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ABSTRACT

The mechanics of flowing granular materials such as coal, sand, fossil-fuel energy recovery, metal ores, etc., and their flow characteristics have received considerable attention in recent years because it has relevance to several technological problems. In a number of instances these materials are also heated prior to processing, or cooled after processing. The governing equations for the flow of granular materials taking into account the heat transfer mechanism are derived using the continuum model proposed by Rajagopal and Massoudi (1990). For a fully developed flow of granular materials down an inclined plane, these equations reduce to a system of coupled ordinary differential equations. The resulting boundary value problem is solved numerically and the results are presented. For a special case, it is possible to obtain an analytic solution; this is given in the appendix A of this report.

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LIST OF SYMBOLS

D	Stretching Tensor
K	Thermal Conductivity
L	Velocity Gradient
T	Cauchy Stress Tensor
b	body force
h	Characteristic Height
q	Heat Flux
r	Radiating Heat
α	Angle of Inclination of the Inclined Plane with the Horizontal
ε	Specific Internal Energy
γ	Distributed Mass Density
Θ	Temperature
ν	Volume Fraction
ρ	Bulk Mass Density

1.0 INTRODUCTION

In recent years there has been considerable interest in understanding the behavior of granular materials because of its relevance to several technological problems. This includes the handling of such substances as coal, agricultural products, fossil-fuel energy recovery, metal ores, crushed oil shale, dry chemicals, rocket propellants, fertilizers, cement, sand and other particulate solids; the process of fluidization of coal particles and its effect on combustion; and the mechanics of avalanches and other natural disasters that involve the flow of powders and bulk solids. In addition, flowing granular streams are being considered for some advanced concepts for solar power plants and fusion reactor chambers. Many situations, such as discharge through bin outlets, flow through hoppers and chutes, pneumatic transport of coal, fluidized beds, *etc.*, require information on material properties of those particles, flow patterns, concentration profiles, *etc.*, [Rajagopal and Massoudi (1990)]. Extensive research has been devoted to the mechanics of flowing granular materials and their flow characteristics [Savage (1984)].

In a number of applications, these materials are also heated prior to processing, or cooled after processing [Patton, *et al.*, (1986)]. Very little fundamental work, from a mathematical point of view, has been devoted to these types of heat transfer in granular materials. These contact dominated (dense-phase) flows have applications in certain industrial equipment designed to heat, cool, or dry granular materials [Uhl and Root (1967)]. Sullivan and Sabersky (1975) studied the heat transfer from a flat plate to various granular materials. They compared their experimental results to an idealized model that they called the "discrete particle model." Later, Spelt *et al.* (1982) generalized this problem by studying the heat transfer to granular materials flowing along an inclined chute at higher velocities. Based on their experimental results, they speculated that the higher velocities caused a decrease in the density of the material and that decrease in density caused the reduction of the heat transfer. Patton *et al.* (1986) continued this research by extending the range of conditions investigated previously; an experimental technique was developed to allow density measurements, since the density changes are significant in the rapid flows of granular materials. They proposed a model which generalized Sullivan and Sabersky's (1975) correlation to include density variation. Also, a major challenge facing the designers of coal gasification plants is to assure reliable and efficient movement of solids into and out of high-pressure, high-temperature fluidized-bed processing units.

Granular materials are unlike solids in that they conform to the shape of the vessel containing them, thereby exhibiting fluidlike characteristics. On the other hand, they cannot be considered a fluid, because they can be heaped. The characteristics of the particles that constitute the bulk solids are probably of major importance in influencing the characteristics of the bulk solids, both at rest and during flow. Also, characterizing bulk solids, which are composed of a variety of materials, is difficult mainly because small variations in some of the primary properties of the bulk solids such as size, shape, hardness particle density and surface roughness can result in very different behavior. Furthermore, secondary factors such as the presence or absence of moisture, the severity of prior compaction, the ambient temperature *etc.*, which are not directly associated with the particles, can have a significant effect on the behavior of the bulk solids.

The concept of "granular materials" covers the combined range of granular powders and granular solids with components ranging in size from about 10μm up to 3mm. A powder is composed of particles up to 100μm (diameter) with further subdivision into ultrafine (0.1 to 1.0μm), superfine (1 to 10μm), or granular (10 to 100μm) particles. A granular solid consists of materials ranging from about 100 to 3,000μm [Brown (1970)]. Also, little is known of the relationship between particle shape and flow properties in detail although it is observed that smooth spherical particles display more favourable flow conditions than the particles with a sharp angular surface, especially if they have a tendency to interlock. In addition, moisture content of bulk solids is the most important factor controlling the flow properties of the granular materials. Moisture content in bulk solids may be due to inherent moisture in which water is chemically bound to the grains; hygroscopic moisture, which is absorbed by the mass of solids from air; and surface moisture, which forms a wet film on the surface of the particles and fills the spaces between the particles. In fact, moisture content in bulk solids is undesirable, because the surface moisture leads to the appearance of cohesive forces between particles of solids and of adhesive forces between particles and the walls of the container. Both retard the flow of solids and under certain conditions may stop the flow entirely. Since, for the same weight of solid, the total surface of solids is greater for small grains, the surface moisture content increases inversely as the particle diameter. For that reason, fine particles display more cohesive and adhesive forces than the larger grain solids. Furthermore, fine particles, when stored undisturbed for a certain time have a tendency to compact (*i.e.* to reduce the total volume), which creates additional resistance to the flow. Thus, the amount of fine particles and moisture controls the gravity flow of solids. In general, the flow properties of most materials can be expected to decrease drastically as moisture content increases, particularly for finer materials [cf. Stepanoff (1969)].

It is thus apparent that modeling granular materials would require a fusion of the ideas from solid, fluid, and soil mechanics. Moreover, granular materials exhibit phenomena like normal stress differences in simple shear flow, a characteristic of non-Newtonian fluids and non-linearly elastic solids. Thus, modeling granular materials and slurries is very complex and has to draw upon our experiences from non-linear fluid and solid theories. One of the approaches used in the modeling of granular materials is the **continuum approach**, which assumes that the material properties of the ensemble may be represented by continuous functions so that the medium may be divided infinitely without losing any of its defining properties . Continuum plasticity type models based on the Mohr-Coulomb criterion for failure have been proposed by Drucker & Prager (1952), Shield (1955), Drucker, Gibson, & Henkel (1957), Jenike & Shield (1959), Spencer (1964) and Jenike (1964). A continuum model for flowing granular materials was proposed by Goodman & Cowin (1971, 1972) and later theories developed by other investigators in this area [cf. Cowin (1974[a,b]), Savage (1979, 1983b), Jenkins (1975, 1980), Ahmadi (1982a, 1982b), McTigue (1982), Nunziato, *et al.* (1980), and Passman, *et al.* (1980), Massoudi *et al.* (1991)] use the basic ideas suggested by Goodman & Cowin (1971, 1972) with modifications, improvements, and generalizations. The other approach used in the modelling of granular materials is the **kinetic theory approach**, which is generally used in the modelling of rapidily flowing granular materials [cf. Ackerman and Shen (1981), Jenkins and Savage (1984), Ahmadi and Shahinpoor (1984), Hutter (1986), Boyle and Massoudi (1989, 1990), Jenkins and Richman (1985)].

2.0 GOVERNING EQUATIONS

The granular material is treated as a continuum and its stress tensor is modelled as proposed by Rajagopal and Massoudi (1990). This model was originally proposed by Goodman & Cowin (1971), Cowin (1974) and Savage (1979), and has been modified by Rajagopal and Massoudi (1990). The model is

$$\mathbf{T} = \{ \beta_0(\nu) + \beta_1(\nu)\, \nabla\nu \cdot \nabla\nu + \beta_2(\nu)\, tr\mathbf{D} \}\, \mathbf{1} \\ + \beta_4(\nu)\, \nabla\nu \otimes \nabla\nu + \beta_3(\nu)\, \mathbf{D}. \tag{2-1}$$

In the above equation, **T** denotes the Cauchy stress tensor, ν the volume fraction of the solid, **D** denotes the stretching tensor associated with the solid motion, $\beta_0(\nu)$ is similar to pressure in a compressible fluid and is given by an equation of state, $\beta_2(\nu)$ is akin to the second coefficient of viscosity in a compressible fluid, $\beta_1(\nu)$ and $\beta_4(\nu)$ are the material parameters that reflect the distribution of the granular materials, and $\beta_3(\nu)$ is the viscosity of the granular materials. The above model allows for normal-stress differences, a feature observed in granular materials. In general, the material properties β_0 through β_4 are functions of the density (*or* volume fraction ν), temperature, and the principal invariants of the stretching tensor **D**, given by

$$\mathbf{D} = \frac{1}{2}\,[\,(grad\mathbf{u}) + (grad\mathbf{u})^T\,], \tag{2-2}$$

where **u** is the velocity of the particles. In equation (2-1), **1** is the identity tensor, ∇ the gradient operator, $\otimes$ indicates the outer (dyadic) product of two vectors, and *tr* designates the trace of a tensor. Furthermore, ν is related to the bulk density of the material ρ, through

$$\rho = \gamma\,\nu, \tag{2-3}$$

where γ is the actual density of the grains at the place $\mathbf{x}$ and time $\mathbf{t}$ and the field ν is called the volume fraction (*or* the volume distribution) and is related to the porosity n *or* the void ratio e by

$$\nu = 1 - n = \frac{1}{1+e}; \qquad \textit{with } 0 \leq \nu < 1 \tag{2-4}$$

Rajagopal and Massoudi (1990) have shown that in general the material parameters can have the structure

$$\begin{aligned} \beta_0(\nu) &= k\nu \\ \beta_1(\nu) &= \beta_{10} + \beta_{11}\nu + \beta_{12}\nu^2 \\ \beta_2(\nu) &= \beta_{20} + \beta_{21}\nu + \beta_{22}\nu^2 \\ \beta_3(\nu) &= \beta_{30} + \beta_{31}\nu + \beta_{32}\nu^2 \\ \beta_4(\nu) &= \beta_{40} + \beta_{41}\nu + \beta_{42}\nu^2 \end{aligned} \tag{2-5}$$

