_2 &= (A \cdot A + C \cdot C)(x^2 + y^2) + 2(A \cdot B + C \cdot D)x + \\ &\quad 2(A \times B + C \times D)y + (B \cdot B + D \cdot D); \end{aligned}$$

$$\begin{aligned} Q_3 &= (C \cdot A)(x^2 + y^2) + (B \cdot C + D \cdot A)x - \\ &\quad (B \times C + D \times A)y + D \cdot B, \end{aligned} \quad (13)$$

$$\begin{aligned} Q_4 &= (A \cdot A - C \cdot C)(x^2 + y^2) + 2(A \cdot B - C \cdot D)x + \\ &\quad 2(A \times B - C \times D)y + (B \cdot B - D \cdot D), \end{aligned}$$

where \mathbf{A} , \mathbf{B} , \mathbf{C} , and \mathbf{D} are the elements of the Jones matrix considered as vectors in the complex plane. The operations of the dot and cross products have the same meaning as defined for ordinary vectors. However, only the magnitude (with proper

sign) of the cross product is to be used in Eqs. (12) and (13). Note the symmetrical forms of Q_1 and Q_3 , Q_2 and Q_4 . The expressions of the Q 's are such that $Q = 0$ represents a circle in the complex plane.

If Eqs. (12) and (13) are substituted into Eqs. (10) and (11), respectively, it is seen that the latter are both quartic in x and y of the form

$$\lambda_1 x^4 + \lambda_2 y^4 + \dots + \lambda_{14} y + \lambda_{15} = 0. \quad (14)$$

Of the five coefficients in the fourth-power terms in Eq. (14) (λ_1 to λ_5), three are equal and two are zeros. The remaining ten coefficients (λ_6 to λ_{15}) are still not all independent. This is applicable for any optical system with further simplification expected when special cases are considered. Unless the point at infinity belongs to \mathcal{E}_i or \mathcal{A}_i , Eq. (14) represents a closed contour in the complex plane. The latter condition can be seen from Eq. (5) to be equivalent to stating that A/C is neither pure real nor pure imaginary.

A note of caution regarding the meaning of the locus \mathcal{A}_i of the IAS is in order. If $\bar{\chi}$ and $\bar{\xi}$ are defined with respect to parallel coordinate systems at the input and output of the optical system, the direction of the major axis of the elliptic vibration corresponding to a point on \mathcal{A}_i [defined by Eq. (7)] would either remain unchanged or is rotated by $\pi/2$ after propagation through the system. This indeterminacy of $\pi/2$ is related to the fact that one circle through the points $R(0,1)$ and $L(0,-1)$ consists of two branches or arcs with the points on one arc corresponding to a given azimuth and on the other to the orthogonal azimuth.³ If $\bar{\chi}$ and $\bar{\xi}$ are referenced to nonparallel coordinate systems, Eq. (7) gives the locus of vibrations whose major axes are rotated by a fixed amount around the beam direction. This amount either equals the angle between the input and output coordinate systems or this angle plus or minus $\pi/2$. Obviously \mathcal{E}_i , the locus of the IES as given by Eq. (6) does not depend on the choice of coordinate systems.

So far we have referred to \mathcal{E}_i and \mathcal{A}_i , the loci of IES and IAS at the input of the system, respectively. Except for the two points that represent the system's eigenpolarizations, each point on \mathcal{E}_i is carried over by the system onto another point which, although of the same ellipticity, has a different azimuth. Thus \mathcal{E}_i is mapped by the system into another contour \mathcal{E}_0 in the complex plane, with both \mathcal{E}_i and \mathcal{E}_0 intersecting at the two eigenpolarizations. Similarly \mathcal{A}_i gives rise to \mathcal{A}_0 at the output of the system both also intersecting at the two eigenpolarizations. This is schematically shown in Fig. 2. The equations of \mathcal{E}_0 and \mathcal{A}_0 can be found by inverting Eq. (5)

$$\bar{\chi} = (D\bar{\xi} - B)/(-C\bar{\xi} + A), \quad (15)$$

substituting $\bar{\xi} = x + jy$ and $A, B, C,$ and D from Eq. (9) into Eqs. (6) and (7). It is interesting to note that \mathcal{E}_0 and \mathcal{A}_0 are also described by quartic equations.

