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Non-Decreasing Sequences

Amy E. Klass
Valparaiso University, amy.klass@valpo.edu

Alexander Bruno
Valparaiso University, alexander.bruno@valpo.edu

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Alexander Bruno and Amy Klass
Advisor: Dr. Jon Beagley

Abstract

Non-decreasing sequences are a generalization of binary covering arrays, which has made research on non-decreasing sequences important in both math and computer science. The goal of this research is to find properties of these non-decreasing sequences as the variables d, s, and t change. The goal is also to explore methods for creating a maximum length non-decreasing sequence for a given strength and size set. Through our research, we discovered and proved basic properties of these non-decreasing sequences. In addition to this, we can describe a method we used while trying to find the maximum length of a sequence.

Definitions and Notation

- Let $\boldsymbol{S}$ be a set of $\boldsymbol{s}$ elements
- The **strength** of non-decreasing sequence is the amount of subsets whose union we consider, and is represented using $\boldsymbol{d}$
- A **non-decreasing sequence** of strength d is a sequence of non-empty subsets, $\{S_1, S_2, \dots, S_t\}$, where the union of any d previous subsets does not contain any subsequent subset
- The number of subsets in a non-decreasing sequence is called the **length**, $\boldsymbol{t}$
- $\boldsymbol{NDS(d,s,t)}$ is the set of non-decreasing sequences with strength d, s elements and length t
- $\boldsymbol{NDST(d,s)}$ is the maximum t such that $NDS(d,s,t)$ is non-empty
- Let r_j be the number of elements in the subset S_j

Binary Arrays

- Represent a non-decreasing sequence using an $s \times t$ binary array
- Rows represent elements of S
- Columns represent subsets of non-decreasing sequence

	S_i
1	0
⋮	⋮
k	1
⋮	⋮
s	0

	S_1	S_2	S_3	S_4	S_5
1	1	0	0	0	1
2	0	1	0	0	1
3	0	0	1	0	1
4	0	0	0	1	0

Basic Results

Theorem 1-Permuting rows in a binary array gives another $NDS(d,s,t)$.

Theorem 2-If the union of any d subsets contain all elements in S, no subsets can be added to the sequence.

Theorem 3-Every subset in $NDS(d,s,t)$ must be distinct for $d \geq 1$.

Theorem 4-$NDS(d,s,t) \subseteq NDS(d,s+1,t)$

Corollary 5-$NDST(d,ks) \geq kNDST(d,s)$, where $k \in \mathbb{Z}$.

	S_1	S_2	S_3	S_4	S_5
1	1	0	0	0	1
2	0	1	0	0	1
3	0	0	1	0	1
4	0	0	0	1	0

A

A	0	0
0	A	0
0	0	A

Block array

Standard Sequence

Theorem 6-There exists an $NDS(d,s,t)$ where the first s subsets are of size 1. We call this a **standard non-decreasing sequence**.

	1	**2**	**3**	**4**	**5**	**6**
1	1	0	0	0	0	...
2	0	1	0	0	0	...
3	0	0	1	0	0	...
4	0	0	0	1	0	...
5	0	0	0	0	1	...
⋮	⋮	⋮	⋮	⋮	⋮	⋱

Theorem 7-A standard non-decreasing sequence of strength d does not have any subsets of size $1 < r \leq d$.

Theorem 8- In a standard non-decreasing sequence, any subset S_j of size $r_j = d + 1$, may contain at most 1 element from any previous subset S_i.

Theorem 9-In a standard non-decreasing sequence, any subset S_j of size $r_j \geq d + 1$ must contain at least d elements that differ from any previous subset S_i.

Bounds

- Gives range for $NDST(d,s)$
- Lower bound is the length of sequence constructed for a given d, s
- Upper bound initially $2^s - 1$, number of nonempty subsets possible for any set S with s elements
- Upper bound decreased using Theorems 7, 8, and 9

s	Lower Bound	Found Upper Bound	$2^s - 1$
1	1	1	1
2	2	2	3
3	4	4	7
4	5	5	15
5	7	7	31
6	11	13	63
7	15	20	127

Table 1: Bounds for $d = 2$

Future Work

- Find exact formula for $NDST(d,s)$
- Find different computational methods
- Find relation to binary covering arrays
- Effect of permuting columns
- Find bounds for larger d and s values

References

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