A streamlined method for chiral fermions on the lattice*

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We discuss the use of renormalization counterterms to restore the chiral gauge symmetry in a lattice theory of Wilson fermions. We show that a large class of counterterms can be implemented automatically by making a simple modification to the fermion determinant.

Some time ago we presented a lattice method for chiral gauge theories that involves the introduction of auxiliary Dirac species [1, 2]. Here we elaborate on an alternative approach [2] that achieves the effects of the auxiliary species through a direct modification of the fermion determinant. This alternative method has the advantage that the computational algorithm is simpler, involving two determinants instead of three. It also eliminates ambiguous square roots of determinants that arise in the previous method.

1. BASIC STRATEGY

Our approach is similar in general philosophy to that of the Rome group [3]. However, as we shall see, it differs significantly in detail.

We begin by introducing a *Dirac* particle via the "naive" lattice action:

$$S_N = a^4 \sum_{x,\mu} \overline{\psi}(x) \gamma_\mu \frac{1}{2a} [\psi(x + a_\mu) - \psi(x - a_\mu)].$$
(1)

However, we couple the gauge field only to the part of the Dirac field which, in the continuum limit of the action, would be the left-handed component:

$$S_{NI} = a^4 \sum_{x,\mu} \overline{\psi}(x) \gamma_\mu P_L \frac{1}{2a} [(U_\mu(x) - 1)\psi(x + a_\mu)$$
$$-(U_\mu^\dagger(x - a_\mu) - 1)\psi(x - a_\mu)],$$
(2)

where $P_{R/L} = (1/2)(1 \pm \gamma_5)$. The Feynman propagator corresponding to the naive action is

$$S_F^N(p_\mu) = [(1/a) \sum_\mu \gamma_\mu \sin(p_\mu a)]^{-1},$$
(3)

which, in addition to the usual pole at $p = 0$, has extra poles when one or more momentum components are equal to π/a. It can be seen that half of the poles have positive chiral charge and half have negative chiral charge [4], so, contrary to our initial expectation, this doubling phenomenon leads to gauge-field couplings to both left- and right-handed species.

We follow the standard approach of eliminating the doublers by including a Wilson mass term [5] in the action:

$$S_W = a^4 \sum_{x,\mu} \overline{\psi}(x) \frac{1}{2a} [\psi(x + a_\mu)$$
$$+ \psi(x - a_\mu) - 2\psi(x)].$$
(4)

We can gauge the Wilson term by adding to the action

$$S_{WI} = a^4 \sum_{x,\mu} \overline{\psi}(x) \frac{1}{2a} [(U_\mu(x) - 1)\psi(x + a_\mu)$$
$$+ (U_\mu^\dagger(x - a_\mu) - 1)\psi(x - a_\mu)].$$
(5)

(As we shall see, it may sometimes be convenient to drop this coupling of the Wilson term to the gauge field.) Now the propagator has a pole only at $p = 0$:

$$S_F^W = \{(1/a) \sum_\mu \gamma_\mu \sin(p_\mu a)$$
$$+ (2/a) \sum_\mu [1 - \cos(p_\mu a)]\}^{-1},$$
(6)

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which would seem to leave us, as desired, with a single Dirac particle with only left-handed couplings to the gauge field. Unfortunately, the Wilson terms S_W and S_{WI}, having the Dirac structures of masses, break the chiral gauge invariance and couple the right-handed component of the Dirac field back into the theory. Specifically, the difficulty is that γ_5 commutes, rather than anticommutes, with the (identity) Dirac matrices in S_W and S_{WI}. As a consequence, the chiral gauge current is no longer conserved.

Such violations of chiral current conservation are unacceptable in a chiral gauge theory since they jeopardize the decoupling of ghost fields and, hence, unitarity. Furthermore, current conservation is an important ingredient in the standard renormalization program. Without it, there is an explosion of new counterterms, whose coefficients must each be tuned in order to obtain a satisfactory theory. For example, in the absence of current conservation, the vacuum polarization can generate a quadratically divergent gauge-boson mass, the light-by-light graph requires counterterms, Lorentz-noncovariant counterterms can arise on the lattice, and, in non-Abelian theories, the fermion–gauge-boson coupling can become different from the triple gauge-boson coupling.

A key idea in our proposal (and in that of the Rome group), is that, by tuning a suitable set of counterterms, one can restore chiral current conservation in the continuum limit. A heuristic argument in support of this idea is the following. We can regard the lattice formulation as a UV regularization of the theory. By definition, the difference between the lattice regularization and any other UV regularization resides at large loop momentum ($p_\mu \sim 1/a$). Because the Wilson term eliminates the poles at $p_\mu = \pi/a$, large Euclidean loop momentum implies that propagators are far off their mass shells. Then the corresponding subdiagram is equivalent to a local interaction. Thus, if there exists a satisfactory UV regularization of the chiral theory (that is, one that respects the chiral gauge symmetry), then it is equivalent to the Wilson lattice regularization plus local counterterms.

