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REPORT

IDEAL BALLOONING STABILITY NEAR AN EQUILIBRIUM MAGNETIC ISLAND

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Ideal Ballooning Stability Near an Equilibrium Magnetic Island

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Abstract

The stability properties of ideal ballooning modes on toroidal flux surfaces near a quasistatic magnetic island is examined. On these surfaces, magnetic field-line trajectories tend to bunch on that part of the magnetic surface closest to the X-point of the magnetic island. Because of this preferential bunching, the stabilizing effect of field-line bending due to magnetic shear can be reduced. Eigenfunctions localized in helical angle near the X-point and in poloidal angle on the bad curvature side of the tokamak are more susceptible to ballooning instability than are modes in corresponding equilibria without the magnetic island. For a slowly growing island, a growing number of flux surfaces located near the separatrix become ballooning unstable. Secondary ballooning instabilities may play a part in the crash phase of sawteeth or macroscopic island dynamics.

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I. Introduction

Understanding the stability properties of three-dimensional magnetostatic equilibria is a problem of considerable interest and difficulty. Equilibria of magnetic confinement systems are usually described by a set of nested, toroidal flux surfaces.[1] However, these surfaces are only guaranteed to exist for magnetic geometries that contain a continuous symmetry[2] (i. e., axisymmetry in a tokamak). More generally, the magnetic topology may change through the formation of magnetic islands and regions of magnetic stochasticity. These magnetic field modifications can occur because of the presence of field errors[3] or inherent three-dimensionality, or appear as a quasistatic succession of equilibria resulting from slowly growing resistive instabilities (e. g., nonlinear growth of constant-ψ tearing modes[4]). The loss of nested, toroidal flux surfaces seriously complicates the issue of ideal stability analysis since one needs to consider three-dimensional instabilities growing in a three-dimensional equilibrium. Most present day stability codes are incapable of analyzing this problem since the assumption of well-defined toroidal surfaces as found from the equilibrium Grad-Shafarnov equation is usually made.

In this work, we examine the stability properties of an equilibrium that contains a quasistatic magnetic island (the island growth time is long compared to the ideal growth time). In particular, ideal ballooning modes are examined on the toroidal flux surfaces just outside the separatrix of the magnetic island. Since the pressure profile is flat inside a magnetic island (assuming that there are no pressure sources inside the island separatrix), it is sufficient to examine the toroidal surfaces nearest the island where the effect of the island is most dramatically felt on the pressure-driven modes. We also note that three-dimensional effects can cause the flux surfaces very close to the separatrix to be destroyed and cause the pressure profile to flattened in the stochastic region. However, this effect is small for low mode number magnetic islands,[5] so we will not consider this effect on the surfaces near the island and assume that pressure gradients exist on every toroidal flux surface on either side of the island.

It has been suggested that secondary pressure-driven instabilities play a role in the crash phase of sawteeth.[6,7] In ref. (6), a two step process for internal disruptions is put forth. First, an m = 1 internal kink is excited and evolves nonlinearly until a helical, kinked neighboring equilibrium that is accessible from the initial equilibrium is found.[8] This helical neighboring equilibrium also occurs for ideal m = 1, n = 1 modes in tokamaks with nonmonotonic q profiles[9] and for m = 1 kink-tearing modes.[10] This new equilibrium is then unstable to ballooning modes due to an enhancement in the pressure gradient. In ref. 7, it is suggested that a steep pressure gradient builds up in a boundary layer near the surface of reconnection. This enhanced pressure gradient causes a magnetohydrodynamic (MHD) instability. More recently, simulations of three-dimensional MHD indicate that toroidal effects are important for understanding the m = 1 resistive mode.[11]

In this work, we hypothesize that the presence of a magnetic island changes the MHD stability properties and introduce a general formalism to treat the effects of a magnetic island on large-n ballooning instabilities in tokamak plasmas. The effect of the island enters in the metric elements that are needed in the ballooning mode δW. These metric elements basically describe the fact that magnetic field-lines tend to "spend more time" on that part of the magnetic surface that is nearest to the X-point of the magnetic island where the "effective shear" is small. Because of this bunching of the field-lines, the stabilizing effect of field-line bending due to magnetic shear can be reduced. As discussed above, the compression of the magnetic surfaces caused by the island formation can enhance the pressure gradient near the island. The inherent three-dimensionality of the equilibrium affords the ideal mode greater freedom in finding an unstable perturbation. The most unstable (or least stable) eigenfunction is found to be localized in poloidal angle on the bad curvature side of the tokamak, and in helical angle (the angle associated with the resonant magnetic perturbation) on that part of the flux surface nearest the X-point of the magnetic island. For a slowly growing magnetic island, a growing region of flux surfaces located near the separatrix are more susceptible to ballooning instabilities than the associated two-dimensional equilibrium without the magnetic island.

We would also like to point out that a similar set of calculations was performed by Bishop and co-workers to examine the effect of an axisymmetric magnetic separatrix due to a magnetic divertor on ideal ballooning modes.[12,13] It was found that the poloidal location of the X-point was crucial to the stability analysis, with the greatest stability occuring when the X-point was located on the good curvature side of the tokamak so that the bunching of the field-lines near the X-point amplifies the stabilizing effect of the local curvature. The major difference between the work of Bishop and our work is that the divertor geometry considered retains the axisymmetry property of the equilibrium, so that the eigenfunction is only localized in the poloidal angle. In the present work, the equilibrium is inherently three-dimensional and this allows the mode to find a more unstable perturbation.

