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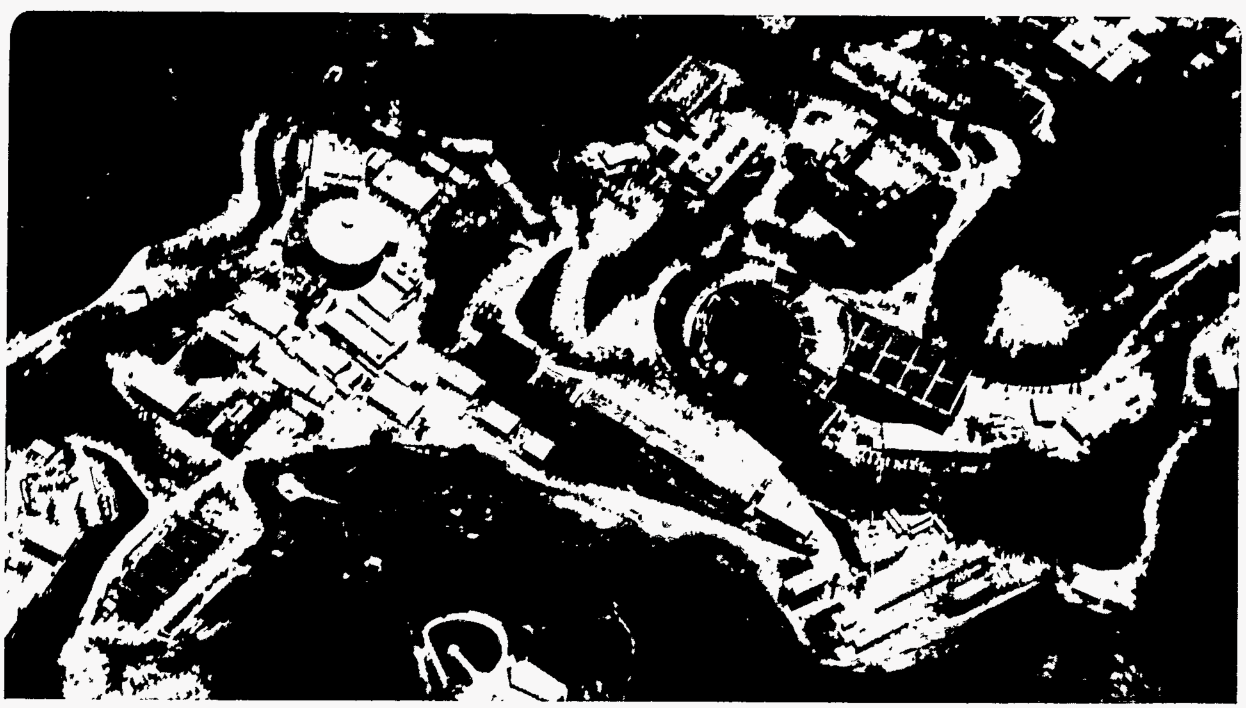

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REALIZATION OF VECTOR FIELDS FOR QUANTUM GROUPS AS PSEUDODIFFERENTIAL OPERATORS ON QUANTUM SPACES *

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Abstract

The vector fields of the quantum Lie algebra are described for the quantum groups $GL_q(N), SL_q(N)$ and $SO_q(N)$ as pseudodifferential operators on the linear quantum spaces covariant under the corresponding quantum group. Their expressions are simple and compact. It is pointed out that these vector fields satisfy certain characteristic polynomial identities. The real forms $SU_q(N)$ and $SO_q(N, R)$ are discussed in detail.

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1 Vector Fields for Quantum Groups

A quantum group can be described[1] in terms of matrices A with noncommuting elements satisfying the equation

$$\hat{R}_{12}A_1A_2 = A_1A_2\hat{R}_{12}, \tag{1}$$

with the $\hat{R}$ matrix appropriate to the particular quantum group. The matrix elements generate the algebra of functions on the group. Here we have used a well known standard notation: for instance, Eq.(1) written explicitly, takes the form

$$\hat{R}^{ij}_{kl}A^k_mA^l_n = A^i_kA^j_l\hat{R}^{kl}_{mn}. \tag{2}$$

The vector fields on the quantum group can be described [2, 3, 4, 5] by the matrix elements of a matrix Y satisfying the commutation relation