Therefore, we attempt to restore the chiral cur-

rent conservation for the Wilson action by adding to it local counterterms. If this procedure succeeds, then the resulting theory is unique, up to a coupling-constant renormalization. The reason is that one has freedom only to alter the coefficients of the gauge-invariant counterterms, and those counterterms correspond to coupling-constant renormalization.

2. THE STREAMLINED METHOD

2.1. General Considerations

The counterterms of concern to us correspond to local parts of the divergent subgraphs involving the fermion. In four dimensions, these subgraphs are the closed fermion loops with up to four external gauge bosons, the fermion self-energy correction, and the fermion–gauge-boson vertex correction. In general, there are counterterms corresponding to every operator that is consistent with the symmetries of the lattice theory and has dimension less than or equal to four. For the gauged Wilson theory, there are eleven such counterterms in the Abelian case and more in the non-Abelian case. Four of these can be absorbed into two additional coupling constant renormalizations, but the rest must be dealt with separately.

Clearly it would be awkward to tune so many coefficients in simulation. Therefore, we will try to find rules for computation that *automatically* implement at least some of the counterterms. Our approach will be to modify amplitudes in ways that correspond to adding local contributions, with the goal of producing a final expression that respects conservation of the chiral gauge current.

2.2. Closed Loops

Let us focus first on the divergent subgraphs involving closed fermion loops. We set aside for now the self-energy and vertex corrections.

By examining the Feynman identity, we can see at the graphical level how the violations of current conservation occur. For simplicity, we give only a schematic form, which exhibits the essential features of the full lattice expression:

$$\not{k}P_L = (\not{p} + \not{k} + M)P_L - P_R(\not{p} + M) + M\gamma_5. \quad (7)$$

On the right side of (7), the first term cancels a

propagator on the left and the second term cancels a propagator on the right. If we were to apply the Feynman identity to a set of diagrams containing all permutations of the fermion–gauge-field vertices, then the contributions of the first two terms on the right side of (7) would exhibit the usual pair-wise cancellations that appear in the textbook proofs of current conservation. (In the case of the lattice theory, a few complications, which are irrelevant for our purposes, arise because of seagull vertices.) However, the contributions corresponding to the last term of (7) would remain and would correspond to a violation of chiral current conservation.

The last term in (7) appears because γ_5 commutes with M. This suggests that we try to restore lattice current conservation by modifying the computational rules in such a way that γ_5 effectively anticommutes with M_W. Such a modification would, of course, change the amplitude. However, the change would be proportional to M_W. Now $M_W = \sum_\mu (2/a)[1 - \cos(p_\mu a)]$ vanishes as $a \to 0$ unless $p_\mu \sim 1/a$. Thus, such a change in the amplitude is a purely local contribution in the continuum limit and corresponds to the addition of a local counterterm to the action. Since loop momenta of order $1/a$ can give important contributions only in divergent subdiagrams, this modification would leave the continuum limits of convergent subdiagrams unchanged. A similar procedure is often used in the continuum in dealing with dimensionally regulated graphs involving γ_5. There the prescription is to anticommute γ_5's *before* continuing away from $d = 4$.

We can exploit this anticommutation trick in order to eliminate the γ_5's in closed fermion loops in all terms containing an even number of γ_5's. We anti-commute the γ_5's through all gamma matrices *and* through the Wilson mass. Schematically, we have

$$\left[\cdots \gamma_\mu P_L \frac{1}{\not{p}_1 + M} \gamma_\nu P_L \frac{1}{\not{p}_2 + M} \gamma_\rho P_L \right]_{\substack{\text{even no.} \\ \text{of } \gamma_5\text{'s}}}$$

$$\longrightarrow \cdots \gamma_\mu \frac{1}{\not{p}_1 + M} \gamma_\nu \frac{1}{\not{p}_2 + M} \gamma_\rho \left(\frac{1}{2} \right). \quad (8)$$

(Vertices arising from (5) complicate the analysis slightly, but do not change the conclusions.) The

resulting expression is vectorlike, so the gauge field couples to a conserved current, as required. Because of the factor $1/2$ on the right side of (8), such contributions correspond to the *square root* of the determinant of the Wilson-Dirac operator for a fermion with vector-like couplings to the gauge field. (The action is given by $S_N + S_{NI} + S_W + S_{WI}$ with $P_L \to 1$.)

We can implement this trick conveniently in simulations by noting that

$$\{\det [\not{p} + \not{A}P_L + M]\}^* = \det [\not{p} + \not{A}P_R + M]. \tag{9}$$

That is, the part of the determinant that is odd in γ_5 is the phase, and the part that is even in γ_5 is the magnitude. Consequently, in a simulation we can effect (8) by replacing the magnitude of the chiral Wilson-Dirac determinant with the square root of the determinant for Wilson-Dirac particle with vector-like couplings to the gauge field.