Thus the bilinear transformation preserves the degree of the loci \mathcal{E}_i and \mathcal{A}_i .

As an example, consider a simple system composed of a $\pi/4$ rotator followed by a quarter-wave retarder. The input and output coordinate systems are chosen parallel to the fast-slow axes of the retarder. For such a system it is easy to prove that

$$A = -B = -j, C = D = 1, \quad (16)$$

apart from a multiplying factor. Thus, in the complex plane, $A, B, C,$ and D are all unit vectors. A and B are oppositely oriented along the imaginary axis in the negative and positive directions, respectively; whereas C and D are both oriented along the positive direction of the real axis. Using this simple set of vectors in Eqs. (12) and (13) it is easy to find that

$$\begin{aligned} \text{Cut 91} \quad Q_1 &= 1 - x^2 - y^2, \quad Q_2 = 2(1 + x^2 + y^2), \\ Q_3 &= 2y, \quad Q_4 = -4x, \end{aligned} \quad (17)$$

Substituting Q_1 and Q_2 from Eq. (17) into Eq. (10), we obtain

$$(x^2 + y^2 + 1)(x^2 + y^2 + 2y - 1) = 0, \quad (18)$$

which shows that the locus \mathcal{E}_i of the IES is composed of two circles one of which,

$$x^2 + y^2 + 1 = 0 \quad (19a)$$

is imaginary, and the other,

$$x^2 + y^2 + 2y - 1 = 0, \quad (19b)$$

has a radius of $\sqrt{2}$ and center at $(0, -1)$. This circle is drawn in Fig. 3, which also shows the limiting circles $\gamma_1, \gamma_2; \Gamma_1, \Gamma_2$ (see the Introduction). Note that r and l now coincide with the origin and the point at infinity. The ellipticity range is determined from

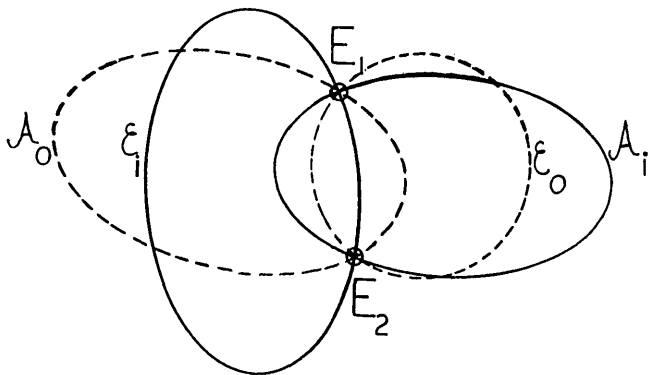


Fig. 2. The loci \mathcal{E}_i and \mathcal{A}_i of the incident IES and IAS, respectively, intersect at two points E_1 and E_2 , representing the two eigenpolarizations of the optical system. The optical system maps \mathcal{E}_i and \mathcal{A}_i into \mathcal{E}_0 and \mathcal{A}_0 which represent the loci of the IES and IAS at the output of the system, respectively. Obviously E_1 and E_2 remain invariant and thus belong to all four loci $\mathcal{E}_i, \mathcal{E}_0, \mathcal{A}_i,$ and \mathcal{A}_0 .