$$\hat{R}_{12}Y_2\hat{R}_{12}Y_2 = Y_2\hat{R}_{12}Y_2\hat{R}_{12}, \tag{3}$$

which corresponds to the Lie algebra relations in the classical case. The action of the vector fields on the group is then given by the commutation relation

$$Y_1A_2 = A_2\hat{R}_{12}Y_2\hat{R}_{12}. \tag{4}$$

The quantum group matrices can coact on a quantum space, for instance by right multiplication. A point of coordinates x_{0i} will be transformed into $x_i = x_{0j}A^j_i$ or, more compactly,

$$x = x_0A. \tag{5}$$

Keeping the original point x_0 fixed, the action of a vector field on the quantum group induces an action on the quantum space

$$Y_1x_2 = x_2\hat{R}_{12}Y_2\hat{R}_{12}, \tag{6}$$

i.e.

$$Y^i_jx_k = x_m\hat{R}^{im}_{ln}Y^n_r\hat{R}^{lr}_{jk}. \tag{7}$$

We shall consider the case when a differential calculus covariant with respect to the coaction of the quantum group exists on the quantum space. In this case it is natural to ask whether it is possible to realize the vector fields Y as pseudodifferential operators satisfying Eqs.(3) and (6). We shall show that this can be done for the quantum groups $GL_q(N), SL_q(N)$ and $SO_q(N)$. Their real forms are also considered.

2 $GL_q(N), SL_q(N)$ and $SU_q(N)$

The calculus for the quantum plane covariant under $GL_q(N)$ is well known[6]. The coordinates x_i in the plane satisfy the commutation relations

$$x_1 x_2 = q^{-1} x_1 x_2 \hat{R}_{12} \tag{8}$$

and the derivatives ∂^i satisfy

$$\partial^i x_j = \delta^i_j + q \hat{R}^{ik}_{jl} x_k \partial^l \tag{9}$$

and

$$\partial_2 \partial_1 = q^{-1} \hat{R}_{12} \partial_2 \partial_1 . \tag{10}$$

All indices run from 1 to N and $\hat{R}$ is the $GL_q(N)$ matrix, which satisfies the characteristic equation

$$\hat{R}^2 = 1 + \lambda \hat{R}, \quad \lambda = q - q^{-1} . \tag{11}$$

Using Eq.(11) and the above commutation relations, it is easy to verify that the differential operator

$$Y^i_j = q^{-2} \delta^i_j + q^{-1} \lambda \partial^i x_j \tag{12}$$

satisfies Eq.(6), which we repeat here,

$$Y_1 x_2 = x_2 \hat{R}_{12} Y_2 \hat{R}_{12} , \tag{13}$$

as well as

$$\partial_2 Y_1 = \hat{R}_{12} Y_2 \hat{R}_{12} \partial_2. \tag{14}$$

Combining these two, one finds that the matrix Y satisfies also Eq.(3).

The quantum subgroup $SL_q(N)$ can be obtained from $GL_q(N)$ as follows[4]. For the quantum matrices one uses the standard quantum determinant $det_q A$ and defines a new matrix

$$T = (det_q A)^{-1/N} A \tag{15}$$

having quantum determinant equal to one. For the vector fields, one defines an appropriate determinant $DetY$ and defines a new matrix of vector fields[4, 5]

$$Z = (DetY)^{-1/N} Y \tag{16}$$

having determinant one. The number of independent elements of the matrix Z is $N^2 - 1$, as in the classical case. For the particular representation Eq.(12) of the Y matrix, it is possible to show that

$$DetY = \mu, \tag{17}$$

where μ is the rescaling operator in the plane

$$\mu = 1 + q\lambda x_i \partial^i, \tag{18}$$

which satisfies

$$\mu x_i = q^2 x_i \mu, \quad \partial^i \mu = q^2 \mu \partial^i. \tag{19}$$

Thus here

$$Z = \mu^{-1/N} Y \tag{20}$$

realizes the $SL_q(N)$ vector fields as pseudodifferential operators in the quantum plane. Note that μ commutes with the elements of Y.

It is very easy to verify that the matrix given by Eq.(12) satisfies the identity

$$(Y - \mu)(Y - q^{-2}) = 0, \tag{21}$$

where matrix multiplication is implied. This is a special example of polynomial characteristic equations satisfied by quantum vector fields[7]. In general these equations are of higher order but for the realization Eq.(12) we see that the polynomial is quadratic in Y. We intend to come back to a general treatment of these characteristic equations in a forthcoming publication.