In the following section, the equilibrium with a magnetic island is described. The stability analysis of this equilibrium is studied in Sec. III. A discussion of this model and its ramifications for tokamak phenomenolgy is presented in Sec. IV.

II. Equilibrium with a magnetic island

First, let's consider an equilibrium with nested, toroidal flux surfaces, which are labeled by the flux function Φ. The toroidal and poloidal angles are given by ζ and $\bar{\theta}$, respectively. We write an equilibrium magnetic field with toroidal flux surfaces

$$\mathbf{B}_0 = q\nabla\Phi \times \nabla\bar{\theta} + \nabla\zeta \times \nabla\Phi \ , \qquad (1)$$

where $q = q(\Phi)$ is the inverse rotational transform. A magnetic island forms when a resonant, symmetry-breaking magnetic field perturbation is present at rational values of q. Consider an island forming at the surface $q = q_0 = m_0/n_0$ caused by the presence of a magnetic perturbation of the form

$$\mathbf{B}_1 = \nabla\bar{\theta} \times \nabla A(\Phi,u) \ , \qquad (2)$$

where $u = \zeta - q_o\bar{\theta}$ is the resonant angle. It is assumed that A depends on the resonant angle u and the radial variable Φ alone in the neighborhood of the rational surface and is given by

$$A(\Phi,u) = \sum_{p=1} \frac{A_p(\Phi)}{n_o p} \cos(pn_o u + \phi_p) \ . \tag{3}$$

Changing to the coordinates $u = \zeta - q_o\bar{\theta}$, and $\theta = \bar{\theta}$, the total magnetic field is given by

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 = \nabla\psi \times \nabla\theta + \nabla u \times \nabla\Phi \ , \tag{4}$$

where

$$\psi = \int d\Phi \, [q(\Phi) - q_o] - A \cong \frac{q_o' x^2}{2} - A \ , \tag{5}$$

where $x = \Phi - \Phi_o$, $q(\Phi_o) = q_o$, and $q_o' = dq/d\Phi$ evaluated at $\Phi = \Phi_o$. The second form for ψ is valid in the vicinity of the rational surface and we will use this form of ψ to describe the magnetic surfaces near the island. If we assume that the radial variation of A is weak (the constant-ψ approximation of tearing mode theory[14]) and assume that the $p = 1$ helicity dominates in Eq. (3), $A = \psi_{sx}\cos(n_o u)$, where ψ_{sx} is the value of ψ on the separatrix. This form of A describes a magnetic island with half-width $w = 2\sqrt{|\psi_{sx}/q_o'|}$.

Since we demanded that $A = A(\Phi,u)$, the magnetic field in the vicinity of the rational surface satisfies $\mathbf{B}\cdot\nabla\psi = 0$. Therefore, it is convenient to transform from the magnetic coordinates Φ and u to a new set of coordinates given by

$$\int d\Phi^* = \int \frac{d\psi}{\Omega(\psi)} \ , \tag{6}$$

$$\int du^* = \int n_o du \, \frac{\Omega(\psi)}{\partial_\Phi \psi(\Phi^*,u)} \ , \tag{7}$$

where $\Phi^* = \Phi^*(\psi)$ serves as a label of the magnetic surfaces modified by the magnetic island formation and u^* is an angle-like variable on the modified surfaces. The function $\Omega(\Phi^*)$ serves as the rotational transform for the magnetic surfaces near the magnetic island.

In order that u and u* have the same periodicity properties (i. e., as n_0u goes from 0 to 2π on a surfaces outside the separatrix, u* goes from 0 to 2π), the island rotational transform is given by

$$\Omega(\psi) = \frac{1}{\oint \frac{du}{2\pi/n_0} \frac{1}{\partial_\Phi \psi(\Phi^*,u)}} . \tag{8}$$

It is assumed that the magnetic island width is small compared to macroscopic length scales, so $q_0'w$ is a small quantity. Therefore, the value of Ω is small since it goes as $\Omega \sim O(q_0'w)$; however, $d\Omega/d\Phi^* \sim O(q_0')$ since $\Phi^* \sim O(w)$. In addition, the value of Ω goes to zero as the separatrix is approached. This describes the fact that field-lines on the separatrix stagnate near the X-point. Explicit forms for Φ^*, u* and Ω appear in the Appendix where the assumption that $A = \psi_{sx}\cos(n_0u)$ is made.