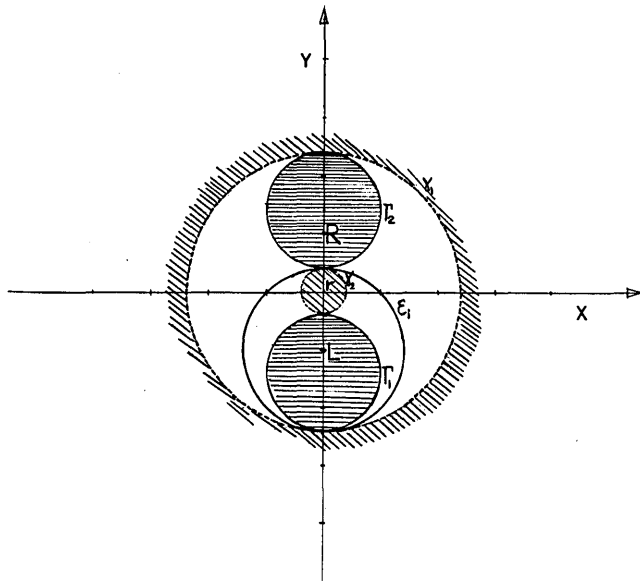


Fig. 3. The locus ε_i of the incident IES for a system composed of a $\pi/4$ rotator followed by a QW retarder. Also shown are the limiting circles $\gamma_1, \gamma_2; \Gamma_1$ and Γ_2 (see Fig. 1). The points r and l coincide with the origin and the point at infinity of the complex plane, respectively. ε_i is given by Eq. (19b).

the points of intersection of the circle in Eq. (19b) with the imaginary axis. The range is

$$-0.414 \leq \tan \alpha \leq 0.414.$$

The locus α_i of the IAS is obtained by substituting Q_3 and Q_4 from Eq. (17) into Eq. (11). This yields

$$y^3 + yx^2 + 2x^2 - y = 0,$$

or

$$x^2 = y(1 - y)(1 + y)/(2 + y). \quad (20)$$

Equation (20) shows that α_i is symmetrical about the y axis and has two separate branches $\alpha_i^{(1)}, \alpha_i^{(2)}$ which are confined to the two strips of the complex plane defined by $0 \leq y \leq 1$ and $-2 \leq y \leq -1$, respectively. (Outside these two intervals x^2 is negative.) A graph of Eq. (20) appears in Fig. 4. $\alpha_i^{(1)}$ is closed and touches the bounding lines $y = 0$ and $y = 1$ at r and R , respectively. $\alpha_i^{(2)}$ is tangent to the line $y = -1$ at L and approaches $y = -2$ asymptotically. The point at infinity belongs to $\alpha_i^{(2)}$ and is expected since A/C is imaginary. Note that for every azimuth value there are two vibrations for which this azimuth is either retained or changed by $\pi/2$, after propagation through the system. This follows because every straight line through r (the origin) and l (at infinity) is transformed by the system into a circle through R and L , both intersecting at two points belonging to α_i .

It is interesting to examine the intersection of ε_i and α_i . As shown in Fig. 5, there are four points of

intersection. These can be found by solving Eqs. (19b) and (20) simultaneously. This gives

$$\begin{aligned} \bar{x}(E_1) &= (0.366, 0.366), \bar{x}(S_1) = (-0.366, 0.366), \\ \bar{x}(S_2) &= (1.366, -1.366), \bar{x}(E_2) = (-1.366, -1.366), \end{aligned}$$

of which E_1 and E_2 represent the system's eigenpolarizations and S_1 and S_2 represent vibrations whose

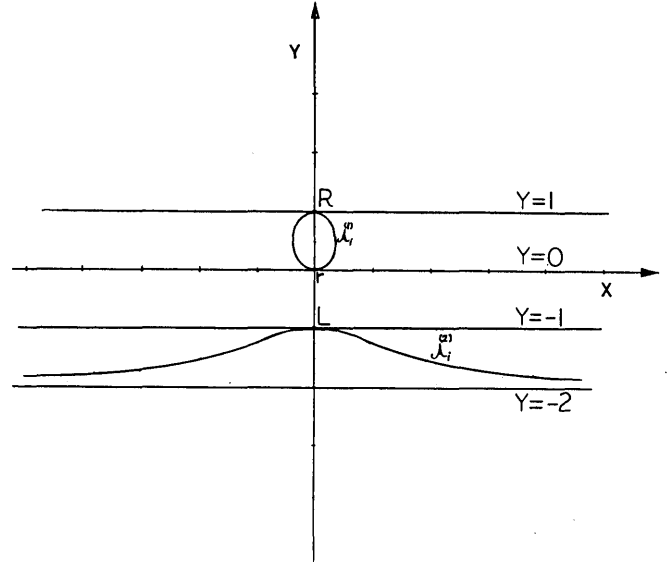


Fig. 4. The locus α_i of the incident IAS for a system of a $\pi/4$ rotator followed by a QW retarder has two separate branches $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$. α_i is defined by Eq. (20).