The invariant quantum trace of the k-th power of the matrix Y is defined as

$$t_k = TrD^{-1}Y^k, \tag{22}$$

where D is the diagonal matrix $(1, q^2, ..., q^{2(N-1)})$. The t_k commute with the matrix elements of Y. In general only the first r $(k = 1, 2, ..., r)$ are independent, where r is the rank of the group[1], a fact which is related to the existence of the characteristic polynomial equations for Y mentioned above. For Y given by Eq.(12) all t_k are simply functions of μ. For instance,

$$t_1 = [N] - 1 + \mu = q^{-2}t_0 - q^{-2N} + \mu, \tag{23}$$

$$t_2 = q^{-2}t_1 - \mu q^{-2N} + \mu^2, \tag{24}$$

$$t_3 = q^{-2}t_2 - \mu^2 q^{-2N} + \mu^3, \tag{25}$$

etc., where

$$[N] = 1 + q^{-2} + q^{-4} + ... + q^{-2(N-1)}. \tag{26}$$

If $|q| = 1$ the calculus given by Eqs.(8-10) for the quantum plane can be given a reality structure[6, 8] by requiring x_i to be real

$$\overline{x_i} = x_i \tag{27}$$

and by defining conjugate derivatives as

$$\overline{\partial^i} = -q^{2i'}\partial^i, \tag{28}$$

where we have introduced the notation

$$i' = N + 1 - i, \quad i = 1, 2, ..., N. \tag{29}$$

Here we consider instead the case when q is real and the complex conjugates of x_i and of ∂^i are new independent variables. It will be convenient to give them new names, i.e. we set

$$\overline{x_i} = \hat{x}^i \tag{30}$$

and

$$\overline{\partial^i} = -\hat{\partial}_i. \tag{31}$$

The commutation relations of these new variables can be obtained immediately from Eqs.(8-10) by complex conjugation (remember that this is an involution which inverts the order of factors in a product). Using the symmetry property

$$\hat{R}^{ij}_{kl} = \hat{R}^{kl}_{ij}, \tag{32}$$

we see that

$$\hat{x}_2\hat{x}_1 = q^{-1}\hat{R}_{12}\hat{x}_2\hat{x}_1, \tag{33}$$

$$\hat{x}^j\hat{\partial}_i = -\delta^j_i + q\hat{R}^{jl}_{ik}\hat{\partial}_l\hat{x}^k \tag{34}$$

and

$$\hat{\partial}_1\hat{\partial}_2 = q^{-1}\hat{\partial}_1\hat{\partial}_2\hat{R}_{12}. \tag{35}$$

Eq.(34) can be written in a form more analogous to Eq.(9) if one introduces the matrix

$$\Psi^{ir}_{js} = (\hat{R}^{-1})^{ri}_{sj}q^{2(j-r)} = (\hat{R}^{-1})^{ri}_{sj}q^{2(i-s)}, \tag{36}$$

which satisfies

$$\hat{R}^{kj}_{li}\Psi^{ir}_{js} = \Psi^{kj}_{li}\hat{R}^{ir}_{js} = \delta^k_s\delta^r_l \tag{37}$$

and

$$\Psi^{ri}_{si} = \delta^r_s q^{-2(N-r)-1}, \tag{38}$$

$$\Psi^{ir}_{is} = \delta^r_s q^{-2(r-1)-1}. \tag{39}$$

It takes the form

$$\hat{\partial}_i\hat{x}^j = \delta^j_i q^{-2i'} + q^{-1}\Psi^{jl}_{ik}\hat{x}^k\hat{\partial}_l, \tag{40}$$

where i' is given by Eq.(29).

To complete the algebra of the complex calculus, we must now give commutation relations between the variables x_i, ∂^i and their conjugate $\hat{x}^i, \hat{\partial}_i$. A consistent set is given by

$$\hat{x}^i x_j = q(\hat{R}^{-1})^{ik}_{jl} x_k \hat{x}^l, \tag{41}$$

$$\partial^i \hat{x}^j = q(\hat{R}^{-1})^{ji}_{lk} \hat{x}^k \partial^l, \tag{42}$$

$$\hat{\partial}_i x_j = q^{-1} \hat{R}^{kl}_{ij} x_k \hat{\partial}_l \tag{43}$$

and

$$\partial^i \hat{\partial}_j = q^{-1} \hat{R}^{ik}_{jl} \hat{\partial}_k \partial^l. \tag{44}$$

Consistency can be checked by verifying that all these relations braid correctly with each other.

