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REPORT

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Pressure Profiles, Resonant Pfirsch-Schlüter Currents, Thermal Instabilities and Magnetic Island Formation

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Abstract

A prescription for constructing the plasma pressure profile in the vicinity of an equilibrium magnetic island is derived by solving a sourced pressure diffusion equation near the island region. For pressure sources and sinks that are relatively constant in space, it is found that the plasma pressure profile is insensitive to pressure sources; thus the pressure profile can be constructed by assuming that the net pressure flux across any topologically toroidal magnetic surface is constant. This construction of the pressure profile is also valid for magnetic islands that are slowly evolving in time. By coupling the pressure evolution equation with the magnetostatic equilibrium equations, the theory is applied to the case of self-consistent construction of pressure-gradient-driven magnetic islands. In particular, we address the question of resonant Pfirsch-Schlüter current induced magnetic islands and the role of thermal effects on nonlinear magnetic islands.

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I. INTRODUCTION

Magnetic island formation is important in a variety of plasma physics phenomena in toroidal confinement devices. In particular, field errors and inherent three-dimensionality cause the destruction of magnetic surfaces in stellarator and tokamak equilibria.[1] Plasma pressure can have a dramatic effect on magnetostatic equilibria by causing the formation of magnetic islands in stellarators even when the vacuum fields have well-defined magnetic surfaces.[2-5] In addition, resistive magnetohydrodynamic (MHD)[6,7] and neoclassical MHD[8,9] pressure driven instabilities can produce slowly growing magnetic islands in the nonlinear Rutherford regime[10] of tearing mode theory. More recently, it has been suggested that pressure driven micro-magnetic island formation may play a role in understanding transport processes in tokamak plasmas.[11-13] Clearly, it is desirable to understand how the island formation affects the plasma pressure profile so that a self-consistent construction of the plasma equilibrium including the effects on the plasma current and magnetic fields can be understood.

A standard prescription for modeling the effect of a magnetic island on the plasma pressure profile is to flatten the profile inside the island separatrix.[2-9,13,14] This describes the fact that the plasma inside the island separatrix is thermally insulated from the plasma outside the separatrix. This solution can be derived from a diffusion equation for the pressure by assuming that the sources for the plasma pressure are located far from the island region.[3,4,6,8,13] This is tantamount to demanding that the thermal flux across a topologically toroidal magnetic surface is constant, so that while pressure gradients are zero inside the separatrix they are non-zero on all magnetic surfaces outside the separatrix. More generally, pressure sources and sinks are present at all points in the plasma volume. By introducing a plasma pressure source inside a magnetic island separatrix, plasma gradients can arise on magnetic surfaces inside the island separatrix. Once this interior gradient appears, the question of how this additional gradient self-consistently affects the island formation through the various mechanisms of pressure induced islands described above must be addressed.

In this paper, we solve a sourced pressure diffusion equation in the vicinity of an equilibrium magnetic island. A general prescription is developed that can be used in both analytic and computational models[15,16] for general three-dimensional MHD equilibria. These studies are also useful for developing micro-magnetic island models for plasma transport. In particular, thermal effects can produce magnetic islands when radiation losses dominate over plasma heating in the vicinity of a magnetic island.[11,12,17] More recently, a model was introduced that describes the state of the magnetic topology as an evolving collection of islands, stochastic regions and toroidal flux surfaces (*"magnetic bubbling"*).[13] Important in this description of the plasma was the self-consistent interaction of the magnetic topology with the plasma profiles.

In the next section, a theory is developed to treat the effects of plasma pressure sources and sinks in the vicinity of a magnetic island on the pressure profile. Although the calculation proceeds for an island in magnetostatic equilibrium, the construction is also shown to be valid for slowly growing magnetic islands. In Sec. III, a self-consistent calculation is presented to construct the effect of the pressure profile on the equilibria by using Ampere's Law. In particular the effects of resonant Pfirsch-Schlüter currents and thermal effects are discussed.

II. PRESSURE PROFILE NEAR A MAGNETIC ISLAND

A. Magnetic Configuration

As a simple model for the equilibrium near an island, assume that in the limit that the island width disappears the flux surfaces are labeled by a radial-like variable Φ, which has the physical interpretation as the toroidal flux through a magnetic surface. The island width is assumed to be small compared to equilibrium length scales. The equilibrium magnetic field with the island present is given by

$$\mathbf{B} = \nabla\Phi \times \nabla\alpha + \nabla\zeta \times \nabla\psi \ , \tag{1}$$

where ζ is the toroidal angle, $\alpha \equiv \theta - \iota_0\zeta$ is the helical, resonant angle, θ is the poloidal angle, and $\nabla\Phi \times \nabla\alpha$ ($\equiv I\nabla\zeta$) is the toroidal magnetic field at the rational surface $\iota(\Phi_0) = \iota_0 = n_0/m_0$, where $\iota = \iota(\Phi)$ is the rotational transform. The helical flux function ψ is given by

$$\psi = \int d\Phi\,(\iota - \iota_0) - \psi_{sx}\cos(m_0\alpha) \ , \tag{2}$$

where $x = \Phi - \Phi_0$. The coordinate system defined by Φ, α and ζ will be used to describe the magnetic configuration throughout this paper. The symmetry-breaking magnetic field responsible for the magnetic island is given by $\psi_{sx}\cos(m_0\alpha)$ where we assume ψ_{sx} varies weakly with Φ near the rational surface (the constant-ψ approximation of tearing mode theory[18]) and a single harmonic approximation is used. Since ψ is independent of the toroidal angle ζ in this coordinate system, $\mathbf{B}\cdot\nabla\psi = 0$; ψ labels the magnetic surfaces. By Taylor expanding the expression in the integral, an approximate analytic form for the magnetic surfaces near the rational surface is given by

$$\psi \cong \frac{\iota_0' x^2}{2} - \psi_{sx}\cos(m_0\alpha) \ , \tag{3}$$

where ι_0' is $d\iota/d\Phi$ evaluated at $\Phi = \Phi_0$. This representation of the magnetic flux function describes a magnetic island at $\Phi = \Phi_0$ whose half-width in the flux coordinate Φ is given by $w = 2(|\psi_{sx}/\iota_0'|)^{1/2}$. We will take ψ_{sx} to have the same sign as ι_0', so that the O-point of the island is located at $\alpha = 0$, and the X-point of the island is located at $m_0\alpha = \pm\pi$.