Further restrictions on the coefficients can be obtained by using the following argument, that the stress should vanish as $\nu \rightarrow 0$, thus

$$\beta_{30} = \beta_{20} = 0 \tag{2-6}$$

Also, the special case of

$$\beta_0(\nu) = 0$$

and $\beta_1(\nu)$ through $\beta_4(\nu)$ being constants, amends itself an exact solution. This solution is provided in the Appendix A of this report. In this study, we assume a special case of equation (2-5) where β_1 through β_4 are assumed to be constants.

We now consider the flow of granular material modeled by the above continuum model down an inclined plane (cf. Figure A) due to the action of gravity. This problem has been studied extensively [cf. Goodman & Cowin (1971), Savage (1979), Johnson and

Jackson (1987) and Richman and Marciniec (1991)]. In this problem, we consider steady one dimensional flow of incompressible granular materials (*i.e.* γ = constant) down an inclined plane, where the angle of inclination is α. Let us further suppose that the inclined plane is maintained at a constant temperature Θ_w, which is at a higher temperature than the temperature of the surrounding environment Θ_∞, and, as a result, there is transfer of heat. We assume that the heat flux vector **q** satisfies Fourier's law with constant thermal conductivity, *i.e.*,

$$\mathbf{q} = -K\,\nabla\Theta, \tag{2-7}$$

where Θ is the temperature and K is the thermal conductivity, which in general is a function of ν, but in the present problem it is assumed to be constant. The effect of radiation is neglected. Also, we assume that β_1 through β_4 are constants and

$$\beta_0 = k\,\nu \tag{2-8}$$

The governing equations of motion are the conservation of mass, momentum, and energy. The conservation of mass is

$$\frac{\partial \rho}{\partial t} + div(\rho\,\mathbf{u}) = 0, \tag{2-9}$$

where $\frac{\partial}{\partial t}$ is the partial derivative with respect to time. The balance of linear momentum is

$$div\mathbf{T} + \rho\,\mathbf{b} = \rho\,\frac{d\mathbf{v}}{dt}, \tag{2-10}$$

where $\frac{d}{dt}$ is the material time derivative and **b** is the body force. The energy equation in its general form is

$$\mathbf{T.L} - div\mathbf{q} + \rho\,r = \rho\,\frac{d\varepsilon}{dt}, \tag{2-11}$$

where ε denotes the specific internal energy, **q** the heat flux vector, r is the radiating heat, and **L** is the velocity gradient. For the problem under consideration, we make the following assumptions:

- Steady motion
- Incompressible granular materials, *i.e.*, γ = constant
- Negligible radiant heating, *i.e.*, $r = 0$
- The material properties β_1 through β_4^* are constants and β_0 is given by equation (2-8).
- The constitutive equation for the stress tensor is given by equation (2-1), and the constituitve equation for the heat flux vector is that of the Fourier's law, given by equation (2-7), with constant thermal conductivity.*
- The density, velocity, and temperature fields are assumed to be of the form

$$\begin{aligned} \nu &= \nu(y) \\ u &= u(y) \\ \Theta &= \Theta(y) \end{aligned} \tag{2-12}$$

With the above assumptions, the conservation of mass is identically satisfied and the balance of linear momentum reduces to

$$k\frac{d\nu}{dy} + 2\,(\beta_1 + \beta_4)\frac{d\nu}{dy}\frac{d^2\nu}{dy^2} = \gamma\, g\, \nu\, cos\alpha, \tag{2-13}$$

and,

$$\beta_3\frac{d^2u}{dy^2} = -2\,\gamma\, g\, \nu\, sin\alpha, \tag{2-14}$$

where g denotes the acceleration due to gravity and α is the angle of inclination of the plane. Also the energy equation becomes

$$2\,K\frac{d^2\Theta}{dy^2} = -\beta_3\left\{\frac{du}{dy}\right\}^2 \tag{2-15}$$

We need to solve equations (2-13), (2-14), and (2-15) subject to the appropriate boundary conditions. These boundary conditions are:

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*In a related work Gudhe *et al.* (1992) are studying this problem when β_3 and K are quadratic functions of ν

$$u = 0$$

$$\nu = \nu_0 \qquad at\ y = 0\ (on\ inclined\ plane) \qquad (2\text{-}16)$$

$$\Theta = \Theta_w$$

and,

$$\frac{du}{dy} = 0$$

$$k\,\nu + (\beta_1 + \beta_4)\left\{\frac{d\nu}{dy}\right\}^2 = 0 \qquad at\ y = h\ (at\ the\ free\ surface) \qquad (2\text{-}17)$$

$$\Theta = \Theta_\infty$$

Notice, the boundary condition at $y = 0$ for the volume fraction of the solid has to be prescribed, and this can be done on the basis of an experimental measurement or using kinetic theories of granular material [Hui, *et al.* (1984), Jenkins and Richman (1986), Johnson and Jackson (1987), Johnson, *et al.* (1990) and Jenkins (1991)]. On the other hand, we could also specify an average volume fraction defined through

$$Q = \int_0^h \nu\, dy\,. \qquad (2\text{-}18)$$

This quantity is easier to measure than the value ν_0 at $y = 0$. Equation $(2\text{-}16)_2$ and (2-18) are different for it is possible that there is more than one value of ν_0 and distribution $\nu(y)$ which corresponds to Q. Thus, the more general prescription would be by (2-18). Also, notice that equations $(2\text{-}17)_{1,2}$ are the stress free conditions and equation $(2\text{-}16)_1$ indicates the no-slip condition (rough wall) assumption.* Further, Rajagopal and Massoudi (1990) and Rajagopal *et al.* (1992) have shown that

$$k < 0. \qquad (2\text{-}19)$$

Now, the system of equations (2-13), (2-14), and (2-15) subject to the boundary conditions $(2\text{-}16)_1$, $(2\text{-}16)_3$, (2-18), and (2-17) are non-dimensionalized by

*In the present problem we are assuming the no-slip condition at the wall; however, some experiments indicate that there is indeed slip at the wall. In a related work Gudhe *et al.* (1992), we will address this point.

$$\bar{y} = \frac{y}{h}; \qquad \bar{u} = \frac{u}{U}; \qquad \bar{\Theta} = \frac{\Theta - \Theta_\infty}{\Theta_w - \Theta_\infty}, \tag{2-20}$$

where h is a characteristic length and U is a reference velocity. Now, the above system of equations reduce to

$$R_1 \frac{d\nu}{d\bar{y}} + R_2 \frac{d\nu}{d\bar{y}} \frac{d^2\nu}{d\bar{y}^2} = \nu \cos\alpha \tag{2-21}$$

$$R_3 \frac{d^2\bar{u}}{d\bar{y}^2} = -\nu \sin\alpha \tag{2-22}$$

$$\frac{d^2\bar{\Theta}}{d\bar{y}^2} = -R_4 \left\{\frac{d\bar{u}}{d\bar{y}}\right\}^2, \tag{2-23}$$

and the boundary conditions become

$$\bar{u} = 0$$

$$\bar{\Theta} = 1 \qquad at\ \bar{y} = 0\ (on\ inclined\ plane) \tag{2-24}$$

$$N = \int_0^1 \nu \, d\bar{y} \tag{2-25}$$

and,

$$\frac{d\bar{u}}{d\bar{y}} = 0$$

$$R_1 \nu + \frac{R_2}{2} \left\{\frac{d\nu}{d\bar{y}}\right\}^2 = 0 \qquad at\ \bar{y} = 1\ (at\ the\ free\ surface) \tag{2-26}$$

$$\bar{\Theta} = 0$$

Now, the non-dimensional parameters R_1, R_2, R_3, and R_4 are given by

$$R_1 = \frac{k}{h\gamma g}; \qquad R_2 = \frac{2(\beta_1 + \beta_4)}{h^3 \gamma g}$$

$$R_3 = \frac{\beta_3 U}{2h^2 \gamma g}; \qquad R_4 = \frac{\beta_3 U^2}{2K(\Theta_w - \Theta_\infty)} \qquad (2\text{-}27)$$

These dimensionless parameters do have physical interpretations. R_1 could be thought of as the ratio of the pressure force to the gravity force. R_2 is the ratio of volume distribution force to the gravity force. R_3 is the ratio of the viscous force to the gravity force (related to Reynolds number) and R_4 is the product of the Prandtl number and the Eckert number. Since $k < 0$, R_1 can only have negative values, and since $\beta_3 > 0$, R_3 and R_4 are only given positive values.