Having the complex calculus we can now ask how the vector field realization of Eq.(12) acts on the conjugate variables. It is not hard to verify that

$$\hat{x}_2 Y_1 = \hat{R}_{12} Y_2 \hat{R}^{-1}_{12} \hat{x}_2 \tag{45}$$

and

$$Y_1 \hat{\partial}_2 = \hat{\partial}_2 \hat{R}_{12} Y_2 \hat{R}^{-1}_{12}. \tag{46}$$

On the other hand, by complex conjugation, Eqs.(6),(14), (45) and (46) give

$$\hat{x}_2 Y_1^\dagger = \hat{R}_{12} Y_2^\dagger \hat{R}_{12} \hat{x}_2, \tag{47}$$

$$Y_1^\dagger \hat{\partial}_2 = \hat{\partial}_2 \hat{R}_{12} Y_2^\dagger \hat{R}_{12}, \tag{48}$$

$$Y_1^\dagger x_2 = x_2 \hat{R}^{-1}_{12} Y_2^\dagger \hat{R}_{12} \tag{49}$$

and

$$\partial_2 Y_1^\dagger = \hat{R}^{-1}_{12} Y_2^\dagger \hat{R}_{12} \partial_2, \tag{50}$$

where $Y^\dagger$ is the hermitian conjugate of the matrix Y

$$(Y^\dagger)^i_j = \overline{Y^j_i} = q^{-2} \delta^i_j - q^{-1} \lambda \hat{x}^i \hat{\partial}_j, \tag{51}$$

which satisfies the equation conjugate of Eq.(3)

$$\hat{R}_{12}Y_2^\dagger\hat{R}_{12}Y_2^\dagger = Y_2^\dagger\hat{R}_{12}Y_2^\dagger\hat{R}_{12}, \tag{52}$$

as well as the commutation relation with Y

$$\hat{R}_{12}Y_2\hat{R}_{12}^{-1}Y_2^\dagger = Y_2^\dagger\hat{R}_{12}Y_2\hat{R}_{12}^{-1}. \tag{53}$$

Until now, we have considered two $GL_q(N)$ groups complex conjugate of each other, i.e. a truly complex $GL_q(N)$[9, 10, 11]. The quantum group can be restricted to $U_q(N)$ by imposing on its matrices the unitarity condition

$$A^\dagger = A^{-1} \tag{54}$$

and to $SU_q(N)$ by further normalizing the matrices as in Eq.(15) so that they have quantum determinant equal to one.

The vector fields of the $U_q(N)$ subgroup can be defined as the elements of the Hermitian matrix

$$U = YY^\dagger. \tag{55}$$

Indeed, it is very easy to check that U commutes with the Hermitian length

$$\mathcal{L} = x_i\hat{x}^i = x_i\overline{x_i} \tag{56}$$

(Y and $Y^\dagger$ separately do not), i.e. the U vector fields leave $\mathcal{L}$ invariant. U is a perfectly good matrix of vector fields and satisfies equations similar to Eq.(3) and Eq.(6)

$$\hat{R}_{12}U_2\hat{R}_{12}U_2 = U_2\hat{R}_{12}U_2\hat{R}_{12}, \tag{57}$$

$$U_1x_2 = x_2\hat{R}_{12}U_2\hat{R}_{12} \tag{58}$$

and

$$\hat{x}_2U_1 = \hat{R}_{12}U_2\hat{R}_{12}\hat{x}_2, \tag{59}$$

as a consequence of equations for Y and $Y^\dagger$ given above. Notice that

$$q^2U^i_j = q^{-2}\delta^i_j + q^{-1}\lambda\partial^i x_j - q^{-1}\lambda\hat{x}^i\hat{\partial}_j - \lambda^2\partial^i\mathcal{L}\hat{\partial}_j \tag{60}$$

which will be useful to us later.