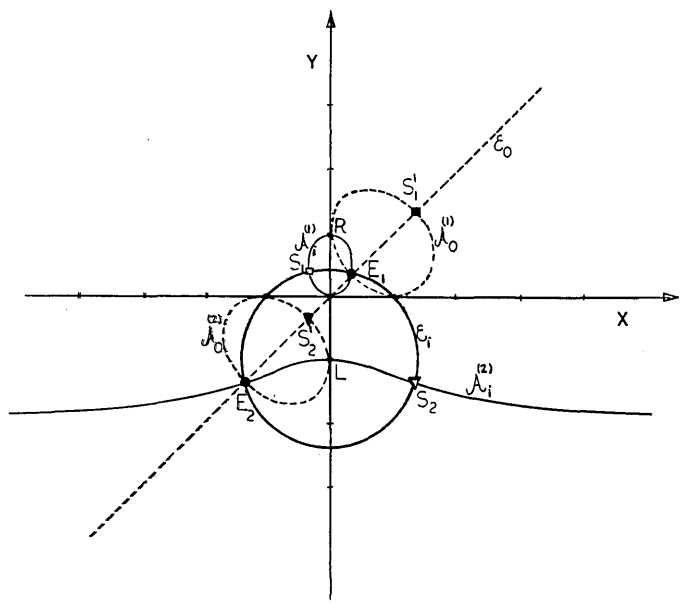


Fig. 5. The intersection of the loci $\varepsilon_i, \alpha_0; \alpha_i$ and α_0 for the system of Figs. 3 and 4. Note that only E_1 (0.366, 0.366) and E_2 (-1.366, -1.366) are common to all four loci and hence represent the true eigenpolarizations of the optical system. S_1 and S_2 are common to ε_i and α_i , but they are mapped by the system into S_1' and S_2' , respectively. They represent incident vibrations whose ellipticities are preserved but whose major axes are rotated by $\pi/2$ after propagation through the system.

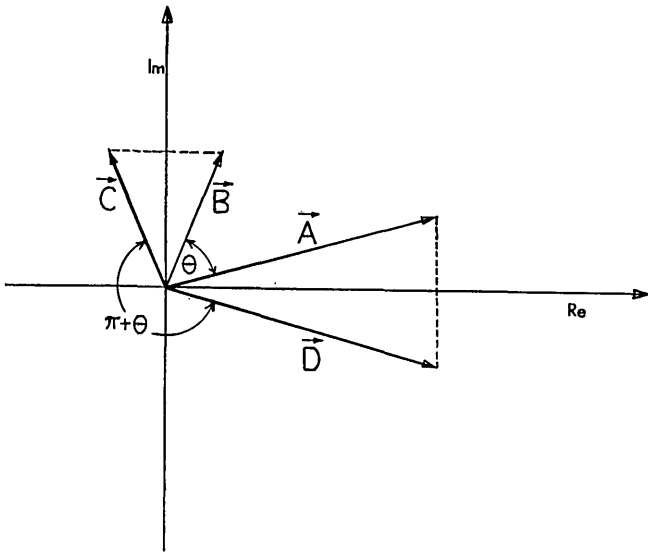


Fig. 6. Shows the set of four vectors A , B , C , and D [corresponding to the four complex coefficients of the polarization transfer function in Eq. (5)] in the case of a system exhibiting pure phase anisotropy and whose Jones matrix is unitary. These vectors determine the loci \mathcal{E}_i and \mathcal{G}_i of the incident IES and IAS according to Eqs. (10)–(13).

ellipticities remain unchanged with major axes rotated through $\pi/2$ by the system. For completeness, Fig. 5 also shows the two loci \mathcal{E}_0 and \mathcal{G}_0 representing the IES and IAS at the output of the system, i.e., the system's response to \mathcal{E}_i and \mathcal{G}_i .