B. Pressure Diffusion Equation

The requirement that the plasma pressure p be in MHD equilibrium $\mathbf{B}\cdot\nabla p = 0$, demands that the pressure equilibrates on magnetic surfaces, $p = p(\psi)$. To determine the form of the pressure profile, an equilibrium pressure diffusion equation is introduced

$$\nabla\cdot D_p \nabla p = -S \ , \tag{4}$$

where S is the sum of the pressure sources (Ohmic or auxiliary heating and particle sources), pressure sinks (radiation) and any other non-diffusive transport process. The diffusive part of the pressure flux is written $q = -D_p \nabla p$, where all that we require of the diffusion coefficient D_p is that it be a function of the flux surface and not vary greatly in magnitude in the vicinity of the magnetic island. Since, we would expect the plasma density, temperatures and current to be flux functions near an equilibrium magnetic island, this requirement is met even if D_p is a nonlinear functional of the plasma profiles. Integrating Eq. (4) over a volume bounded by magnetic surfaces results in the condition

$$[D_p p'(\psi) \oint ds \cdot \nabla \psi]_{\partial V} = - \int_V S(\mathbf{x}') \, d^3\mathbf{x}' \ , \tag{5}$$

where

$$d^3\mathbf{x}' = \mathcal{J} \frac{d\psi \, d\alpha \, d\zeta}{\iota_o' x} \cong \pm \frac{R_o}{B_o} \frac{d\psi \, d\alpha \, d\zeta}{\sqrt{2\iota_o'[\psi + \psi_{sx}\cos(m_o\alpha)]}} \ , \tag{6}$$

$$ds = \frac{\nabla\psi \mathcal{J} \, d\alpha \, d\zeta}{\iota_o' x} \cong \pm \frac{R_o}{B_o} \frac{\nabla\psi \, d\alpha \, d\zeta}{\sqrt{2\iota_o'[\psi + \psi_{sx}\cos(m_o\alpha)]}} , \tag{7}$$

and $\mathcal{J} = (\nabla\Phi \times \nabla\theta \cdot \nabla\zeta)^{-1} = (\nabla x \times \nabla\alpha \cdot \nabla\zeta)^{-1} \cong R_o/B_o$ is the Jacobian and is assumed to vary weakly. The plus-minus sign refers to the sign of x (plus sign for $x > 0$).

Equation (5) is solved in three different regions: I) on toroidal flux surfaces located radially inside the island ($|\psi| > |\psi_{sx}|$ and $x < 0$); II) on flux surfaces located inside the separatrix ($|\psi| < |\psi_{sx}|$); and III) on toroidal flux surfaces located radially outside the magnetic island ($|\psi| > |\psi_{sx}|$ and $x > 0$). For each region a boundary condition on the pressure profile is introduced.

In region I, on a flux surface very far from the island region we prescribe the value of the pressure gradient. For surfaces located far from the island region ($x >> w$), the flux surfaces are basically described by the radial variable Φ. Therefore, we prescribe $\nabla p/|\nabla\Phi| \to dp/d\Phi|_\infty \equiv p_\infty'$. This can be viewed as the exterior solution of the pressure profile in a boundary layer theory for a small island.[3,4] Equation (5) now gives an equation for $p'(\psi)$ in region I,

$$D_p g^{\Phi\Phi}\left[\frac{dp_I}{d\psi}\int_0^{2\pi}\frac{d\alpha}{2\pi}\,\iota_o' x(\psi,\alpha) - p_\infty'\right] = -\int_{\psi_\infty}^{\psi} d\psi' \int_0^{2\pi}\frac{d\alpha}{2\pi}\,\frac{S(\psi',\alpha)}{\iota_o' x(\psi',\alpha)} \ , \tag{8}$$

where $g^{\Phi\Phi} = \nabla\Phi\cdot\nabla\Phi$ at the rational surface and ψ_∞ is the location of the flux surface of the boundary condition in the coordinate ψ. The source term S is assumed to be in independent of toroidal angle ζ. We define the coordinate k which also labels the helical flux surfaces near the island by

$$k^2 = \frac{2\psi_{sx}}{\psi + \psi_{sx}}\,\frac{\iota_o'}{|\iota_o'|} \ , \tag{9}$$

where k = 1 is the separatrix of the island and $k \to 0$ as the island width goes to zero. The factor $\iota_o'/|\iota_o'|$ is added so that $k^2 > 0$. Using this coordinate, Eq. (8) is rewritten

$$\frac{dp_I}{d\psi} = \frac{p_\infty'}{\int_0^{2\pi}\frac{d\alpha}{2\pi}\,\iota_o' x(k',\alpha)} + \frac{\iota_o' w^2}{D_p g^{\Phi\Phi}}\,\frac{\int_{\varepsilon_k}^{k} dk' \frac{1}{k'^3}\int_0^{2\pi}\frac{d\alpha}{2\pi}\,\frac{S}{\iota_o' x(k',\alpha)}}{\int_0^{2\pi}\frac{d\alpha}{2\pi}\,\iota_o' x(k',\alpha)} \ ,$$