Finally we observe that, if we want to reduce the vector fields to the number appropriate to $SU_q(N)$, we must normalize U, i.e., take the matrix

$$ZZ^{\dagger} = U/(\mu\bar{\mu})^{1/N}. \tag{61}$$

In addition to commuting with Y^i_j, the rescaling operator μ in Eq.(18) commutes with $\hat{x}^i, \hat{\partial}_i$ and therefore with $(Y^{\dagger})^i_j$ and

$$\bar{\mu} = 1 - q\lambda\hat{\partial}_i\hat{x}^i. \tag{62}$$

On the other hand $\bar{\mu}$ commutes with $(Y^{\dagger})^i_j, x_i, \partial^i, Y^i_j$ and satisfies

$$\bar{\mu}\hat{x}^i = q^{-2}\hat{x}^i\bar{\mu}, \quad \hat{\partial}_i\bar{\mu} = q^{-2}\bar{\mu}\hat{\partial}_i. \tag{63}$$

Clearly $\mu\bar{\mu}$ commutes with $\mathcal{L}$, therefore so does $ZZ^{\dagger}$. Z and $Z^{\dagger}$ satisfy equations analogous to Eq.(3),(52),(53). Using this fact one can show that

$$DetZZ^{\dagger} = (DetZ)(DetZ^{\dagger}) = 1. \tag{64}$$

Notice that the vector field matrix $ZZ^{\dagger}$ is Hermitian, which is the natural reality condition for $SU_q(N)$.

3 $SO_q(N)$ and $SO_q(N,R)$

We shall call T the quantum matrices of $SO_q(N)$, instead of A. In addition to

$$\hat{R}_{12}T_1T_2 = T_1T_2\hat{R}_{12}, \tag{65}$$

they satisfies the orthogonality relations[1]

$$T^t gT = g, \quad Tg^{-1}T^t = g^{-1}, \tag{66}$$

where the numerical quantum metric matrices $g = g_{ij}$ and $g^{-1} = g^{ij}$ can be chosen to be equal $g_{ij} = g^{ij}$. The $SO_q(N)$ $\hat{R}$ matrix satisfies also orthogonality conditions

$$(\hat{R}^{-1})^{ij}_{kl} = g^{im}\hat{R}^{jn}_{mk}g_{nl} = g_{km}\hat{R}^{mi}_{ln}g^{nj}, \tag{67}$$

as well as the usual symmetry relations

$$\hat{R}^{ij}_{kl} = \hat{R}^{kl}_{ij}. \tag{68}$$

The $SO_q(N)$ vector field matrix, which we shall call Z, satisfies

$$\hat{R}_{12}Z_2\hat{R}_{12}Z_2 = Z_2\hat{R}_{12}Z_2\hat{R}_{12}, \tag{69}$$

$$Z_1T_2 = T_2\hat{R}_{12}Z_2\hat{R}_{12}, \tag{70}$$

as well as an orthogonality constraint in one of the two equivalent forms[3, 5]

$$g_{ij}(Z_2\hat{R}_{12}Z_2)^{ij}_{kl} = q^{1-N}g_{kl}, \tag{71}$$

$$(Z_2\hat{R}_{12}Z_2)^{ij}_{kl}g^{kl} = q^{1-N}g^{ij}. \tag{72}$$

Eq.(71) or (72) reduces the number of independent vector fields from N^2 to $N(N-1)/2$ as in the classical case.

The projector decomposition of the $\hat{R}$ matrix for $SO_q(N)$ is

$$\hat{R} = qP^+ - q^{-1}P^- + q^{1-N}P^0. \tag{73}$$

Here P^+ is the traceless part of the symmetriser, P^- is the antisymmetriser and P^0 is the trace operator. It is related to the metric by

$$(P^0)^{ij}_{kl} = \nu g^{ij}g_{kl}, \quad \nu = \frac{\lambda}{(q^N-1)(q^{1-N}+q^{-1})}. \tag{74}$$

The coordinates x_i of the quantum Euclidean space satisfy the commutation relations

$$x_kx_l(P^-)^{kl}_{ij} = 0, \tag{75}$$

or equivalently

$$x_kx_l\hat{R}^{kl}_{ij} = qx_ix_j - \lambda\alpha(x\cdot x)g_{ij}, \tag{76}$$

where $x\cdot x = x_kx_lg^{kl} = x_kx^k$ and

$$\alpha = \frac{1}{1+q^{N-2}}. \tag{77}$$

As a consequence the length

$$L = \alpha x \cdot x \tag{78}$$

commutes with all the coordinates, $Lx_i = x_i L$.