2.1 NUMERICAL RESULTS AND CONCLUSIONS

The system of equations (2-21), (2-22), and (2-23) with the boundary conditions (2-24), (2-25), and (2-26) and subject to the restriction (2-19) are solved numerically using a collocation method. It follows from Rajagopal and Massoudi (1990) that R_1 must always be less than zero. Furthermore, R_3 and R_4 are always positive, since the viscosity β_3 is a positive quantity. The paramater of interest here is R_2. Rajagopal, *et al.* (1992) have been able to prove the existence of solutions to the system of equations (2-21), (2-22), and (2-23) subject to boundary conditions (2-24), (2-25), and (2-26), only when $R_2 > 0$. They find that these equations admit non-unique solutions, one in which the volume fraction monotonically increases and the other in which it monotonically decreases from the inclined plane to the free surface. By observing the exact solution obtained for the volume fraction given in the appendix A, we can see that the volume fraction profile increases monotonically from the inclined plane to the free surface. This solution is not physically expected, but by carrying out the stability analysis, we can check whether this solution is stable or not. A parametric study of the numerical solutions of volume fraction, velocity and temperature profiles is carried out and the results are presented in the form of graphs.

The parameter R_1 affects the volume fraction, velocity, and temperature profiles as shown in Figures 1, 3, and 5. Notice, the volume fraction profile decreases from the surface of the plane to the free surface, which is physically expected. Increasing values of R_1 results in an increase of velocity and temperature. The profiles of gradient of volume fraction, velocity gradient and temperature gradient are shown in Figures 2, 4 and 6.

The parameter R_2 affects the volume fraction, velocity, and temperature profiles as shown in Figures 7, 9, and 11. Increasing values of R_2 results in an increase in velocity and temperature. The profiles of gradient of volume fraction, velocity gradient and temperature gradient are shown in Figures 8, 10, and 12. It should be noted that the parameters R_1 and R_2 do not directly influence the velocity and temperature profiles, as a glance at equations (22) and (23) reveals. That is, varying R_1 and R_2 changes the volume fraction profiles, which in turn influences the velocity and temperature profiles. The affect of α on the volume fraction shown in Figure 13 is not that pronounced. It appears that due to the volume fraction integral condition [equation (2-25)], the particles re-adjust themselves to the flow. However, the affect of α on velocity and temperature profiles is very significant, as shown in Figures 14 and 16. It can be seen as α increases the velocity and temperature increases. The velocity gradient profile is shown in Figure 15.

The parameter R_3 affects the velocity profile as shown in Figure 17. Notice, that the velocity increases as R_3 decreases. The affect of R_4 on temperature and temperature gradient profiles is shown in Figures 18 and 19. Here, as R_4 increases, the temperature increases and does not have any effect on volume fraction and temperature profiles. The Parameter N affects the volume fraction, velocity and temperature profiles as shown in Figures 20, 21, and 22. Increasing values of N results in an increase in volume fraction, velocity, and temperature.

2.1.1 ACKNOWLEDGEMENT:

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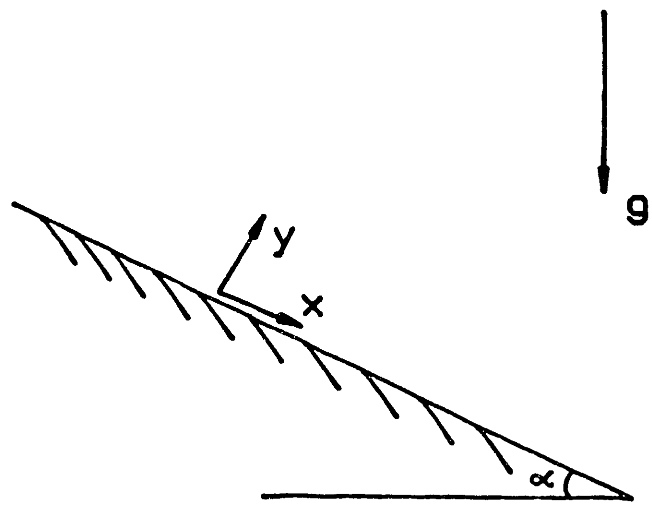