$$= -\frac{p_\infty'}{\iota_o' w\, 2E(k)/\pi k} + \frac{1}{D_p \iota_o' g^{\Phi\Phi}\, 2E(k)/\pi k}\int_{\varepsilon_k}^{k} dk' \frac{2K(k')}{\pi k'^2} S(k') \ , \tag{10}$$

where $\varepsilon_k^2 = |2\psi_{sx}/(\psi_\infty + \psi_{sx})|$, $g^{\Phi\Phi} = \nabla\Phi\cdot\nabla\Phi$ at the rational surface, K(k) and E(k) are elliptic integrals of the first and second kind, respectively, and the second form of Eq. (10) is valid if S is independent of α. If we make a further assumption that S varies slowly in space $[S >> \psi(\partial S/\partial\psi)]$ so that we can take it as some constant S_o, Eq. (10) can be integrated to give

$$\frac{dp_I}{d\psi} = -\frac{p_{sx}'}{\iota_o' w\, 2E(k)/\pi k} - \frac{S_o}{D_p \iota_o' g^{\Phi\Phi}} \ , \tag{11}$$

where

$$p'_{sx} = p'_{\infty} - \lim_{\varepsilon_k\, w \to 0} \frac{wS_o}{D_p g^{\Phi\Phi}} \frac{2E(\varepsilon_k)}{\pi\varepsilon_k} \qquad (12)$$

is dp/dΦ at the rational surface if no island were present. If we further assume that S is constant throughout the plasma cross-section and that the flux surfaces in the limit that the island disappears are concentric circles, $\Phi \cong B_o r^2/2$, so that $p'_{sx} \cong -S_o/2B_oD_p$, we can write Eq. (11) as

$$\frac{dp_I}{d\psi} \cong p'_{sx} \{ \frac{-1}{\iota'_o w\, 2E(k)/\pi k} + \frac{1}{\iota'_o \Phi_o} \} . \qquad (13)$$

Since $w << \Phi_o$ by assumption, the second term in Eq. (13) and the corresponding second term in Eq. (11) is much smaller than the first term. The first term is a generalization of earlier calculations of pressure profiles near magnetic islands[3,4,6,8,13] where the pressure profiles near the island were essentially given as the exterior pressure profile averaged over the modified surfaces. Since, the second term in Eq. (13) due to localized pressure sources and sinks is smaller than the first, the modified pressure profiles to leading order in w/Φ_o can be constructed in region I by assuming a constant thermal flux through each magnetic surface. Thus, near the island the distortion of the profile is dominated by the rearrangement of the field-lines by the insertion of the island and negligibly affected by the corresponding changes in the source profile in the helical coordinates.

In region II, the pressure gradient must go to zero on the magnetic axis of the magnetic island. Using this condition, the pressure profile inside the island separatrix is given by

$$D_p g^{\Phi\Phi} \frac{dp_{II}}{d\psi} \oint \frac{d\alpha}{2\pi} \iota'_o x(\psi,\alpha) = - \int_{-\psi_{sx}}^{\psi} d\psi' \oint \frac{d\alpha}{2\pi} \frac{S(\psi',\alpha)}{\iota'_o x(\psi',\alpha)} . \qquad (14)$$

Recalling the definition of the flux surface label k from Eq. (9) where k = 1 represents the island separatrix and 1/k = 0 is the magnetic axis of the island and if we further assume that $S = S(\psi)$, Eq. (14) can be rewritten as

$$\frac{dp_{II}}{d\psi} = -\frac{1}{D_p g^{\Phi\Phi} \iota_o'} \frac{\int_0^{1/k} dk'^{-1} \frac{K(1/k')S(k')}{k'}}{E(1/k)-(1-k^{-2})K(1/k)} . \tag{15}$$

For $S = S_o$, the pressure gradient in region II is given by

$$\frac{dp_{II}}{d\psi} = -\frac{S_o}{D_p g^{\Phi\Phi} \iota_o'} , \tag{16}$$

which is the same gradient as the second part of Eq. (11), the pressure gradient in region I. Thus, if sources dominate over sinks inside the island separatrix ($S_o > 0$), the pressure profile peaks (slightly) on the axis of the magnetic island.

In region III, the boundary condition used is the requirement that the flux across the island separatrix is continuous.[19]

$$p_{III}'(\psi)|_{sx} = -p_I'(\psi)|_{sx} + 2p_{II}'(\psi)|_{sx} , \tag{17}$$

where the change in sign going from region I to III is because x changes sign across the separatrix and the factor of 2 in the last term comes from the fact that the pressure flux flows out from the inside through the separatrix on both sides of the island. The pressure profile in region III is given by

$$\frac{dp_{III}}{d\psi} = \frac{p_{III}'(\psi)|_{sx}}{\iota_o' w\, 2E(k)/k\pi} - \frac{\int_1^{k} dk' \frac{2K(k')}{\pi k'^2} S(k')}{D_p g^{\Phi\Phi} \iota_o' w\, 2E(k)/k\pi} ,$$

$$= \frac{p_{sx}'}{\iota_o' w\, 2E(k)/\pi k} - \frac{S_o}{D_p \iota_o' g^{\Phi\Phi}} , \tag{18}$$

where the second form of Eq. (18) is evaluated with $S = S_o$. With the exception of a sign on the first term (due to x changing signs) this is exactly the same as the pressure profile in region I, given by Eq. (11).