A calculus on quantum Euclidean space can be obtained by introducing derivatives ∂^i which satisfy

$$\partial^i x_j = \delta^i_j + q\hat{R}^{ik}_{jl} x_k \partial^l \tag{79}$$

and

$$(P^-)^{ij}_{kl} \partial^l \partial^k = 0. \tag{80}$$

The Laplacian

$$\Delta = \alpha g_{ij} \partial^j \partial^i \tag{81}$$

commutes with all derivatives, $\Delta \partial^i = \partial^i \Delta$. One can define a rescaling operator

$$\Lambda = 1 + q\lambda x_i \partial^i + q^N \lambda^2 L \Delta, \tag{82}$$

which satisfies

$$\Lambda x_i = q^2 x_i \Lambda, \quad \partial^i \Lambda = q^2 \Lambda \partial^i. \tag{83}$$

A useful relation is

$$\partial^i L = q^2 L \partial^i + q^{2-N} x^i. \tag{84}$$

The action of the vector fields Z on $SO_q(N)$ induces in the standard way an action on Euclidean space analogous to (6)

$$Z^i_j x_k = x_m \hat{R}^{im}_{ln} Z^n_r \hat{R}^{lr}_{jk}. \tag{85}$$

For q real, the quantum Euclidean space can be endowed with a reality structure as follows. For the coordinates one imposes the reality condition

$$\overline{x_i} = g^{ij} x_j = x^i. \tag{86}$$

Let us now define derivatives $\hat{\partial}_i$ in terms of the conjugate derivatives by

$$\hat{\partial}_i = g_{ij}\hat{\partial}^j = -q^N\overline{\partial^i}. \tag{87}$$

The complex conjugate of Eq.(79) can be transformed to the form

$$\hat{\partial}_i x^j = \delta_i^j + q^{-1}(\hat{R}^{-1})^{jl}_{ik}x^k\hat{\partial}_l. \tag{88}$$

The relation between the derivatives ∂^i and their complex conjugates or the $\hat{\partial}_i$ can be written[12] in the nonlinear form

$$\hat{\partial}^i = \Lambda^{-1}(\delta^i_j + q^{N-1}\lambda\alpha x^i\partial_j)\partial^j, \tag{89}$$

which can be shown to satisfy Eq.(88). Using Eq.(89), one can show that

$$\hat{\partial}^i\partial^j = q\hat{R}^{ji}_{lk}\partial^k\hat{\partial}^l. \tag{90}$$

We wish to find a realization for the vector fields Z of $SO_q(N,R)$ as pseudodifferential operators on Euclidean space. One way to find the appropriate expression is to proceed in analogy with Eq.(60) by writing similar terms but adjusting the coefficients so that all relations required of Z are satisfied. It turns out that the correct formula is

$$Z^i_j = q^{-2}\delta^i_j + q^{-1}\lambda\partial^i x_j - q^{1-N}\lambda x^i\hat{\partial}_j - \lambda^2 L\partial^i\hat{\partial}_j. \tag{91}$$

Using the relations given above for the calculus on Euclidean space, one can verify that Z^i_j satisfies Eq.(85) as well as

$$\partial^i Z^j_k = \hat{R}^{ji}_{lm}Z^m_n\hat{R}^{ln}_{kr}\partial^r, \tag{92}$$

and

$$\hat{\partial}^i Z^j_k = \hat{R}^{ji}_{lm}Z^m_n\hat{R}^{ln}_{kr}\hat{\partial}^r. \tag{93}$$

Combining Eqs.(85), (92) and (93), one finds that Z satisfies also Eq.(69). It is remarkable that Z, as given by Eq.(91) satisfies even the orthogonality relations Eqs.(71) and (72), without need for any further normalization as was necessary in Eqs.(20) and (61). This can be verified by direct computation and is due, apparently, to the fact that the $SO_q(N)$ $\hat{R}$ matrix already satisfies orthogonality relations.

Finally we may ask whether Z, as given by Eq.(91) satisfies the natural reality condition for $SO_q(N, R)$ which is

$$Z^\dagger = Z. \tag{94}$$

It is very easy to see that this is indeed the case if one observes that Eq.(91) can be written in the more symmetric form

$$q^2 Z^i_j = \delta^i_j + q\lambda\partial^i x_j + q\lambda\overline{x_i}\overline{\partial^j} + \alpha q^N \lambda^2 \partial^i x_k \overline{x_k}\overline{\partial^j}, \tag{95}$$

using Eq.(84).