Fig. A. Flow Down Inclined Plane

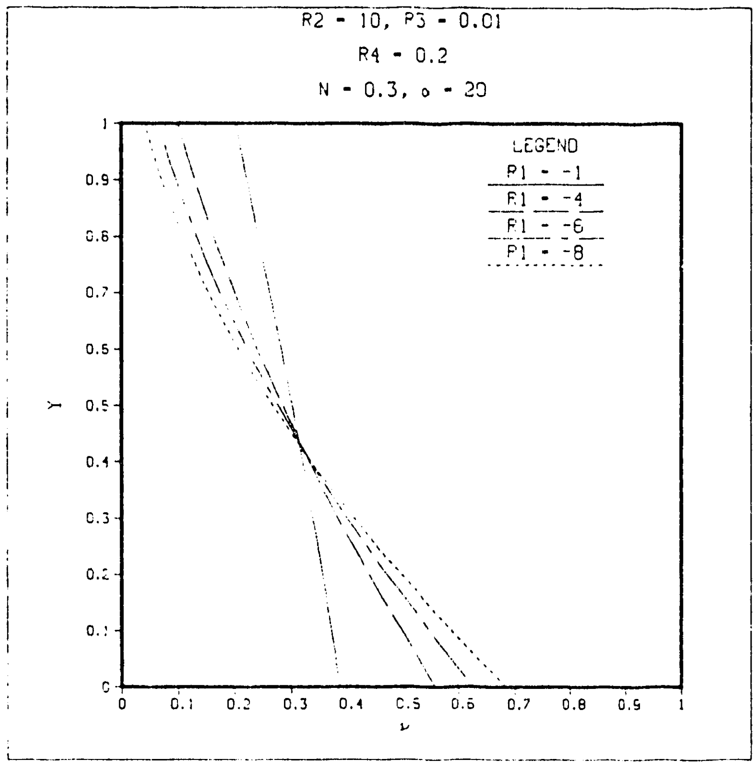

Figure 1. Effect of R1 on Volume Fraction

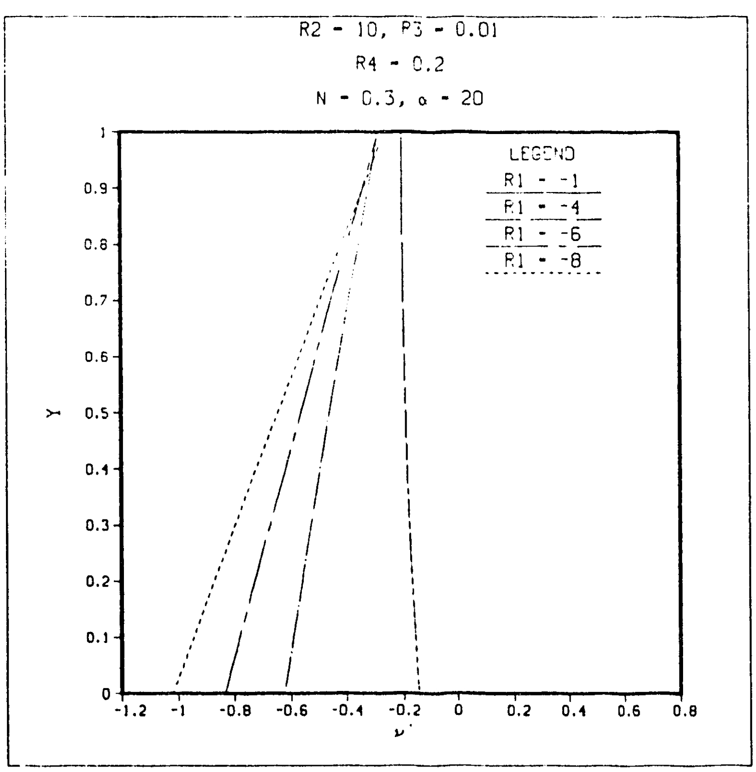

Figure 2. Effect of R1 on Gradient of Volume Fraction

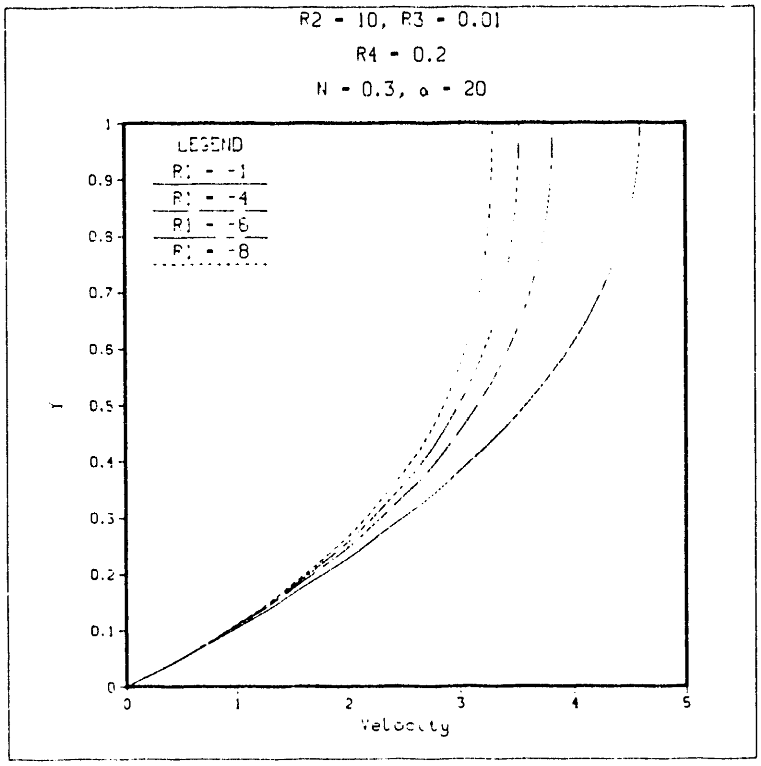

Figure 3. Effect of R1 on Velocity Profile

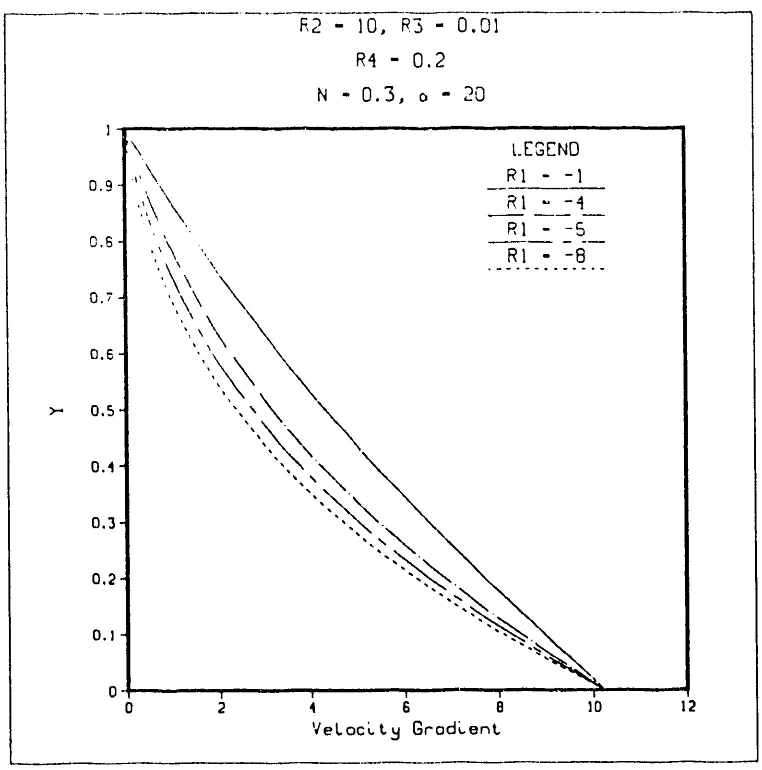

Figure 4. Effect of R1 on Velocity Gradient

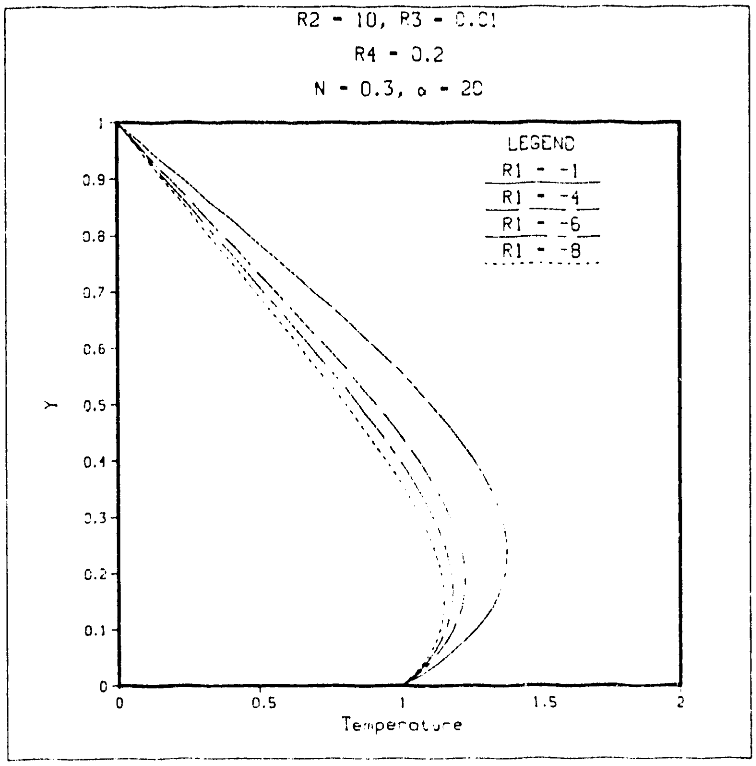

Figure 5. Effect of R1 on Temperature Profile

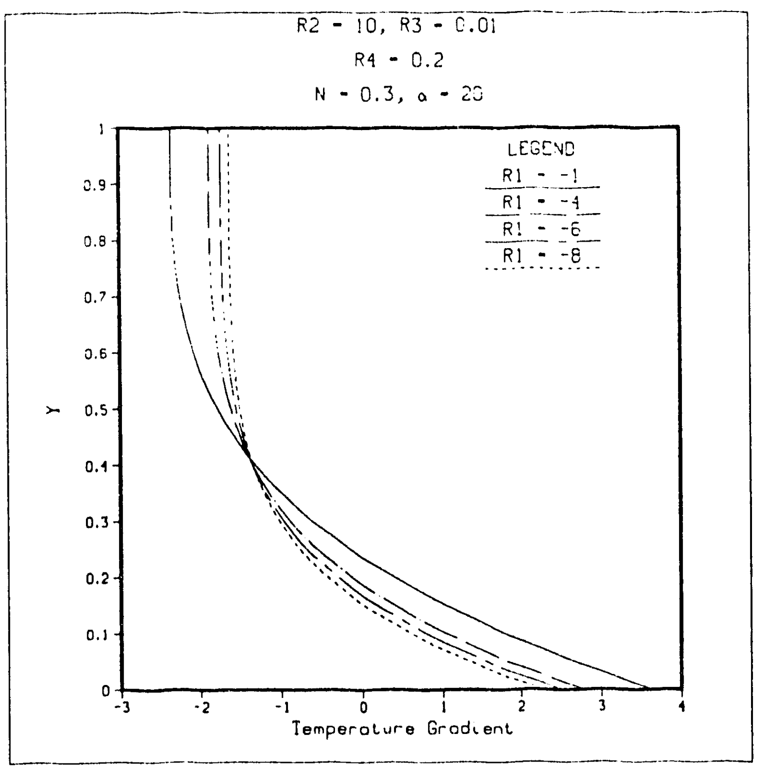

Figure 6. Effect of R1 on Temperature Gradient

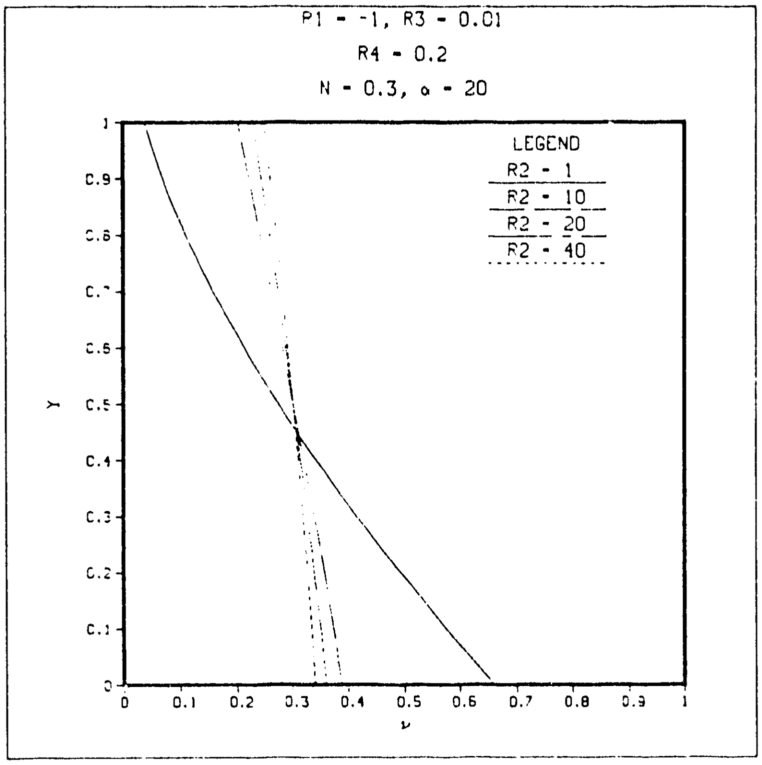

Figure 7. Effect of R2 on Volume Fraction

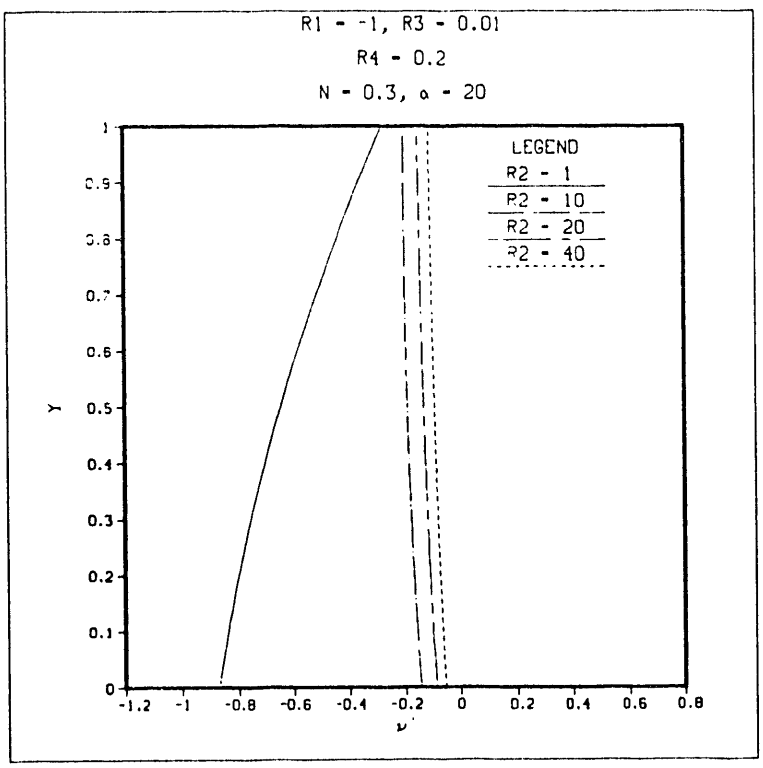

Figure 8. Effect of R2 on Gradient of Volume Fraction

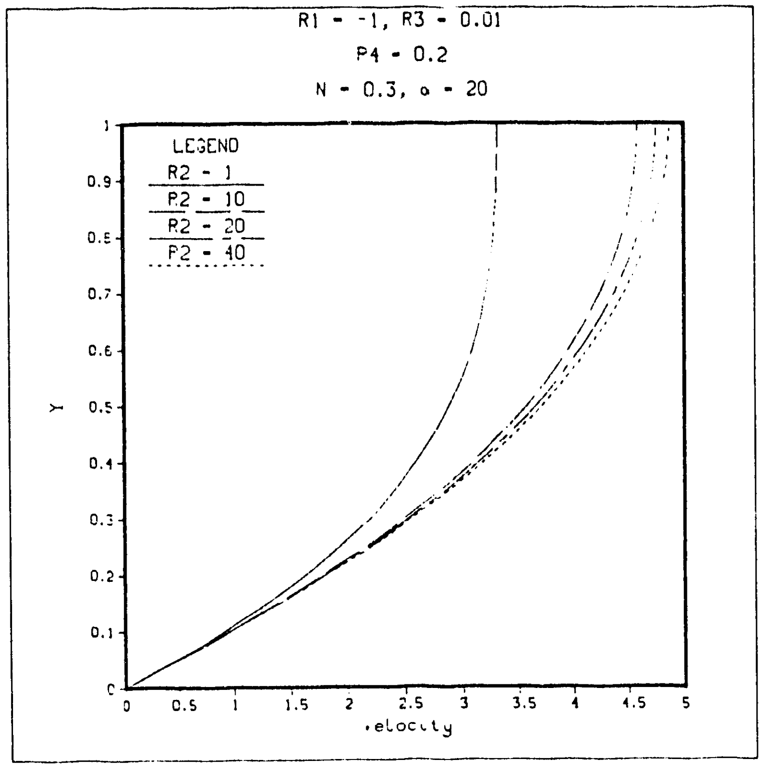

Figure 9. Effect of R2 on Velocity Profile

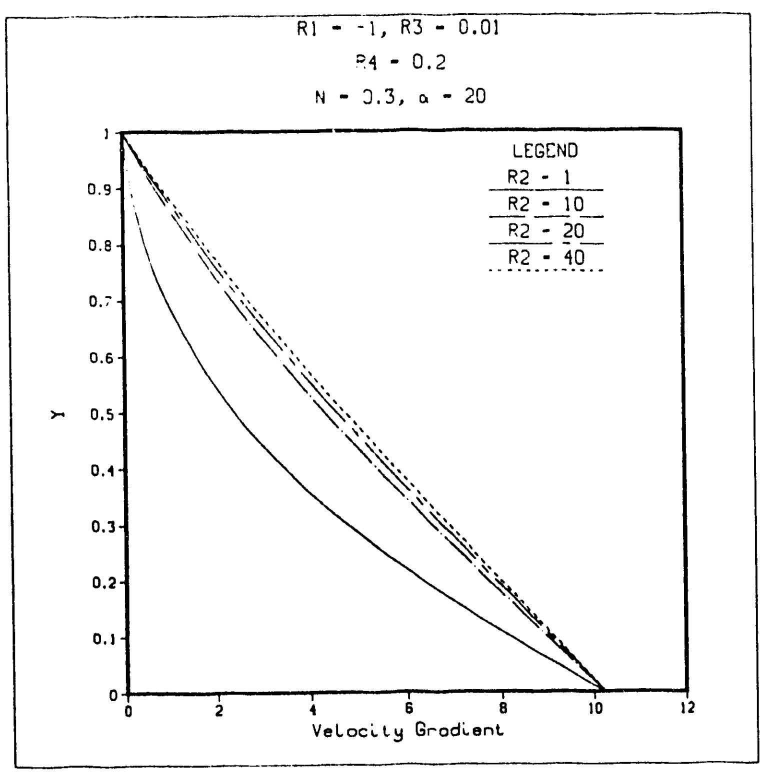

Figure 10. Effect of R2 on Velocity Gradient

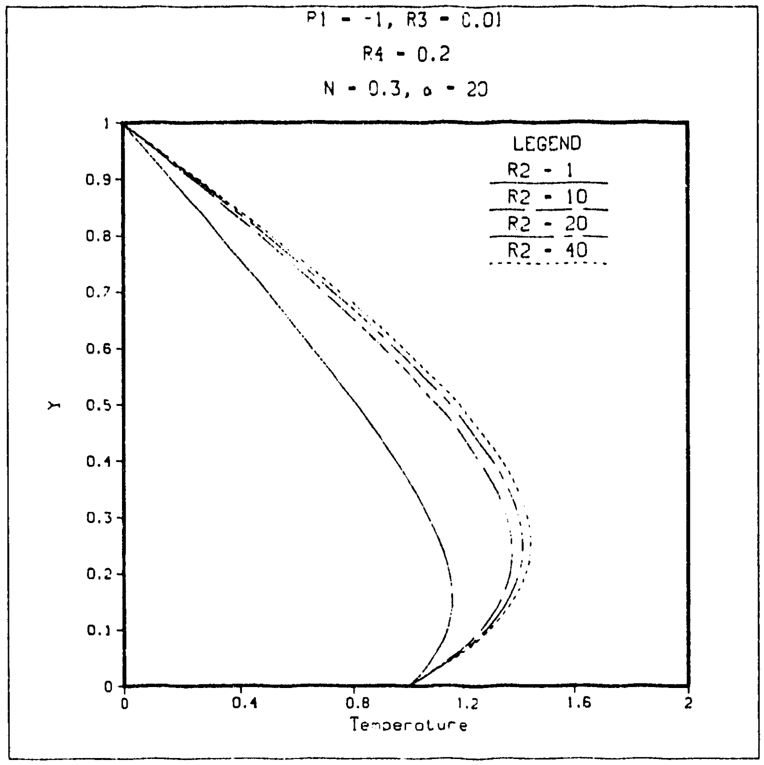

Figure 11. Effect of R2 on Temperature Profile

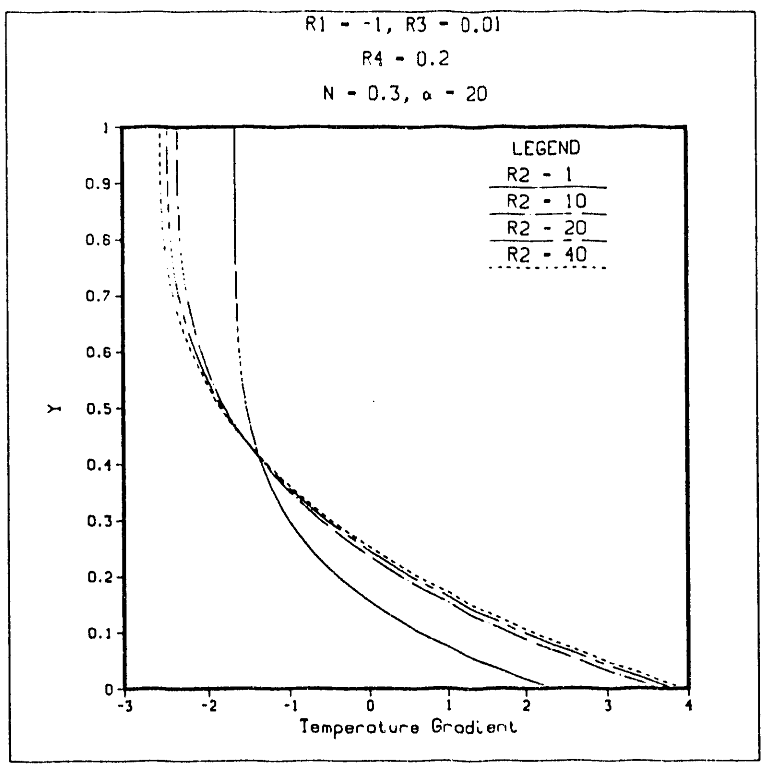

Figure 12. Effect of R2 on Temperature Gradient

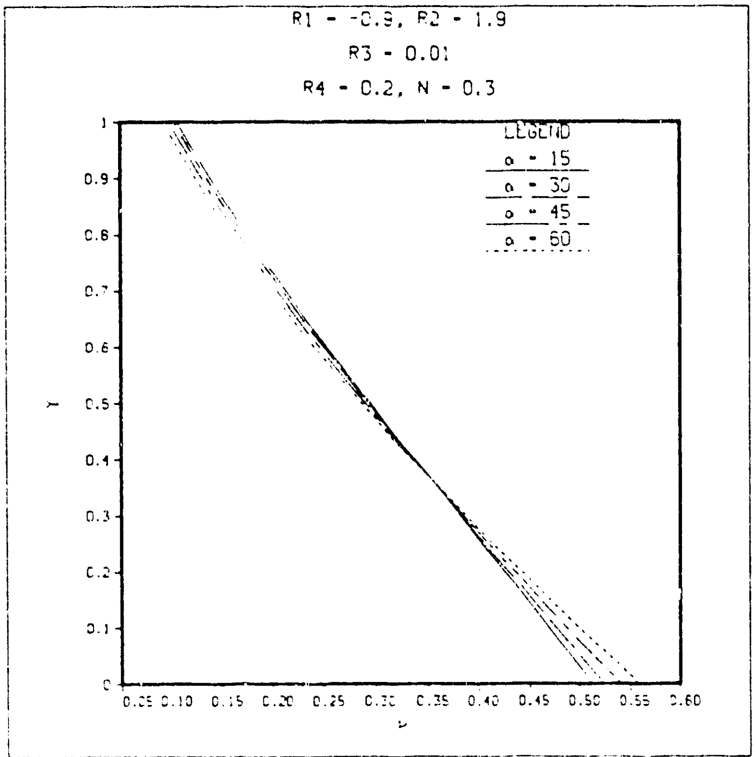