In the above example of a $\pi/4$ rotator followed by a QWP we have seen that \mathcal{E}_i degenerates to a circle. This result will now be proved to apply to any optical system exhibiting pure phase anisotropy. Such a system is characterized by a Jones matrix that can be put in the form of a constant complex multiplier times a unitary matrix.² The constant multiplier accounts for any over-all (isotropic) phase delay or absorption (positive or negative) in the system. The unitary matrix represents the pure phase anisotropy and has two orthogonal eigenpolarization states that are retarded differently but attenuated equally by the system. In this case the elements of the Jones matrix satisfy the following conditions⁵

$$D = A^*, C = -B^*. \quad (21)$$

The set of vectors A , B , C , and D for this system are now arranged as seen in Fig. 6. From Fig. 6 it can be easily proved that

$$\begin{aligned} A \cdot B + C \cdot D &= 0, \\ A \times B + C \times D &= 0. \end{aligned} \quad (22)$$

Substituting Eq. (22) into Eq. (12), the expression of Q_2 simplifies to

$$Q_2 = (|A|^2 + |B|^2)(x^2 + y^2 + 1). \quad (23)$$

Using Eq. (23), Eq. (10) (defining the locus \mathcal{E}_i of the IES) can be factored in the form

$$(x^2 + y^2 + 1)[Q_1 - (|A|^2 + |B|^2)y] = 0, \quad (24)$$

which describes the combination of the imaginary circle

$$x^2 + y^2 + 1 = 0$$

and the real circle

$$Q_1 - (|A|^2 + |B|^2)y = 0. \quad (25)$$

Thus for an optical system exhibiting phase anisotropy alone (and thus has two orthogonal eigenpolarizations that are retarded differently but are attenuated equally) the locus \mathcal{E}_i of the incident invariant ellipticity states is a circle in the complex plane. \mathcal{E}_0 which is obtained from \mathcal{E}_i by a bilinear transformation is also a circle.

In the general case of an optical system whose eigenpolarizations are not orthogonal or whose orthogonal eigenpolarizations suffer different amounts of phase delay and attenuation the elements A , B , C , and D are not interrelated, and a variety of shapes for the loci of IES and IAS can be expected.⁶

III. Conclusions

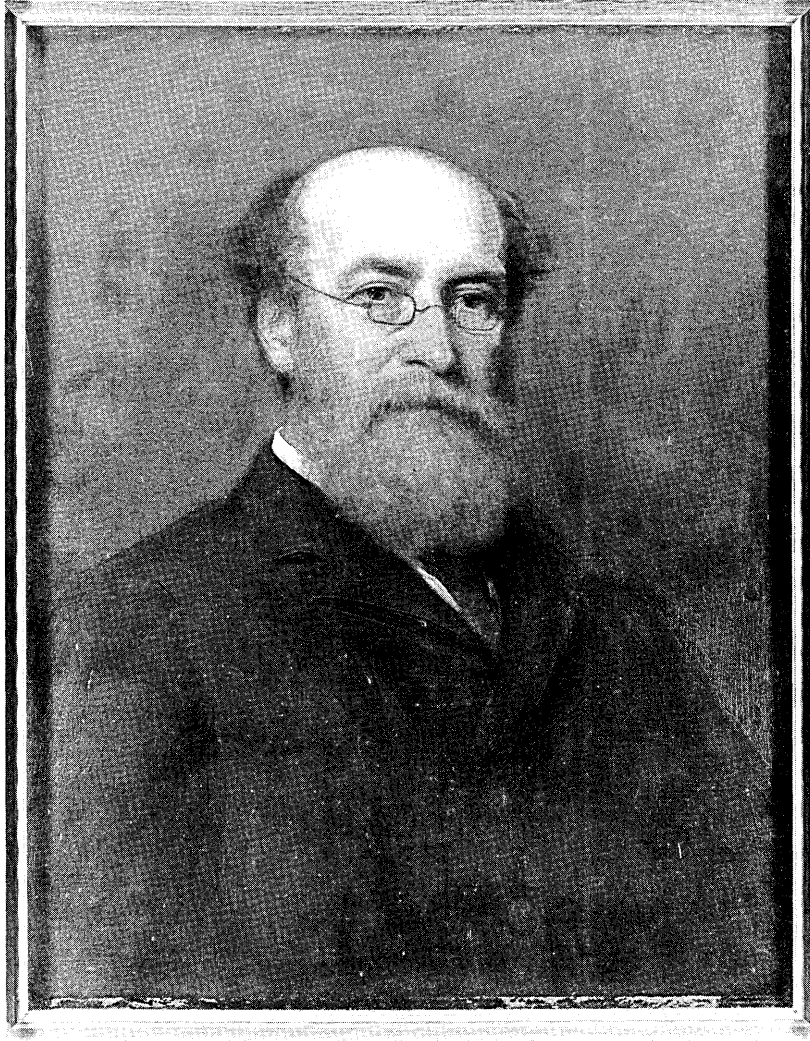
For each linear nondepolarizing optical system there is one set of polarization states that propagate through the system with their ellipticity preserved and another set that pass through the system with the orientation of the major axis of the polarization ellipse unchanged. The locus of the set of the invariant-ellipticity states (IES) and that of the invariant-azimuth states (IAS) always intersect at two points representing the two eigenpolarization states of the optical system. After introducing these loci we derived their cartesian equations in the complex plane in Sec. II. Equations (10) and (12) give the locus \mathcal{E}_i of the incident IES, and Eqs. (11) and (13) determine the locus \mathcal{G}_i of the incident IAS. These equations are conveniently expressed in terms of four vectors A , B , C , and D in the complex plane corresponding to the four elements of the system's Jones matrix. Because of the quartic nature of the equations described these loci [Eq. (14)], a variety of shapes is expected. For systems exhibiting pure phase anisotropy (with a Jones matrix equal to a complex multiplier times a unitary matrix) the locus \mathcal{E}_i of the IES is a circle [Eq. (25)]. In this case the locus \mathcal{G}_i of the IAS is not necessarily a simple one as can be seen from the example of a $\pi/4$ rotator followed by a QW retarder. The loci \mathcal{G}_i and \mathcal{E}_i refer to polarization states at the input of the optical system. These are