On the other hand, if one does not impose Eq.(86) and doesn't identify $\hat{\partial}_i$, as given in Eq.(89), with the complex conjugate derivative $\overline{\partial^i}$ by Eq.(87), then (94) will not be true. However, Eq.(91) would still give a realization of vector fields for the complex quantum group $SO_q(N)$ on Euclidean space.

In the differential calculus on a quantum space, one naturally introduces the differentials of the coordinates

$$\xi_i = dx_i. \tag{96}$$

For quantum Euclidean space, they satisfy the commutation relations

$$\xi_k \xi_l (P^+)^{kl}_{ij} = 0, \quad \xi_k \xi_l (P^0)^{kl}_{ij} = 0, \tag{97}$$

$$x_i \xi_j = q\xi_k x_l \hat{R}^{kl}_{ij}, \tag{98}$$

$$\partial^i \xi_j = q^{-1} (\hat{R}^{-1})^{ik}_{jl} \xi_k \partial^l. \tag{99}$$

According to Eq.(86) it is natural to introduce variables $\hat{\xi}_i$ related to $\overline{\xi_i}$ by

$$\overline{\xi_i} = g^{ij} \hat{\xi}_j = \hat{\xi}^i. \tag{100}$$

The complex conjugate of Eq.(98) can be written as

$$\hat{\xi}_i x_j = q x_k \hat{\xi}_l \hat{R}^{kl}_{ij}. \tag{101}$$

It was shown[12] that the $\hat{\xi}_i$ can be related to ξ_i by a (nonlinear) transformation which was given explicitly there. It turns out that that transformation can be written very compactly as

$$\hat{\xi}_i = \sigma q^N \Lambda \xi_k Z^k_i, \tag{102}$$

where Λ is given by Eq.(82). In this form one can easily verify that $\hat{\xi}$ satisfies all desired relations. For instance Eq.(101) follows immediately from Eqs.(83), (85) and (98). The requirement that complex conjugation be an involution restricts σ to be a phase, $|\sigma| = 1$. Vice versa, if one knows the correct expression for $\hat{\xi}_i$, one can infer from it the formula for Z^k_i.

4 Conclusion

All above equations are "covariant". This means thay they go into themselves by coaction transformations. For instance, for all equations for $GL_q(N)$ from Eq.(1) to (14), it is easy to see that the transformation

$$A \to AB, \quad x \to xB, \quad \partial \to B^{-1}\partial, \tag{103}$$

$$Y \to B^{-1} Y B, \quad x_i \partial^i \to x_i \partial^i \tag{104}$$

leaves them invariant. Here the matrix elements of B are taken to commute with everything (which is the reason for using the word co action) but B is itself a quantum matrix, satisfying the analogue of Eq.(1). It holds similarly for the complex conjugate sector of $GL_q(N)$,

$$A^\dagger \to B^\dagger A^\dagger, \quad \hat{x} \to B^\dagger \hat{x}, \quad \hat{\partial} \to \hat{\partial}(B^\dagger)^{-1}, \tag{105}$$

$$Y^\dagger \to B^\dagger Y^\dagger (B^\dagger)^{-1}, \quad \hat{\partial}_i \hat{x}^i \to \hat{\partial}_i \hat{x}^i \tag{106}$$

(the relation $(B^\dagger)^{-1} = (B^{-1})^\dagger$ is used). Analogous transformation laws leave invariant the $SL_q(N), SO_q(N)$ equations as well as their respective real forms.

The realization of vector fields for $GL_q(N)$ and $SL_q(N)$ given in section 2 is equivalent to that given earlier[13]. The formulas given here are simpler because of a more convenient choice of notations and definitions. For instances, we use a right coaction and a corresponding more convenient lower index for the coordinates x_i and upper index for the derivatives ∂^i. The same applies to a comparsion between the formulas written above for $SO_q(N)$ and earlier ones[12]. The reader should have no difficulty in establishing the correspondence between the conventions of these different references.

A realization of vector fields for the orthogonal group in terms of pseudodifferential operators on quantum Euclidean space has been given by Gaetano Fiore[14]. He uses the explicit description of the quantum Lie algebra by Drinfeld and Jimbo, instead of Eqs.(69), (71) and (72) and gives explicit realizations for the vector fields in that basis. Ours is an alternative solution of the same problem which has perhaps the advantage of being more symmetric and also covariant, as explained above.

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