For terms containing an odd number of γ_5's, there is clearly no way to use the preceding trick to eliminate all the γ_5's, and, for such terms, the chiral gauge current is not conserved. This is not too surprising since, if we could have eliminated all of the violations of chiral current conservation, then we would have found a counterterm that eliminates the Adler-Bardeen-Jackiw anomaly, contrary to the proof of Adler and Bardeen [7]. Fortunately, it turns out that *all* of the violations of current conservation are proportional to the anomaly. Thus, the violations cancel if the fermion species in the theory satisfy the anomaly-cancellation condition $\text{Tr} \lambda_a \{\lambda_b, \lambda_c\} = 0$, where the λ's are the flavor matrices (or charges) associated with the fermion species.

2.3. Self-energy and Vertex Corrections

In general, the graphs associated with the fermion self-energy correction and fermion-gauge-boson vertex correction lead to six counterterms: $(Z_1^i - 1)\bar{\psi}\not{A}P_i\psi$, $\delta m^i \bar{\psi}P_i\psi$, and $(Z_2^i - 1)\bar{\psi}((-i\not{\partial})P_i\psi$, where the index i stands for L or R.

In computing the phase of the chiral determinant (that is, the terms containing an odd number of γ_5's), we can choose to drop the part of the action S_{WI} that corresponds to the gauging

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of the Wilson term. (This does not upset the preceding anomaly-cancellation argument.) Then there is a shift symmetry, discussed by Golterman and Petcher [6], which guarantees that $(Z_1^R - 1)$, $(Z_2^R - 1)$, δm^L, and δm^R vanish. However, it turns out that $Z_1^L \neq Z_2^L$, so we need a counterterm $(\tilde{Z}_1^L - 1)S_{NI}$, where $\tilde{Z}_1^L = Z_1^L/Z_2^L$, or, equivalently, a counterterm $(\tilde{Z}_2^L - 1)S_N$, where $\tilde{Z}_2^L = Z_2^L/Z_1^L$.

In a simulation, $\tilde{Z}_1^L$ (or $\tilde{Z}_2^L$) must be tuned so that the renormalized fermion–gauge-boson coupling is the same as the renormalized triple-gauge-boson coupling. The dominant contribution to $\tilde{Z}_1^L$ comes from the region of large Euclidean loop momenta. Hence, for asymptotically free theories, $\tilde{Z}_1^L$ can be computed in perturbation theory, and it is a finite renormalization. Unfortunately, $\tilde{Z}_1^L$ is gauge dependent, so one must gauge fix in simulations. However, because $\tilde{Z}_1^L$ is a local (perturbative) quantity, it should be insensitive to Gribov ambiguities. For the terms containing an odd number of γ_5's, one can prove a version of the Adler-Bardeen no-renormalization theorem [8] to the effect that, if $\tilde{Z}_1^L$ is properly adjusted and $\text{Tr}\, \lambda_a\{\lambda_b, \lambda_c\} = 0$, then the Ward identity for the complete fermion–gauge-boson vertex is non-anomalous. That is, the presence of radiative corrections does not upset the anomaly cancellation.

In computing the vector-like determinant (whose square root replaces the magnitude of the chiral determinant), we *must* gauge the Wilson term in order to maintain current conservation. Hence, there is no Golterman-Petcher symmetry to protect against a mass counterterm. However, for a vector-like theory, $Z_1^L = Z_1^R = Z_2^L = Z_2^R$ and $\delta m^L = \delta m^R = \delta m$. Since there are no $\tilde{Z}_1$ or $\tilde{Z}_2$ counterterms, we need only tune δm (or the hopping parameter) according to the criterion $m_{renorm} = 0$.

3. SUMMARY

One can simulate a chiral gauge theory on the lattice through the following procedure.

1. Start with an anomaly-free complement of Dirac particles ($\text{Tr}\, \lambda_a\{\lambda_b, \lambda_c\} = 0$) with left-handed couplings to the gauge field.

2. Fix to a renormalizable gauge.

3. Compute the determinant of each Dirac operator, including in the Dirac action the naive terms S_N and S_{NI}, an ungauged Wilson mass term S_W, and a counterterm $(\tilde{Z}_1^L - 1)S_{NI}$ (or $(\tilde{Z}_2^L - 1)S_N$).

4. Retain the phase of each determinant, but replace its magnitude with the square root of the determinant of the Dirac operator with a vector-like coupling to the gauge field. The vector-like action includes the naive terms S_N and S_{NI} with $P_L \to 1$, the Wilson term S_W, and its gauging S_{WI}. One must also include a counterterm for δm or a hopping parameter.

5. Tune δm so that the physical mass vanishes; tune $\tilde{Z}_1^L$ (or $\tilde{Z}_2^L$) so that the renormalized fermion–gauge-boson coupling is equal to the renormalized triple-gauge-boson coupling. For an asymptotically free theory, the critical hopping parameter and $\tilde{Z}_1^L$ can be computed in perturbation theory.

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