mapped by the system onto the loci \mathcal{E}_0 and \mathcal{Q}_0 at its output, with \mathcal{E}_i , \mathcal{E}_0 ; \mathcal{Q}_i and \mathcal{Q}_0 all intersecting at the two points representing the system's eigenpolarizations (Fig. 2). The above results should add new insight into the understanding of the polarization-mapping properties of linear nondepolarizing optical systems.

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4. R. Clark Jones, *J. Opt. Soc. Am.* **37**, 110 (1947).
5. The elements of the Jones matrix in the example of a $\pi/4$ rotator and QWP of Eq. (16) satisfy Eq. (21) after multiplication by $\exp(i\pi/4)$.
6. The special case when Eq. (14) is factored in the form $Q_a Q_b = 0$, where Q_a and Q_b are two quadratics, gives rise to a locus composed of two conic sections.



BENJAMIN APTHORP GOULD 1824–1896. When Gould became head of the longitude department of the Coast Survey in 1852, with headquarters at the Harvard Observatory in Cambridge, he became a part of what was called the Cambridge clique of the Lazzaroni (see page A14 of this January issue). Having recently returned from Gottingen boasting a German doctorate, Gould was ready to take on grand enterprises of the kind envisioned by his Cambridge friends in the founding of the Dudley Observatory at Albany. Named Director at the urging of Peirce and Agassiz, Gould attempted to divert the Observatory's building and funds to professional research, much against the wishes of the institution's trustees who preferred a more popularly oriented institution. Irritated by Gould's arrogance and expensive demands, in 1859 the trustees decided that Gould had to go. Gould's program, they believed, had ignored the needs of a democracy. "The time has gone when science or literature is sealed from the common people." Gould had to be removed by force from the building which he and his Lazzaroni supporters believed they controlled. With his dismissal, the Lazzaroni dream of founding a high-ranking research institution with private funds was not realized in Albany. Gould returned to the Coast Survey, where he carried on research of great value. In 1863, he participated with Agassiz, Bache, Davis and Peirce in the founding of the National Academy of Sciences. In 1870, he left the U.S. to direct the national observatory at Cordoba, Argentina. (This portrait was lent to the National Portrait Gallery by Benjamin A. G. Thorndike, Boston.)