Combining the results of sourced diffusion equation calculations of each region, the pressure profile can be succinctly written

$$\frac{dp}{d\psi} = p'_{sx} \frac{\text{sign}(x)\, \Theta(|\psi| - |\psi_{sx}|)}{\iota'_o w\, 2E(k)/\pi k} - \frac{S_o}{D_p \iota'_o g^{\Phi\Phi}} , \tag{19}$$

where Θ is the Heaviside step function. The pressure profile in the vicinity of a magnetic island is given by two pieces. The first piece describes the continuity of thermal flux across magnetic surfaces. This corresponds to the standard picture of the effect of an island on the pressure profile since this portion of the pressure profile is flattened inside the island separtrix. Furthermore, this part of the pressure profile is odd in x across the island separatrix. (Recall, however, that ψ is double valued around the island, so that $dp/d\psi$ changing sign across the separatrix corresponds to ∇p having the same sign on either side of the separatrix.) The second term in Eq. (19) describes the effects of a localized set of sources and sinks near the island. If sources dominate the sinks (plasma heating dominating radiation losses), this term describes a peaking of the pressure profile on the magnetic axis of the island. However, this term is order w/Φ_0 smaller than the first term.

C. Time-Dependent Islands

Up to this point, islands in magnetostatic equilibrium have been discussed. There are many instances in toroidal discharges where the islands are not static; rather they are growing or decaying in time. The question to then address is how much of the previous analysis presented is valid when the island width is changing with time. Now there is a competition between the diffusion process discussed in the last section and convection of the pressure by a growing island. To make a quantitative comparison, we construct the ratio $R = C/D$, where C is the change in the flux of normalized pressure in a volume V due to a growing island and D is the flux of normalized pressure through the volume enclosing area by a diffusion process:

$$C = \int_V d^3x' \, \frac{d\psi_{sx}}{dt} = \int_V d^3x' \, \frac{\iota'_o w}{2} \frac{dw}{dt} \cos(m_o \alpha) , \tag{20}$$

$$D = D_p \int_{\partial V} ds \cdot \nabla \psi \,, \tag{21}$$

where we use a constant-ψ and single harmonic approximation for the island. The volume is picked to be bounded by a surface at fixed k [see Eq. (9)] and at infinity where there is no contribution to D (no perturbed diffusive flux at infinity). The quantity R is given by

$$R = \frac{\iota_o' w^2 \frac{dw}{dt} \frac{2}{3\pi k^2} [E(k)(1-k^2/2) - (1-k^2)K(k)]}{D_p g^{\Phi\Phi} \iota_o' w \, 2E(k)/k\pi} \,. \tag{22}$$

We can examine this expression asymptotically. Very far from the island ($k \rightarrow 0$) R becomes approximately

$$R \cong \frac{w\frac{dw}{dt}}{D_p g^{\Phi\Phi}} \left(\frac{w}{x}\right)^2 , \tag{23}$$

so that very far from the island ($x >> w$), diffusion dominates. This just means that very far from the island the island producing magnetic field doesn't distort the magnetic surfaces appreciably. Closer to the vicinity of the island, R becomes

$$R \cong \frac{w\frac{dw}{dt}}{D_p g^{\Phi\Phi}} = \frac{\delta r \frac{d\delta r}{dt}}{D_p} \,. \tag{24}$$

where $\delta r = w/|\nabla\Phi|$ is the radial extent of the magnetic island. The requirement that diffusion processes dominate the pressure evolution is given by $R << 1$. For constant–ψ magnetic islands that grow at the Rutherford rate,[10] $R \cong (\Delta' w)\eta/\mu_o D_p$ where $\Delta'$is the tearing mode matching parameter, and η/μ_o is the magnetic diffusion coefficient. The constant-ψ assumption demands that $\Delta' w$ be smaller than unity. Thus, so long as $\eta/\mu_o D_p$ is not too large ($\eta/\mu_o D_p$ is generally smaller than unity in present day tokamak plasmas), $R << 1$ for tearing modes in the Rutherford regime. For islands produced by the fluctuating bootstrap current[8,9,13], $R \cong \beta_p \eta/\mu_o D_p$ where β_p is the poloidal beta. As long as the pressure diffusion is anomalous (larger than the neoclassical prediction for electron-ion particle diffusion in the banana regime), $R < 1$ and pressure diffusion dominates over convection due to a growing island.

For islands that grow at the Sweet-Parker timescale ($dw/dt \cong r_0 S_M^{1/2}/\tau_R$, where S_M is the magnetic Reynolds number and τ_r is the resistive diffusion time), $R \cong S_M^{1/2}(w/r_0)(\eta/\mu_0 D_p)$. For fast growing islands with $R >> 1$, the procedure used in the previous section is invalid. Instead of diffusing, the pressure is fixed to a magnetic surface and moves with the surface as the island grows. In this case, the gradient of the pressure on that part of the surface that is closest to the X-point of the island is fixed. Using the notation previously used, the pressure gradient in the vicinity of the island is then given by

$$\frac{dp}{d\psi} = p_{sx}' \frac{\Theta(|\psi|-|\psi_{sx}|)}{\iota_0' x_{min}} = \pm p_{sx}' \,\Theta(|\psi|-|\psi_{sx}|) \sqrt{\frac{\iota_0'}{2(\psi-\psi_{sx})}} \quad , \tag{25}$$

where x_{min} is the smallest distance in magnitude on a particular flux surface from a point on that surface to the X-point of the island, and the sign is associated with the sign of x. This process predicts a dramatic steepening of the pressure profile in the vicinity of the magnetic island. Consequently, fast growing magnetic islands may be susceptible to secondary ideally growing pressure driven instabilities.[20]