Using these coordinates, the equilibrium magnetic field in the presence of the magnetic island, Eq. (4), is given by

$$\mathbf{B} = \nabla u^* \times \nabla\Phi^* + \Omega(\Phi^*)\, \nabla\Phi^* \times \nabla\theta . \tag{9}$$

The Jacobian constructed from the magnetic coordinates is preserved in this transformation

$$\mathrm{J} = \frac{1}{\nabla u^* \times \nabla\Phi^* \cdot \nabla\theta} = \frac{1}{\nabla u \times \nabla\Phi \cdot \nabla\theta} , \tag{10}$$

and the magnetic differential equation $\mathbf{B}\cdot\nabla M = N$ is given by

$$\frac{1}{\mathrm{J}}\left(\frac{\partial M}{\partial\theta} + \Omega\frac{\partial M}{\partial u^*}\right) = N . \tag{11}$$

Finally, we note that since this is the equilibrium magnetic field, the condition $\mathbf{J} \times \mathbf{B} = \nabla p$ holds. This condition is also approximately true for slowly growing (as compared to Alfvenic and sound wave propagation times) magnetic islands. From magnetostatic equilibrium, we get the restriction that the pressure must be constant on a magnetic surface, so $p = p(\Phi^*)$.

III. Linear Stability of Ballooning Modes

The stability against large-n ballooning-type perturbations is addressed in this section. The mode number of the magnetic island is not connected to the mode number of the ideal mode. We use the equilibrium and magnetic coordinates introduced in the last section to describe the magnetic configuration.

The linearized ideal magnetohydrodynamic equations are

$$-\mathbf{B}\cdot\nabla\varphi_1 - \frac{B}{c}\frac{\partial A_{\parallel}}{\partial t} = 0 , \tag{12}$$

$$\mathbf{v}_1 = c\frac{\mathbf{B}\times\nabla\varphi_1}{B^2} , \tag{13}$$

$$\nabla_{\perp}^2 A_{\parallel} = -\frac{4\pi}{c} J_{\parallel} , \tag{14}$$

$$\mathbf{B}\cdot\nabla\frac{J_{\parallel}}{B} = -\, c\nabla\cdot\frac{\mathbf{B}\times\nabla p_1}{B^2} - c\nabla\cdot\frac{\mathbf{B}\times\rho\partial_t\mathbf{v}_1}{B^2} , \tag{15}$$

$$\frac{\partial p_1}{\partial t} = -\mathbf{v}_1\cdot\nabla p_0, \tag{16}$$

where $A_{\parallel} = \mathbf{A}_1\cdot\mathbf{B}/B$, $J_{\parallel} = \mathbf{J}_1\cdot\mathbf{B}/B$, and p_0 is the equilibrium pressure profile. Assuming B is inhomogeneous, a linear equation for φ_1 can be obtained

$$\frac{1}{v_a^2}\nabla_{\perp}^2\frac{\partial^2\varphi_1}{\partial^2 t} = \nabla_{\parallel}\nabla_{\perp}^2\nabla_{\parallel}\varphi_1 - \frac{8\pi}{B^3}\mathbf{B}\cdot\nabla B\times\nabla\left(\frac{\mathbf{B}\cdot\nabla\varphi_1\times\nabla p}{B^2}\right) , \tag{17}$$

where the 0 subscript has been dropped from the equilibrium pressure, $v_a = B/\sqrt{4\pi\rho}$ is the Alfven velocity and the parallel gradient in the vicinity of the island is given by

$$\nabla_{\parallel} = \frac{1}{qR_o}\left(\frac{\partial}{\partial\theta} + \Omega\frac{\partial}{\partial u^*}\right) \tag{18}$$

from Eq. (11) where the assumption $\mathcal{J} = qR_o/B_o$ is made. By making the usual eikonal representation, $\varphi_1 = \tilde{\varphi}\exp(inS)$, where $n \to \infty$ and $S = \int\Omega d\theta - u^*$, the ballooning mode δW is given by

$$\delta W = \int d\mathbf{r} \; \{|\nabla S|^2 |\nabla_{\|}\tilde{\varphi}|^2 + \frac{4\pi}{|\nabla\Phi^*|} \frac{dp}{d\Phi^*} (\kappa_n^* - \kappa_g^* \frac{\nabla S \cdot \nabla \Phi^*}{B}) |\tilde{\varphi}|^2\} \;, \tag{19}$$

where the curvature κ is decomposed into its normal and geodesic contributions

$$\kappa = \frac{\kappa_n \nabla\Phi + \kappa_g \mathbf{b}\times\nabla\Phi}{|\nabla\Phi|} = \frac{\kappa_n^* \nabla\Phi^* + \kappa_g^* \mathbf{b}\times\nabla\Phi^*}{|\nabla\Phi^*|} \;, \tag{20}$$

and the pressure $p = p(\Phi^*)$. To leading order in the small quantity $q_0'w$, $\kappa_n^* = \kappa_n$ and $\kappa_g^* = \kappa_g$.

For tokamak equilibria, the metric elements that enter into δW would be functions of the poloidal angle alone, so that minimizing the potential energy results from finding an appropriate eigenfunction that is a function of the poloidal angle alone. The problem of incompatibility between periodicity (in the poloidal angle) and magnetic shear is resolved by constructing the eigenfunction from a series of nonperiodic quasimodes.[15] In the present study the metric elements in Eq. (19) are functions of both θ and u^*, so that it is now necessary to construct eigenmodes in the following way

$$\tilde{\varphi} = \sum_{l_1 l_2 = -\infty}^{\infty} \varphi(\Phi^*, \theta + 2\pi l_1, u^* + 2\pi l_2) \;, \tag{21}$$

so that φ_1 is periodic in θ and u^*.