Figure 13. Effect of α on Volume Fraction

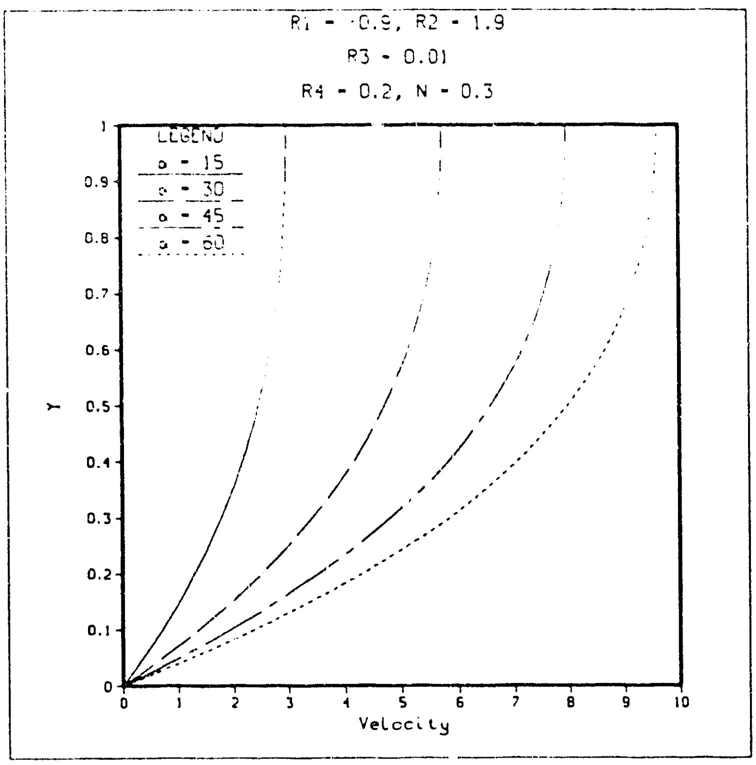

Figure 14. Effect of α on Velocity Profile

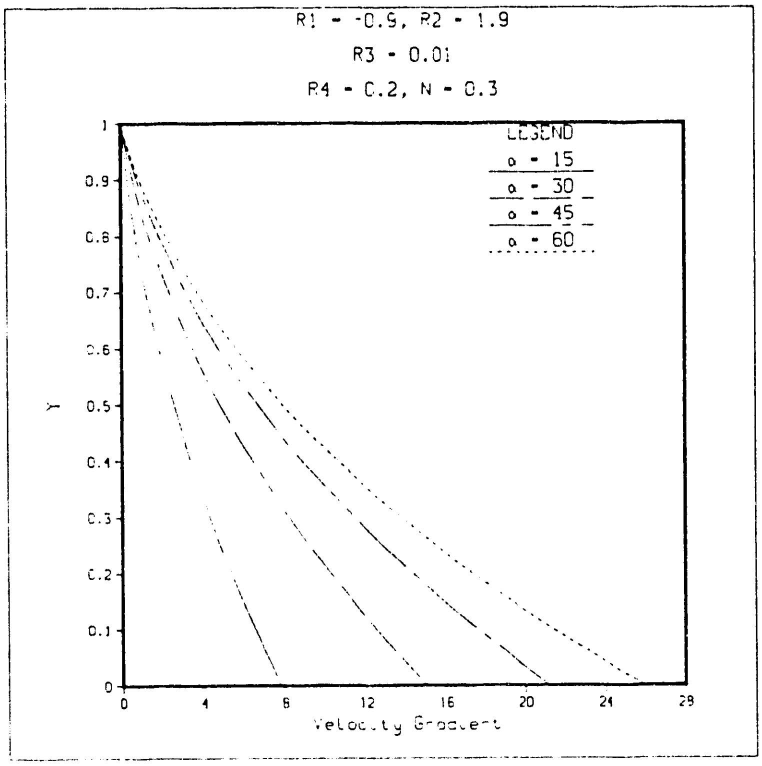

Figure 15. Effect of α on Velocity Gradient

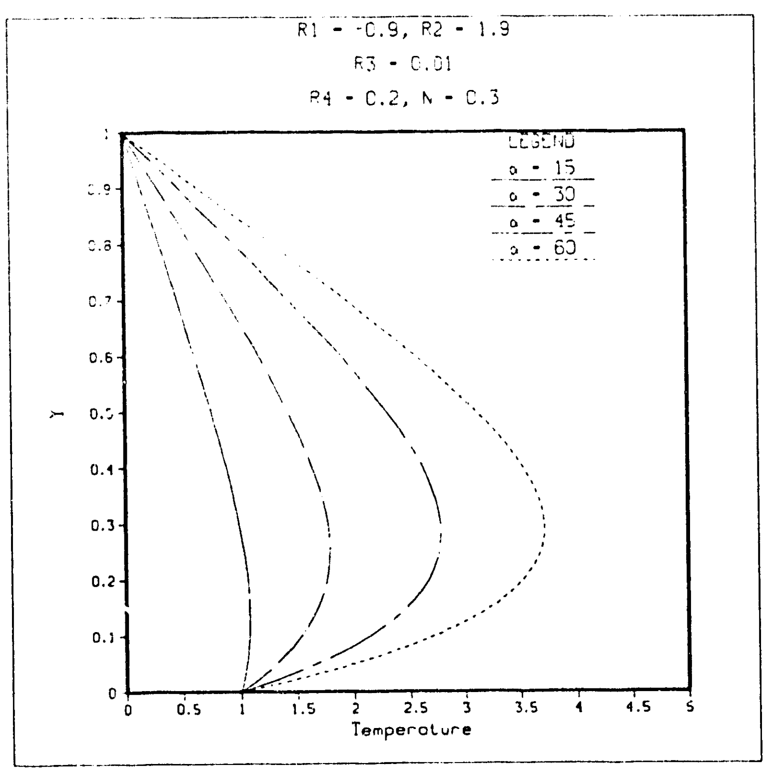

Figure 16. Effect of α on Temperature Profile

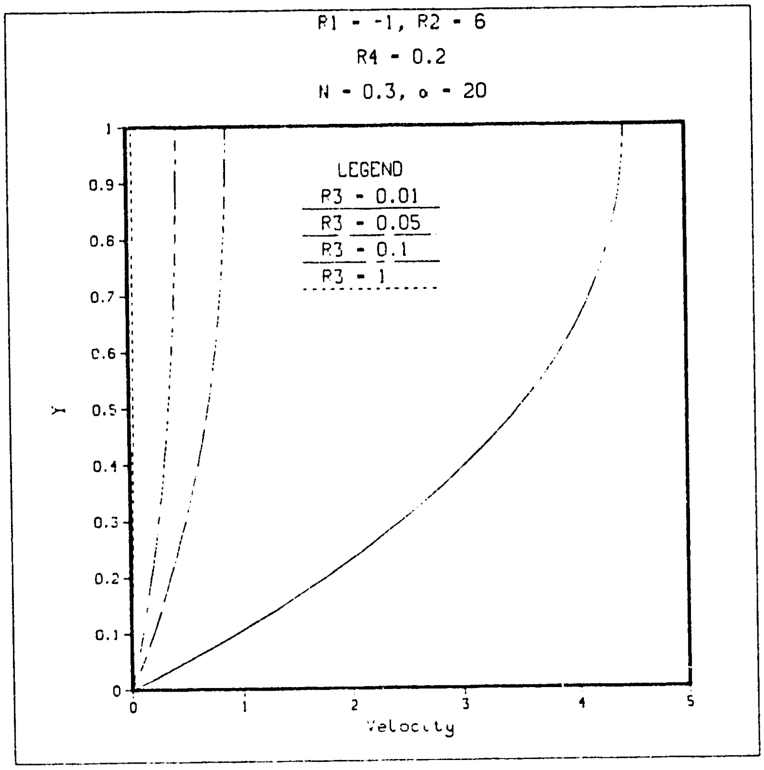

Figure 17. Effect of R3 on Velocity Profile

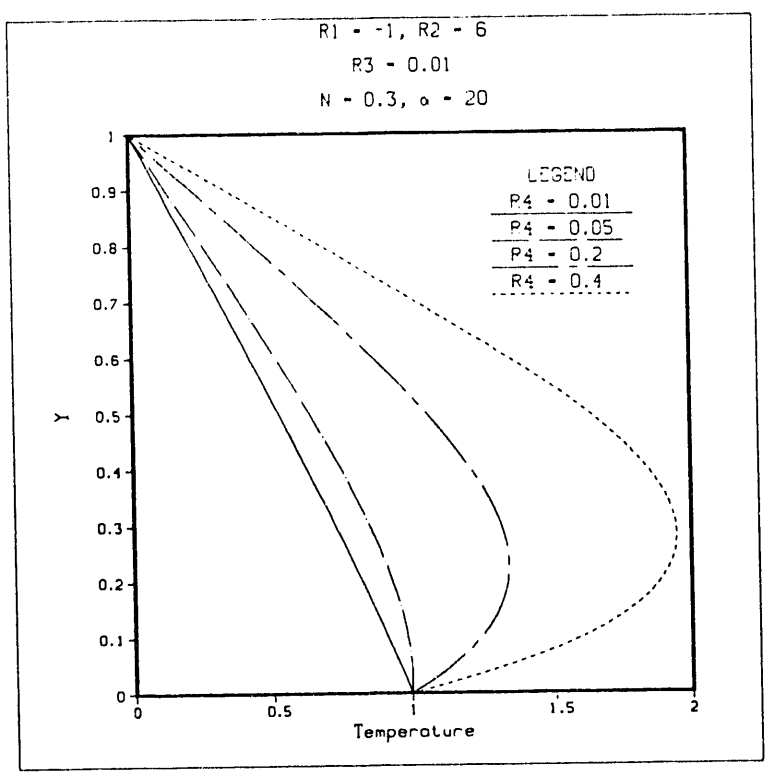

Figure 18. Effect of R4 on Temperature Profile

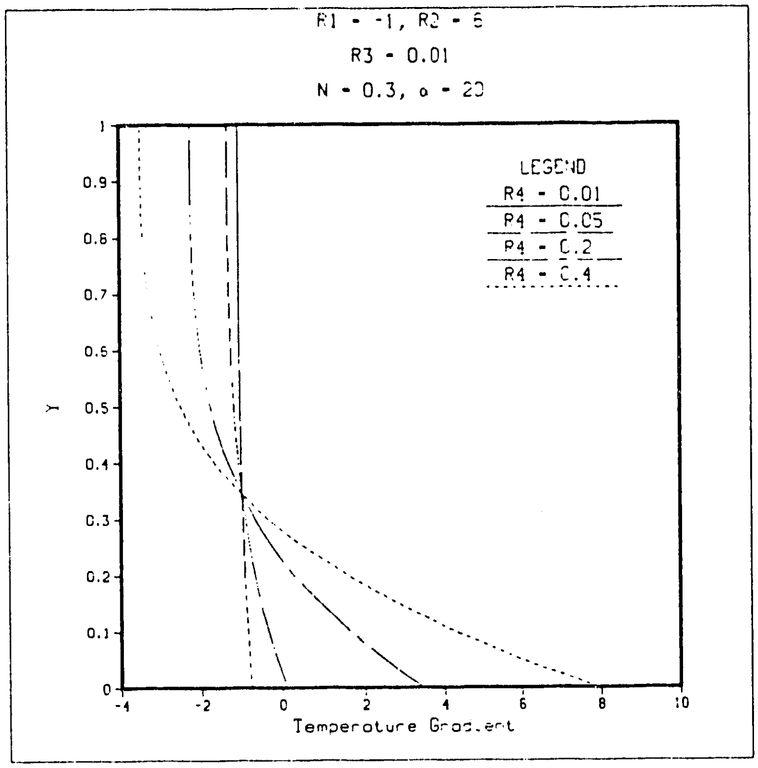

Figure 19. Effect of R4 on Temperature Gradient

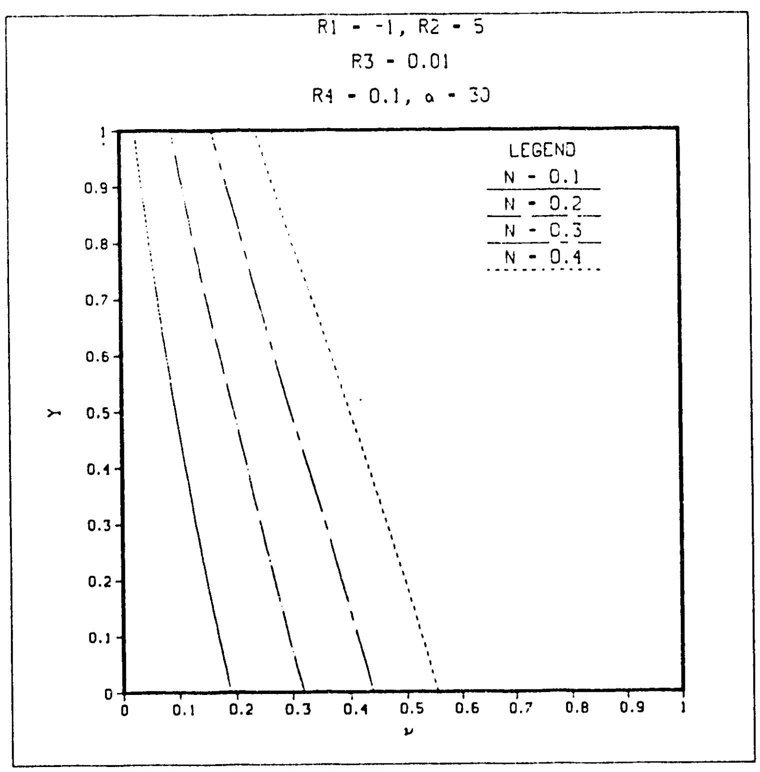

Figure 20. Effect of N on Volume Fraction

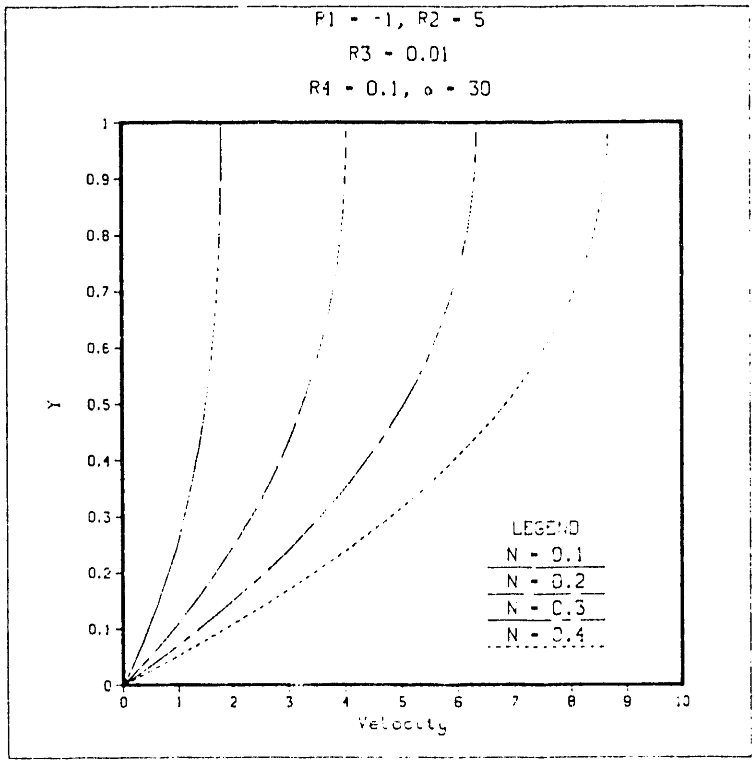

Figure 21. Effect of N on Velocity Profile

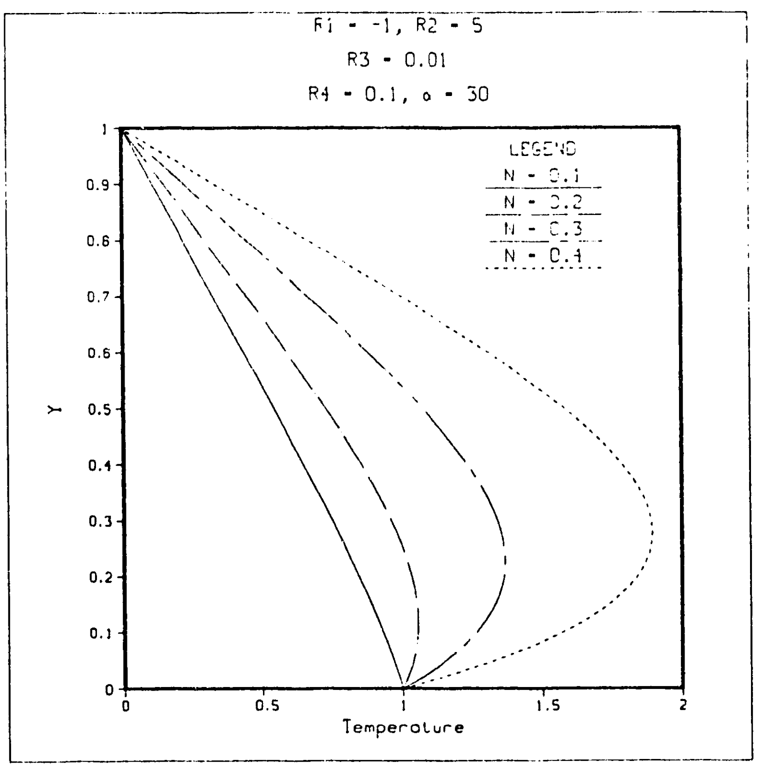

Figure 22. Effect of N on Temperature Profile

APPENDIX A.

APPENDIX A.

Massoudi *et al.* (1989) assumed that the material properties β_1 through β_4 were constant, and as a special case, assumed β_0 to be zero. This simplifies equation (2-13) and makes it possible to obtain an exact solution for the system of equations given below.