III. APPLICATIONS

Up to this point the magnetic islands used to derive the pressure profile have not been derived self-consistently. In this section we couple the pressure evolution equation to the MHD equilibrium equations to derive an island Grad-Shafarnov equation near the magnetic island. In particular, we extend earlier work on Pfirsch-Schlüter current induced magnetic islands[3,4] by accounting for parallel currents resulting from a pressure gradient inside the island separatrix. It is assumed that the vacuum magnetic surfaces are fairly well defined so that the surface breaking is solely due to the plasma pressure. We also examine thermal effects which have been suggested as a mechanism for producing magnetic islands in the edge of tokamaks.[11,12]

A. Pfirsch-Schlüter Current Driven Magnetic Islands

Variations in the magnitude of the magnetic field strength on a magnetic surface cause parallel currents to flow within that flux surface to guarantee quasineutrality. It is important in stellarator designs to try to minimize these currents since unfavorable neoclassical transport scalings and guiding center orbit losses accompany large field strength variations.[21] Additionally, resonant Pfirsch-Schlüter currents can destroy magnetic surfaces through magnetic island formation even when the vacuum magnetic configuration has reasonably well-defined magnetic surfaces.[2-5]

The field strength variation is quantified by the Jacobian for the magnetic coordinates $\mathcal{J} = (\nabla\Phi\times\nabla\alpha\cdot\nabla\zeta)^{-1} = I/B^2$.[3,4,22] Assuming a three-dimensional equilibrium magnetic configuration, the Jacobian is written

$$\mathcal{J} = \sum_{mn} \mathcal{J}_{mn} \exp(im\theta - in\zeta) \ . \tag{26}$$

The Jacobian describes the structure of the equilibrium magnetic field. In particular, the specific volume $dV/d\Phi = V'$ is given by

$$V' = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\zeta}{2\pi} \mathcal{J} = \mathcal{J}_{00}(\Phi) \ . \tag{27}$$

The derivative of the specific volume describes the normal curvature of the configuration, where $V'' > 0$ (< 0) indicates a magnetic hill (well) and determines the low-β stability of plasma in toruses against resistive interchanges. The specific volume of a closed flux tube at the rational surface $\iota = \iota_0 = n_0/m_0$ is given by

$$\oint \frac{dl}{B} = \oint \frac{\mathbf{B}\cdot\partial\mathbf{x}/\partial\zeta}{B^2} d\zeta \equiv \bar{\mathcal{J}} = \sum_l \mathcal{J}_{lm_0 ln_0} \exp(ilm_0\alpha) \ , \tag{28}$$

where the quantity inside the integral is expressed as a function of Φ, $\alpha = \theta - \iota_0\zeta$ and ζ. For general three-dimensional equilibria the $\oint dl/B$ criterion[23] is not satisfied since the quantity in Eq. (28) is not a surface function at the rational surface. For simplicity, in this work we will assume that the resonant value of the Jacobian $\bar{\mathcal{J}}$ is given by the first two terms in the Fourier expansion

$$\bar{\mathcal{J}} = \mathcal{J}_{00}(\Phi) + \mathcal{J}_{m_0 n_0} \cos(m_0\alpha + \phi) \ , \tag{29}$$

where we have imposed the reality condition and ϕ is a constant phase angle.

We follow the calculation of refs. (3) and (4) to determine the currents and magnetic fields near the island. What is different about this calculation is the addition of a pressure gradient inside the island separatrix due to localized sourced and sinks which produces its own Pfirsch-Schlüter current. We will determine to what extent these additional currents change the results of earlier work.

A boundary layer theory is used to determine the self-consistent width of the magnetic island. It was shown in refs. (3) and (4) that in the limit of the island width going to zero, the parallel current profile is singular at the rational surface for a general three-dimensional equilibrium. This singular solution constitutes the exterior solution of the boundary layer theory. The singularity is resolved in the interior solution by allowing the magnetic island width to have finite amplitude. The two solutions are matched by equating the integrated parallel current in the interior region to the amplitude of the current singularity in the exterior region. This calculation is similar to the boundary layer theory used extensively in tearing mode analysis.[18] Using the standard definition of the tearing mode matching parameter Δ', the matching condition is

$$\Delta'\psi_{sx} = -\frac{2I}{g^{\Phi\Phi}} \int_{-\infty}^{\infty} dx \oint \frac{d\alpha}{2\pi} \cos(m_0\alpha)\, Q \ , \tag{30}$$

where $Q = \mathbf{J}\cdot\mathbf{B}/B^2$ is the parallel current profile in the vicinity of the island, and $\Delta'\psi_{sx} = \partial_x\psi|_-^+$ is the radial mismatch of the derivative of the magnetic potential across the rational surface, the magnitude of the current singularity.

The parallel current is determined by solving the MHD equations near the island. The quasineutrality condition $\mathbf{B}\cdot\nabla Q = -\nabla\cdot\mathbf{J}_\perp$ and force balance $\mathbf{J}_\perp = \mathbf{B}\times\nabla p/B^2$ give the equation to leading order in w/Φ_o

$$\nabla\zeta\cdot\nabla\psi\times\nabla Q = -\nabla\zeta\cdot\nabla p\times\nabla\bar{\mathcal{J}} \tag{31}$$

after averaging over the angle ζ. Since $p = p(\psi)$, Eq. (31) has the solution $Q = -p'(\psi)\bar{\mathcal{J}} + f(\psi)$, where $f(\psi)$ is an undetermined function of ψ. The projection of the equilibrium resistive Ohm's Law along the magnetic field gives the condition

$$-\mathbf{B}\cdot\nabla\varphi = \eta\,\mathbf{J}\cdot\mathbf{B}\ , \tag{32}$$

where φ is the electrostatic potential and η is the plasma resistivity. By averaging Eq. (32) over a flux surface, the left hand side is annihilated and a constraint on the parallel current is derived ($\langle Q\rangle = 0$), where the flux surface average is defined by

$$\langle * \rangle = \frac{\oint d\alpha \dfrac{*}{\partial_x\psi(\psi,\alpha)}}{\oint d\alpha \dfrac{1}{\partial_x\psi(\psi,\alpha)}}\ . \tag{33}$$