To simplify the calculation, assume that the flux surfaces in the limit that the island width goes to zero are given by concentric circles; $\Phi = \int dr\, rB_0/q(r)$, $\nabla\theta = \hat{\theta}/r$, $\mathbf{B}_0 \cdot \nabla r = 0$, $\kappa_n = -\cos\theta/R_0$, and $\kappa_g = \sin\theta/R_0$. With the island present, the term $|\nabla S|^2$ to lowest order in the small parameter $q_0'w$ is given by

$$|\nabla S|^2 \cong |\nabla u^*|^2 + \{\int d\theta \, (\frac{\partial\Omega}{\partial\Phi^*}) \, |\nabla\Phi^*|\}^2$$

$$= |\nabla u|^2 \{C^2 + [\int d\theta \, (\frac{<C^2>}{C} s - \frac{dp}{d\Phi^*} \frac{8\pi r R_0 q}{B_0} \cos\theta)]^2\}, \tag{22}$$

where the equilibrium magnetic shear is given by $s = rq^{-1}dq/dr$ evaluated at the rational surface, and the last term in the integral represents the pressure modulation of the local shear that is responsible for the second stability regime. The function C given by

$$C = \frac{\Omega(\Phi^*)}{\partial_\Phi \psi(\Phi^*, u^*)} = \frac{\partial u^*(\Phi^*, u)}{\partial u} \tag{23}$$

describes the effect of the magnetic island. The bracket is defined as an average over the modified magnetic surfaces.

$$\langle A \rangle = \oint \frac{du^*}{2\pi} A(\Phi^*, u^*) = \oint \frac{du}{2\pi} C(\Phi^*, u) A(\Phi^*, u) \; . \tag{24}$$

For constant–ψ islands $C = [x\langle x^{-1}\rangle]^{-1}$, so that it peaks on that point of the magnetic surface that is closest to the X-point of the island and the closer the magnetic surface is to the separatrix the more peaked it gets (see Fig. 1). With no island, C = 1 and one recovers the axisymmetric version of δW. Physically, what C describes is the fact that a field-line trajectory on a particular surface tends to preferentially reside on that part of the magnetic surface that is closest to the X-point of the island. In addition, the effect of the island essentially changes the shear parameter s to the function $s\langle C^2\rangle/C$. The average of this modified shear over a magnetic surface is enhanced by the factor $\langle C^2\rangle$ which is always greater than one. This effect was noted for the axisymmetric divertor separatrix in ref. 12. However, if one instead looks at this local value of shear as a function of the helical angle on a particular surface, it is smallest near the X-point. Since field-lines "spend a long time" near the X-point of the island, the field-line does not experience the global average of the shear on the magnetic surface.

The restriction on the pressure profile is that the pressure equilibrates on the modified surfaces. To construct the relation between the modified pressure gradient to the pressure gradient of the associated equilibrium without the magnetic island, equate the magnitude of ∇p for the two situations near the stagnation point of the magnetic surface (at the minimum value of $\partial_\Phi \psi$). This results in the condition

$$\frac{dp}{d\Phi^*} = \frac{dp_o}{d\Phi}\frac{\Omega}{\partial_\Phi \psi|_{min}} = \frac{dp_o}{dr}\frac{qC_M}{rB_o} , \tag{25}$$

where dp_o/dr is the pressure gradient at the rational surface in the limit that the island disappears and $C_M = C_M(\Phi^*)$ is the maximum of C on a modified magnetic surface with respect to the angle coordinate u*. Physically, this describes the compression of the magnetic surfaces near the magnetic island separatrix and is qualitatively the same effect described in Refs. (6) and (7), where a steep pressure profile is built up in the boundary layer around the separatrix.

The ballooning mode δW, Eq. (19) can now be rewritten using the results of Eqs. (20)–(25)

$$\delta W = \int dl \ \{(C^2 + \Lambda^2)|(\partial_\theta + \Omega\partial_{u^*})\varphi|^2 - \alpha C_M(C\cos\theta + \Lambda\sin\theta)|\varphi|^2\} , \tag{26}$$

where

$$\Lambda = \frac{<C^2>}{C} s(\theta - \theta_o) - \alpha C_M(\sin\theta - \sin\theta_o) , \tag{27}$$

and $\alpha = -(dp_o/dr)(q^2R_o/B_o^2)$. With no island, C = 1 and one recovers the usual ballooning mode δW which can be solved to find the stability boundaries using s-α diagrams.[15]

To construct an analytic solution of the stability boundaries, a trial function for φ is introduced. For the axisymmetric equilibria, consider the form[16]

$$\varphi(\theta) = (1 + \cos\theta)\,\Theta(\pi - |\theta|) , \tag{28}$$

where Θ is the Heaviside step function. Substituting Eq. (28) into (26) results in the expression for δW in the limit C = 1,[17]

$$\delta W = 0.5 + (\frac{\pi^2}{6} - \frac{1}{4}) s^2 - \frac{13}{6} s\alpha + \alpha^2 - \alpha . \tag{29}$$

This analytic solution is qualitatively similar to the numerical solution of the ballooning mode equation except for small values of s and α. Since we are interested in qualitative effects introduced by an equilibrium magnetic island, using an analytic trial function will serve our purposes adequately. Equation (29) gives two solutions for the critical α as a function of s,

$$\alpha_c = 0.5 + 1.08s \pm \sqrt{-0.25 + 1.08s - 0.22s^2}\ , \tag{30}$$

where the minus (plus) sign denotes the first (second) stability boundary (see Fig. 2). The existence of the second stable region occurs because of the pressure modulation of the local shear [the last term in Eq. (22)].[18] If we drop that term from Eq. (22) the resulting δW gives a single root for the critical α[17]

$$\alpha_c = \frac{0.5 + 1.39s^2}{1 + 0.83s}\ . \tag{31}$$

As shown in Fig. 2, this function reproduces the qualitative features of the first stability boundary. The quadratic dependence of s in the numerator of Eq. (31) describes the stabilizing effect of field-line bending, while the s-dependence in the denominator comes from the geodesic curvature.