$$2\,(\beta_1 + \beta_4)\,\frac{d\nu}{dy}\,\frac{d^2\nu}{dy^2} = \gamma\, g\, \nu\, cos\alpha, \tag{A-1}$$

and,

$$\beta_3\,\frac{d^2u}{dy^2} = -2\,\gamma\, g\, \nu\, sin\alpha, \tag{A-2}$$

where g denotes the acceleration due to gravity and α is the angle of inclination of the plane. Also, the energy equation becomes

$$2\,K\,\frac{d^2\Theta}{dy^2} = -\beta_3 \left\{ \frac{du}{dy} \right\}^2 \tag{A-3}$$

We need to solve the equations (A-1), (A-2), and (A-3) subject to the appropriate boundary conditions.

$$\left.\begin{aligned} u &= 0 \\ \nu &= \nu_0 \\ \Theta &= \Theta_0 \end{aligned}\right\} \quad at\ y = 0\ (on\ inclined\ plane) \tag{A-4}$$

and,

$$\left.\begin{aligned} &\frac{du}{dy} = 0 \\ &(\beta_1 + \beta_4)\left\{\frac{d\nu}{dy}\right\}^2 = 0 \\ &\Theta = \Theta_\infty \end{aligned}\right\} \quad at\ y = h\ (at\ the\ free\ surface) \tag{A-5}$$

The exact solution for volume fraction, velocity, and temperature are given by

$$\nu = \frac{F}{18} y^3 + \left(\frac{F^2 \nu_0}{12}\right)^{1/3} y^2 + \left(\frac{3 F \nu_0^2}{2}\right)^{1/3} y + \nu_0 \tag{A-6}$$