Flux surface averaged quantities satisfy $\mathbf{B}\cdot\nabla\langle *\rangle = 0$. Using the constraint derived from Eq. (32), the parallel current profile is given by

$$Q = p'(\psi)[\langle\mathcal{J}\rangle - \mathcal{J}]$$

$$= p'(\psi)\{V''(\langle x\rangle - x) + \mathcal{J}_{m_o n_o}[\langle\cos(m_o\alpha+\phi)\rangle - \cos(m_o\alpha+\phi)]\}\ , \tag{34}$$

where we have used Eq. (29) and expanded $\mathcal{J}_{00} = V' + V''x$ using Eq. (27). The term proportional to V'' in Eq. (34) describes the parallel current arising due to a low-beta resistive interchange perturbation. Note that to this point we have neglected the effects of geodesic curvature which are important in determining the stability of resistive interchange modes.[24] In the guise of magnetic island formation, these effects have been accounted for previously,[4] and although this effect is not derived here, we will include its effect in our final answer. As a practical concern, the E + F criteria of Glasser, et al.,[24] (which includes both the effects of normal and geodesic curvature) should be used to determine the width of the magnetic island.[25] The term proportional to $\mathfrak{J}_{m_o n_o}$ was not included in previous analysis of this problem. However, because of the radial parity of the pressure profiles used in refs. (3) and (4), this term did not affect the matching procedure. This term will contribute if a pressure gradient is present inside the magnetic island separatrix.

The part of the interior parallel current that contributes to the matching condition of Eq. (30) has even radial parity near the island. Using Eqs. (19) and (34), the current with even radial parity is given by

$$Q_{even} = p_{sx}' V'' \frac{\text{sign}(x)\, \Theta(|\psi| - |\psi_{sx}|)}{\iota_o' w\, 2E(k)/\pi k} (<x> - x)$$

$$- \frac{S_o}{D_p \iota_o' g^{\Phi\Phi}} \mathfrak{J}_{m_o n_o} [<\cos(m_o\alpha + \phi)> - \cos(m_o\alpha + \phi)] \ , \qquad (35)$$

where the term proportional to V'' comes from that part of the pressure profile that is odd in x, while the term proportional to $\mathfrak{J}_{m_o n_o}$ results from that part of the pressure profile with even radial parity. Recall from the discussion following Eq. (19) that the part of the pressure that has even radial parity comes from the localized sources and sinks and is w/Φ_o smaller than the pressure profile due to thermal convection across the flux surfaces.

The first term in Eq. (35) is the perturbation resulting from a saturated resistive interchange instability. The reason that such a term appears is that a three-dimensional equilibrium can be viewed as a two-dimensional equilibrium with saturated three-dimensional instabilities. The restriction from the $\oint dl/B$ criteria is not applicable at rational surfaces for a general stellarator equilibrium where the assumption of well-defined rational surfaces does not hold. Consequently the distinction between symmetry breaking magnetic perturbations from instabilities and equilibrium magnetic fields is lost. As mentioned above, a more general description of the resistive interchange criteria must take into account the effects of geodesic curvature in addition to the normal curvature described by V''.[24] Following the derivation of ref. (4), this can be accounted for and the V'' criteria is replaced by the E + F criteria of ref. (24), where E + F > 0 indicates instability to the resistive interchange. To the order of the calculation presented here,

$$E + F = -\frac{p'_{sx} V''}{\iota_o'^2} \frac{I}{g^{\Phi\Phi}}$$

$$+ \frac{p_{sx}'^2}{\iota_o'^2} \frac{I}{g^{\Phi\Phi}} \left[\frac{I}{g^{\Phi\Phi}} \left(\oint d\zeta/2\pi\, \tilde{\mathrm{J}}^2 I/g^{\Phi\Phi}\right) - \left(\oint d\zeta/2\pi\, \tilde{\mathrm{J}} I/g^{\Phi\Phi}\right)^2\right] , \qquad (36)$$

where $\tilde{\mathrm{J}} = \mathrm{J} - \bar{\mathrm{J}}$ are the nonresonant variations of the Jacobian at the rational surface, $I = \mathbf{B}\cdot\partial\mathbf{x}/\partial\zeta$ is the toroidal projection of the magnetic field in the covariant basis and the terms $I/g^{\Phi\Phi}$ appearing outside the integrals are averages of $I/g^{\Phi\Phi}$ over ζ.

The second term in Eq. (35) describes the resonant Pfirsch-Schlüter current flowing because of the localized source. This current flows both inside and outside the island separatrix. Inserting Eq. (35) into Eq. (30), we get

$$\Delta'\psi_{sx} = -0.5\, \iota_o' w[(E + F) + |\mathrm{J}_{m_o n_o} S_o| \frac{4I}{D_p \iota_o'^2 g^{\Phi\Phi 2}}] , \qquad (37)$$

where the absolute value sign indicates that the island will pick the phase of the magnetic island (relative to ϕ) so as to find the most destabilizing perturbation. The ratio of the amplitudes of these two terms is given by $4\mathrm{J}_{m_o n_o}/V''\Phi_o$, assuming $|S_o|\Phi_o/D_p g^{\Phi\Phi} \cong |p'_{sx}|$. For most realistic stellarator designs this is a small number so that the second term makes a small contribution.