In the present study we will consider the effect on the magnetic island on the first stability boundary. An appropriate choice of a trial function will reduce the stabilizing effect of shear. Since the inclusion of the pressure modulation term gives an unrealistic first stability boundary for the trial function we are considering for small values of s, we will drop the pressure modulation term and we will consider the changes of the stability boundary with respect to the critical α as defined by Eq. (31). Therefore, we consider the following δW,

$$\delta W = \int dl\ \{[C^2 + \frac{<C^2>^2}{C^2} s^2(\theta-\theta_0)^2]|(\partial_\theta + \Omega\partial_{u^*})\varphi|^2$$

$$- \alpha C_M[C\cos\theta + \frac{<C^2>}{C} s(\theta-\theta_0)\sin\theta]|\varphi|^2\}\ . \tag{32}$$

To demonstrate the qualitative effect of the island, let's consider the trial function

$$\varphi(\theta,u^*) = C(u^*)\,(1 + \cos\theta)\,\Theta(\pi - |\theta|)\ , \tag{33}$$

where C is defined by Eq. (23). The eigenfunction is now localized in helical angle as well as poloidal angle. Inserting the trial function, Eq. (33) into Eq. (32), the following critical α is derived

$$\alpha_c = \frac{1}{C_M} \frac{0.5<C^4> + 1.39s^2<C^2>^2}{<C^3> + 0.83s<C^2><C>} \quad . \tag{34}$$

Assuming a constant-ψ island, Eq. (34) is plotted for different magnetic surfaces in Fig. (3). The surfaces are labeled by the function k^2, which is given by $k^2 = w^2/(x_k^2 + w^2)$, where w is the magnetic island half-width and x_k is the distance from the X-point of the island to the closest point on the magnetic surface, $x_k^2 = (2q_0/q_0')(\psi - \psi_{sx})$. Figure (3) shows the qualitative effect of the magnetic island through the reduction of the shear stabilization. In addition, those surfaces closest to the island separatrix are the most affected.

IV. Summary and Discussion

An analytic theory has been developed to study ideal ballooning instabilities for quasistatic equilibria with a magnetic island. Because of the special feature the magnetic field-line trajectories have on toroidal magnetic surfaces near the magnetic island, ballooning instabilities are more likely to occur than in the equilibria without the magnetic island. A trial function is used in the ballooning mode δW that is localized near the X-point region of the modified magnetic surface, which reduces the stabilizing effect of field-line bending.

This work suggests that equilibria with magnetic islands are more susceptible to ballooning instabilities than the corresponding equilibria without the island. For a slowly evolving island, a growing region of magnetic surfaces near the magnetic separatrix becomes ballooning unstable. For instance, consider a toroidal equilibrium with a rational surface which is in the first stable region. Now allow an island to slowly grow at that magnetic surface. As can be seen in Fig. 3, if the initial toroidal equilibrium is not too far from the first stability boundary, there will a critical value of k which defines a first stability boundary in the equilibrium with the magnetic island. All surfaces between the separatrix and the surface labeled by k_{crit} will be ballooning unstable. In real space this region is given by $x_{crit} = w\sqrt{k_{crit}^{-2} - 1}$ which grows linearly with the magnetic island width.

These island modification of ballooning instabilities may also be important in understanding the nonlinear evolution of m = 1 modes and sawteeth crashes. A possible scenario for tokamak discharges with q(0) < 1 is that a slowly growing magnetic island forms due to m = 1 resistive instability. As suggested above, this causes a growing region of space around the separatrix of the island that has "secondary" ideal instabilities. Because these are ideal instabilities plasma pressure can be released without a complete reconnection of the magnetic surfaces and q(0) could remain below 1. This explanation is reminiscent of earlier work by Bussac and co-workers which suggested that ballooning modes are important in the nonlinear phase of ideal kinks[6] or following a secondary kink instability in a discharge with an m = 1 resistive mode.[19] What is different here is the suggestion that the presence of a magnetic island in the equilibrium makes ideal ballooning modes more unstable and localizes them near the X-point region in helical angle and on the outside of the torus. This work does not give a complete description of the sawtooth crash phase; however, recent studies on JET[20] and TFTR[21] suggests that some of the elements of the present theory (ballooning-like deformations, a fast instability preceded by a slow precursor instability) are consistent with experimental observations of sawteeth.