$$u = -\frac{2 \gamma g \sin\alpha}{\beta_3} \left\{ \frac{F}{360} y^5 + \frac{1}{12}\left(\frac{F^2 \nu_0}{12}\right)^{1/3} y^4 + \frac{1}{6}\left(\frac{3 F \nu_0^2}{2}\right)^{1/3} y^3 + \frac{\nu_0}{2} y^2 \right\}$$

$$+ \frac{2 \gamma g \sin\alpha}{\beta_3} \left\{ \frac{F}{72} h^4 + \frac{1}{3}\left(\frac{F^2 \nu_0}{12}\right)^{1/3} h^3 + \frac{1}{2}\left(\frac{3 F \nu_0^2}{2}\right)^{1/3} h^2 + \nu_0 h \right\} y \tag{A-7}$$

$$\theta = \frac{b_1}{90} y^{10} + \frac{b_2}{72} y^9 + \frac{b_3}{56} y^8 + \frac{b_4}{42} y^7 + \frac{b_5}{30} y^6 + \frac{b_6}{20} y^5 + \frac{b_7}{12} y^4 + \frac{b_8}{6} y^3$$

$$+ \frac{b_9}{2} y^2 + d_1 y + \theta_0 \tag{A-8}$$

where

$$F = \frac{\gamma g \cos\alpha}{2 (\beta_1 + \beta_2)}$$

$$d_1 = \frac{\theta_\infty - \theta_0}{h} - \frac{b_1}{90} h^9 - \frac{b_2}{72} h^8 - \frac{b_3}{56} h^7 - \frac{b_4}{42} h^6 - \frac{b_5}{30} h^5 - \frac{b_6}{20} h^4$$

$$- \frac{b_7}{12} h^3 - \frac{b_8}{6} h^2 - \frac{b_9}{2} h \tag{A-9}$$

The coefficients b's are given as:

$$b_1 = a_1 \left(\frac{F}{72}\right)$$

$$b_2 = \frac{a_2}{3} \left(\frac{F}{72}\right)\left(\frac{F^2 \nu_0}{12}\right)^{1/3}$$