To find the self-consistent island width, an asymptotic evaluation of Ampere's Law is used. This results in the relation[2-4]

$$\psi_{sx}\left(1 + \frac{|\nabla\Phi| r_0 \Delta'}{2m_0}\right) = \iota_0' C \ , \tag{38}$$

where $C \equiv (\beta g^{\Phi\Phi}/\iota_0'^2 m_0^2 R_0^2)(\mathcal{J}_{m_0 n_0}/\mathcal{J}_{00})$, $\beta = 2p_0/B_0^2$ is the plasma beta, and R_0 is the major radius of the torus. The term C on the right hand side of Eq. (38) is the contribution to the island width from the interaction of the equilibrium pressure profile and resonant Jacobian that does not contribute to the matching procedure.[2] The width of the magnetic island is derived by using Eqs. (37) and (38). This yields the equation

$$\frac{w}{|\nabla\Phi| r_0} = \rho/2 + \sqrt{(\rho/2)^2 + 4|C|} \ , \tag{39}$$

where

$$\rho = \frac{(E + F) + |\mathcal{J}_{m_0 n_0} S_0|(4I/D_p \iota_0'^2 g^{\Phi\Phi 2})}{m_0} \ . \tag{40}$$

Equation (39) is similar to Eq. (72) in ref. (4). The difference between this work and that of ref. (4) is the inclusion of the additional resonant Pfirsch-Schlüter current due to the local sources near the island. One may interpret this additional term in Eq. (40) as a three-dimensional modification to the resistive interchange stability criteria. Since the original work of Glasser, et al.[24] assumed the existence of well-defined flux surfaces *a priori*, this additional piece was not found in the linear stability analysis. As a practical matter, since this term makes a small contribution to Eq. (40), the local sourcing of the plasma pressure profile has little impact on the self-consistent construction of the equilibrium magnetic island. Thus, under normal conditions resonant Pfirsch-Schlüter currents on flux surfaces interior to the island separatrix do not dramatically affect the island itself.

B. Thermal Effects

In Ohmically driven tokamak plasmas, self-consistent temperature variations near a magnetic island cause variations in the parallel current profile through the resistive Ohm's Law:

$$\frac{\delta j_{\|}}{j_{\|o}} = \frac{3}{2}\frac{\delta T}{T_o} \,, \tag{41}$$

where $j_{\|o} = E_{\|}/\eta_o$ is created from an externally applied toroidal loop voltage where η_o is the resistivity evaluated at T_o. Assuming the analysis used in Sec. II for the pressure profile can also be used for the temperature profile, the perturbed temperature is given by

$$\delta T = T'_{sx} \frac{\mathrm{sign}(x)\,\Theta(|\psi| - |\psi_{sx}|)}{\iota'_o w} \int_{\psi_{sx}}^{\psi} \frac{d\psi}{2E(k)/\pi k} - \frac{S_T}{n_{sx}\chi_T \iota'_o g^{\Phi\Phi}} (\psi - \psi_{sx}) \,, \tag{42}$$

where $\delta T = T - T_{sx}$, T_{sx} is the temperature at the island separatrix, the diffusive heat flux is written $\mathbf{q}_T = -n_{sx}\chi_T \nabla T$, where $\nabla \cdot \mathbf{q}_T = S_T$ and $\psi \partial S_T/\partial\psi << S_T$ is assumed. To see the effect of this current perturbation on the magnetic island, we use the same matching procedure used for the Pfirsch-Schlüter current-driven magnetic islands:

$$\Delta'\psi_{sx} = -\frac{2R_o}{g^{\Phi\Phi}} \int_{-\infty}^{\infty} dx \oint \frac{d\alpha}{2\pi} \cos(m_o\alpha)\, \delta j_{\|} \,, \tag{43}$$

where R_o is the major radius of the tokamak. From the radial parity constraint, the only part of $\delta j_{\|}$ that contributes to Eq. (43) comes from the second part of Eq. (42). Therefore,

$$\Delta'\psi_{sx} = \frac{6 j_{\|o} R_o}{n_{sx} T_{sx} g^{\Phi\Phi 2}} \frac{S_T}{\chi_T \iota'_o} \int_{-\psi_{sx}}^{\psi_c} d\psi\, (\psi - \psi_{sx}) \oint \frac{d\alpha}{2\pi} \frac{\cos(m_o\alpha)}{\partial_x \psi(\psi,\alpha)} \,, \tag{44}$$

where ψ_c is some cutoff for the integral. Assuming that $\Phi = B_o r^2/2$ so that the flux surfaces are concentric circles in the limit of the island width going to zero, and $j_{\|o} = 2B_\theta/r_o$, Eq. (44) is rewritten

$$\Delta_I' \psi_{sx} = \frac{3}{n_{sx} T_{sx}} \frac{S_T}{\chi_T \hat{s}} \delta r \, \psi_{sx} \int_{-1}^{\Psi_c} d\Psi \, (1-\Psi) \oint \frac{d\alpha}{2\pi} \frac{\cos(m_o\alpha)}{\sqrt{[\Psi + \cos(m_o\alpha)]/2}} , \tag{45}$$

where $\hat{s} = -\, r\, (d\iota/dr)/\iota$, Δ_I' is the logarithmic derivative of the exterior vector potential with respect to the variable r, $\delta r = w/|\nabla\Phi|$ is the radial extent of the magnetic island, and $\Psi = \psi/\psi_{sx}$.