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Appendix: Explicit Forms for Constant-ψ Islands

For constant-ψ islands, we take $A = \psi_{sx}\cos(n_0 u)$. This gives $\psi = q_0' x^2/2 - \psi_{sx}\cos(n_0 u)$, and $\partial_\Phi \psi = q_0' x(\psi,u) = \pm\sqrt{2q_0'[\psi+\psi_{sx}\cos(n_0 u)]}$, where the plus (minus) sign refers to x > 0 (x < 0). In this calculation we are concerned with the magnetic surfaces outside the separatrix. We introduce the surface label k given by

$$k^2 = \frac{2\psi_{sx}}{\psi + \psi_{sx}} , \tag{A1}$$

where k = 1 is the separatrix and k < 1 away from the separatrix. From Eq. (8), we get the value of the island rotational transform

$$\Omega(k) = \pm\frac{\pi q_0' w}{2kK(k)} , \tag{A2}$$

where K(k) is the complete elliptic integral of the first kind and $w = 2\sqrt{|\psi_{sx}/q_0'|}$ is the island half-width. Since $K(1) = \infty$, Ω goes to zero at the separatrix. The magnetic coordinates are given by

$$\Phi^* = \pm\frac{2wE(k)}{\pi k} , \tag{A3}$$

$$u^* = \frac{\pi}{K(k)} F(n_0 u/2,k) , \tag{A4}$$

where E(k) is the complete elliptic integral of the second kind and $F(n_0 u/2,k)$ is the elliptic integral of the first kind. The island rotational transform has the same sign as Φ^*; the plus (minus) sign in Eqs. (A2) and (A3) refers to x > 0 (x < 0). Notice that as $n_0 u$ goes from 0 to π, u* goes from 0 to π as demanded from the definition of u*. The shear of the island transform is given by

$$\frac{d\Omega}{d\Phi^*} = \frac{q_0'\pi^2 E(k)}{4K^3(1-k^2)} = q_0' \langle C^2 \rangle . \tag{A5}$$

The function C is given by

$$C(k,u) = \frac{\pi}{2K(k)} \frac{1}{\sqrt{1 - k^2 \sin^2(n_o u/2)}} , \qquad (A6)$$

$$C(k,u^*) = \frac{\pi}{2K(k)} \frac{1}{dn(2u^*K/\pi)} ,$$

where $dn(2u^*K/\pi)$ is the Jacobian elliptic function. The averages of integral multiples of C are given by

$$<C^m> = \frac{f_m(k)}{[2K(k)/\pi]^{m+1}(1-k^2)^{m/2}} , \qquad (A7)$$

where $f_m < f_n$ for $m > n$ and $n \geq 0$. The first couple f_m's are given by

$$f_0 = \frac{2}{\pi} K(k) , \; f_1 = 1 , \; f_2 = \frac{2}{\pi} E(k) , \; f_3 = 1 - k^2/2 ,$$

$$f_4 = \frac{2}{\pi} (\frac{4}{3} - \frac{2k^2}{3}) E(k) - \frac{2}{\pi} (\frac{1-k^2}{3}) K(k) \; , f_5 = 1 - k^2 + 3k^4/8 , \qquad (A8)$$

and $C_M = \pi/2K(k)\sqrt{1-k^2}$.

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Figure Captions

Fig. 1 – The function C versus the angle α for different magnetic surfaces given by $k^2 = 0$, 0.5, and 0.9. The horizontal line is $C = 1$ corresponding to $k = 0$ (no equilibrium island). The curve for $k^2 = 0.9$ is the one that peaks the most prominently.

Fig. 2 – The ballooning mode stability boundary in s-α space as given by the analytic expressions Eqs. (30) and (31). The curve with two values for α_c as a function of s represents Eq. (30), with the lower curve indicating the first stability boundary while the upper curve indicates the second stability boundary. The single-valued curve represents Eq. (31), the critical α when the pressure modulation part of the local shear is ignored in δW. This curve reproduces the correct qualitative features of the first stability boundary.

Fig. 3 – The first stability boundary for different the magnetic surfaces labeled by $k^2 = 0$ (no island), 0.5, 0.9, and 0.99. The larger the value of k, the smaller the critical α for instability.