$$b_3 = a_1 \left\{ \frac{1}{9}\left(\frac{F^2 \nu_0}{12}\right)^{2/3} + \frac{F}{72}\left(\frac{3 F \nu_0^2}{2}\right)^{1/3} \right\}$$

$$b_4 = a_1 \left\{ \frac{F}{36} \nu_0 + \frac{1}{3}\left(\frac{F^2 \nu_0}{12}\right)^{2/3}\left(\frac{3 F \nu_0^2}{2}\right)^{1/3} \right\}$$

$$b_5 = a_1 \left\{ \frac{1}{4} \left(\frac{3 F \nu_0^2}{2} \right)^{2/3} + \frac{2 \nu_0}{3} \left(\frac{F^2 \nu_0}{12} \right)^{1/3} \right\} + a_2 \left(\frac{F}{72} \right) \qquad \text{(A-10)}$$

$$b_6 = a_1 \nu_0 \left(\frac{3 F \nu_0^2}{2} \right)^{1/3} + \frac{a_2}{3} \left(\frac{F^2 \nu_0}{12} \right)^{1/3}$$

$$b_7 = a_1 \nu_0^2 + \frac{a_2}{2} \left(\frac{3 F \nu_0^2}{2} \right)^{1/3}$$

$$b_8 = a_2 \nu_0$$

$$b_9 = a_3$$

where

$$a_1 = -\frac{\beta_3}{2K} \left(\frac{2 \gamma g \, sin\alpha}{\beta_3} \right)^2$$

$$a_2 = \frac{2 \gamma g \, sin\alpha}{K} \qquad \text{(A-11)}$$

$$2_3 = -\frac{\beta_3}{2K} C_1^2$$

where

$$C_1 = \frac{2 \gamma g \, sin\alpha}{\beta_3} \left\{ \frac{F}{72} h^4 + \frac{1}{3} \left(\frac{F^2 \nu_0}{12} \right)^{1/3} h^3 + \frac{1}{2} \left(\frac{3 F \nu_0^2}{2} \right)^{1/3} h^2 + \nu_0 h \right\} \qquad \text{(A-12)}$$

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