In the transport models of refs. (11), (12), and (17), the underlying assumption concerning the thermally driven magnetic islands is that these islands are imbedded in a stochastic sea. Consequently, it is reasoned that the magnetic surfaces are destroyed outside the magnetic island separatrix. If we use this assumption, $\Psi_c = 1$, and Eq. (45) becomes

$$\delta r^2 \left(\delta r + m_o \frac{5\pi}{32} \frac{n_{sx} T_{sx}}{r_o} \frac{\chi_T \hat{s}}{S_T}\right) = 0 \ , \tag{46}$$

where we have used the large mode number assumption $\Delta' = -2m_o/r_o$ for the tearing mode matching parameter. In order for Eq. (46) to have a nontrivial root, $\hat{s}/S_T$ has to be negative. For tokamak plasmas with $\hat{s} > 0$, this requires that the plasma temperature profile be hollow within the island separatrix because the temperature sinks dominate sources (radiation losses are greater than plasma heating). If we assume that the equilibrium temperature profile is given by $n_{sx}\chi_H (dT_{sx}/dr) = -\, r_o S_H/2$, where $S_H > 0$ is taken as the heating source and χ_H is the transport coefficient outside the island region, the island width assuming $S_T < 0$ is given by

$$\delta r = -\, m_o \frac{5\pi}{64} L_T \, \hat{s} \, \frac{S_H \chi_T}{S_T \chi_H} , \tag{47}$$

where $L_T = -T_{sx}/(dT_{sx}/dr)$. Clearly, unless $\chi_T/\chi_H \ll 1$, (i. e., the thermal diffusivity is much smaller inside the island than outside of it) this predicts a very large island which causes a breakdown in the theory.

In tokamak plasmas, another potential mechanism for causing micro-magnetic island formation is the fluctuating bootstrap current.[8,9,13] Near a magnetic island the neoclassical pressure gradient driven current in the banana collisionality regime is given by

$$j_{bc} = -1.46\sqrt{\varepsilon}\frac{R_0}{\iota}\frac{dp}{d\psi} <\partial_x\psi> , \tag{48}$$

where recall that $\partial_x\psi = \iota_0' x(\psi,\alpha)$ and ε is the inverse aspect ratio. That part of the pressure profile that produces a current that contributes to Eq. (43) is that part that is odd in x, the first term in Eq. (19). Using only that piece of the pressure gradient, Eq. (48) becomes

$$j_{bc} = -1.46\sqrt{\varepsilon}\frac{R_0}{\iota}p_{sx}' \frac{\Theta(|\psi| - |\psi_{sx}|)}{4E(k)K(k)/\pi^2} . \tag{49}$$

Using $\delta j_{\parallel} = j_{bc} + 3\delta T j_{\parallel 0}/2T_{sx}$ in Eq. (43), the island width equation is given by

$$\Delta' = -2.2\sqrt{\varepsilon}\frac{\beta_p L_q/L_p}{\delta r} + \frac{128}{5\pi}\frac{\delta r}{r_0 \hat{s} L_T}\frac{1}{\tilde{\chi}} , \tag{50}$$

where $\tilde{\chi} = \chi_H S_T/\chi_T S_H$, $L_q = -\iota/(d\iota/dr)$, $L_p = -p/(dp/dr)$, and $\beta_p = 2p/B_\theta^2$ is the poloidal beta. Assuming that $\Delta' = -2m_0/r_0$ and that $\tilde{\chi} > 0$ (thermal effects are stabilizing), Eq. (50) predicts a steady-state island given by

$$\frac{\delta r}{r_0} = \frac{m_0 5\pi L_T\tilde{\chi}}{128 L_q}\{\sqrt{[1 + 17.5\sqrt{\varepsilon}\beta_p L_q^2/L_p L_T m_0^2\tilde{\chi}]} - 1\} . \tag{51}$$

In the limit of $m_0^2\tilde{\chi}$ becoming large, $\delta r = 1.1\, r_0\sqrt{\varepsilon}\,\beta_p L_q/L_p m_0$ which is the result of refs. (8), (9), and (13) where no thermal effects were accounted for. If the transport properties of the plasma within and outside the island separatrix are not too dissimilar so that $\tilde{\chi} \cong 1$, bootstrap current driven magnetic islands with $m_0 \gtrsim 4.2\varepsilon^{1/4}\beta_p^{1/2}L_q/L_T$ are not appreciably affected by the thermal effect.

A number of potentially important effects have been omitted in the single-helicity analysis of this section. In particular, island dynamics have been ignored. Also, since the size of the magnetic islands derived in this section is larger than the average distance between rational surfaces, island interactions must also be accounted for. As pointed out in ref. (13), island growth, interaction and decay may play an important role in the transport properties of tokamak plasmas.

IV. CONCLUSIONS

The plasma pressure profile is insensitive to sources and sinks that are located near magnetic islands. The pressure profile can be computed by flattening the profiles inside the island separatrices and assuming a conservation of thermal flux through each topologically toroidal magnetic surface. Because of this lack of sensitivity to the local pressure gradients inside island separatrices, flattening the profiles inside the island separatrix is an excellent assumption for three-dimensional MHD code work. In particular, the additional pressure gradient does not have much effect on plasma pressure induced magnetic islands in stellarator equilibria or appreciably change the Glasser criteria[24] for resistive MHD modes.

Gradients of the electron temperature inside the island separatrix can affect the formation of micro-magnetic islands in tokamak plasmas through a thermal effect. If the temperature is hollow (temperature is higher at the X-point of the island than at the O-point) due to radiation losses dominating plasma heating, then thermal instabilities can cause the formation of magnetic islands in the absence of any other island producing effect. For medium-m mode number magnetic islands produced by fluctuating bootstrap currents, the thermal effect is small as long as there is not much difference between the transport properties of the plasma within and outside the island separatrtix.

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