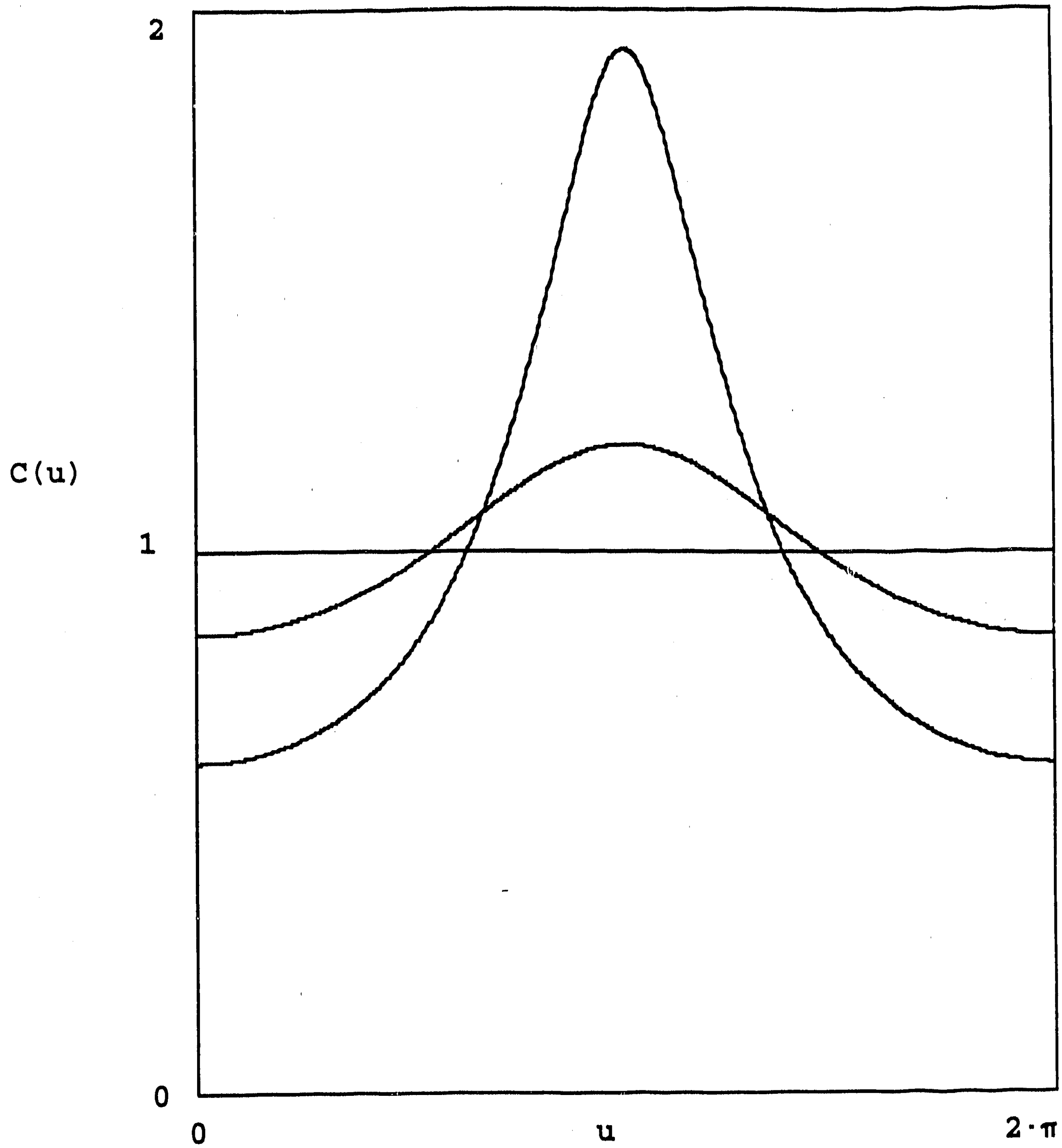

Fig.1

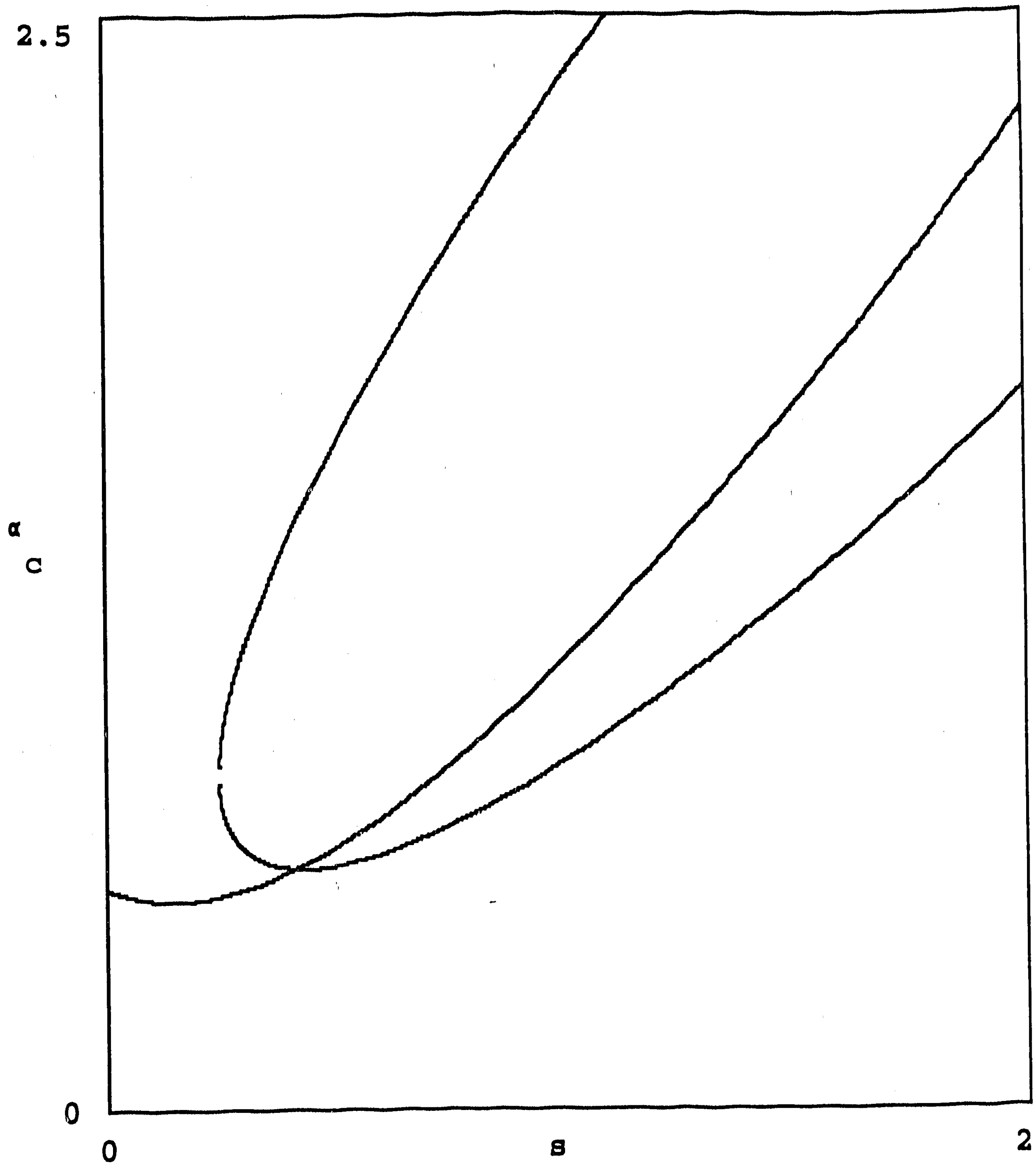

Fig.2

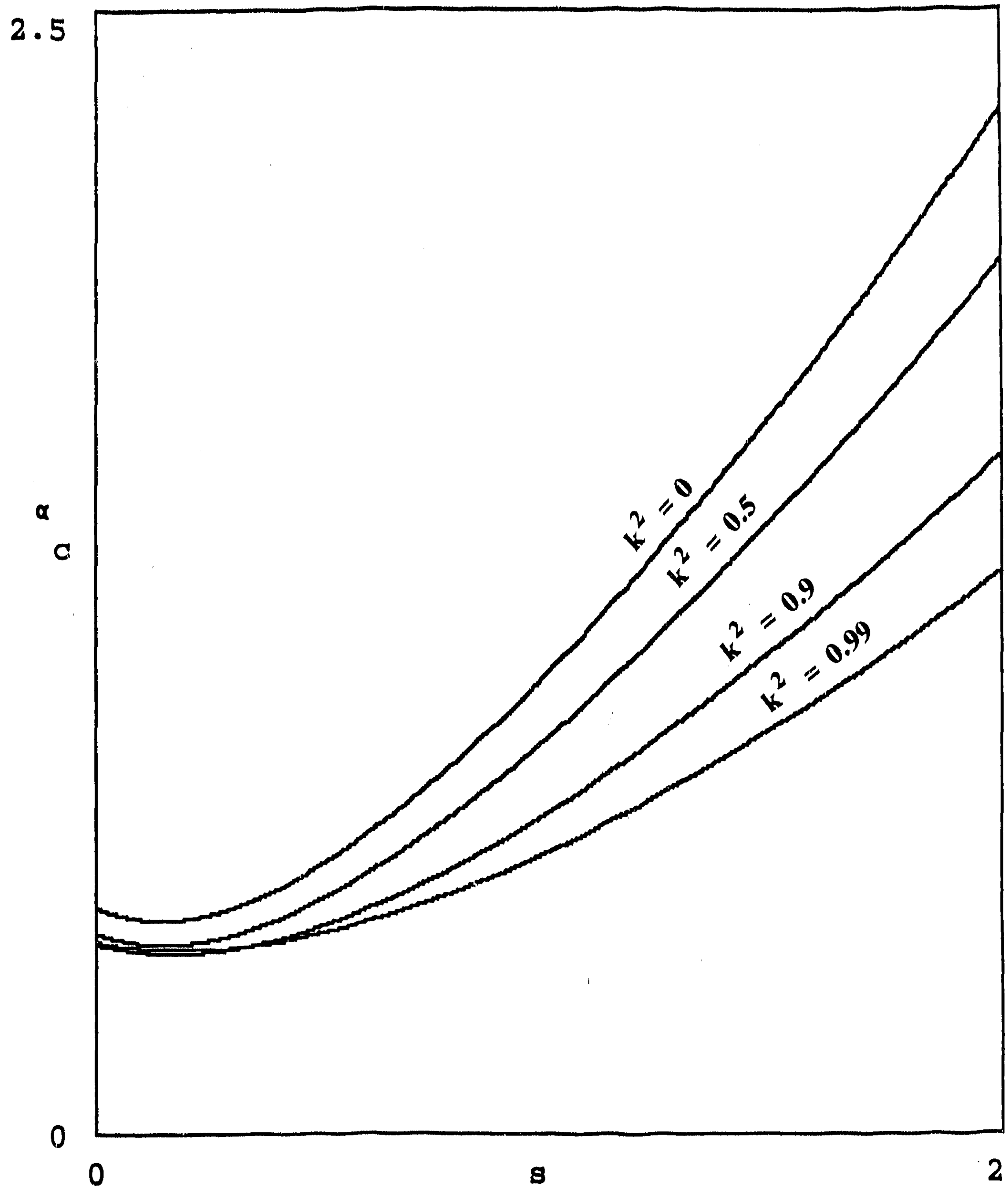

Fig.3

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