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# Discrete Pearson Distributions

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MANAGED BY
MARTIN MARIETTA ENERGY SYSTEMS, INC.
FOR THE UNITED STATES
DEPARTMENT OF ENERGY

ORNL/TM--11899
DE92 004700

Engineering Physics and Mathematics Division

Mathematical Sciences Section

# DISCRETE PEARSON DISTRIBUTIONS

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Date Published: November, 1991

Prepared by the
Oak Ridge National Laboratory
Oak Ridge, Tennessee 37831
managed by
Martin Marietta Energy Systems, Inc.
for the
U.S. DEPARTMENT OF ENERGY
under Contract No. DE-AC05-84OR21400

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## Acknowledgements

A research supported under contract ERD-89-828 with the Center for Indoor Air Research by Martin Marietta Energy Systems, Inc., under contract DE-AC05-84OR21400 with the U.S. Department of Energy.

# DISCRETE PEARSON DISTRIBUTIONS

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**Abstract**

These distributions are generated by a first order recursive scheme which equates the ratio of successive probabilities to the ratio of two corresponding quadratics. The use of a linearized form of this model will produce equations in the unknowns matched by an appropriate set of moments (assumed to exist). Given the moments we may find valid solutions. There are two cases; (a) distributions defined on the non-negative integers (finite or infinite) and (b) distributions defined on negative integers as well. For (a), given the first four moments, it is possible to set this up as equations of finite or infinite degree in the probability of a zero occurrence, the $s$th component being a product of $s$ ratios of linear forms in this probability in general. For (b) the equation for the zero probability is purely linear but may involve slowly converging series; here a particular case is the discrete normal. Regions of validity are being studied.

## 1. INTRODUCTION

Being confronted with several sets of extensive data of a discrete nature organized into cells, we found that none of the classical structures (Poisson, Binomial, Negative Binomial) came near to acceptability from a goodness-of-fit point of view. A search seemed to suggest that even though there are several generalizations, the fitting problem using moments implied fairly serious complications. Since we had 20 or more cells, it appeared that 4 or more parametered discrete distributions would be needed. It was natural to consider the Pearson discrete family generated by the 1st order recurrence

$$y_r = \left\{ \frac{P_s(r)}{Q_t(r)} \right\} y_{r-1}, \qquad (r = 1, 2, \cdots) \tag{1}$$

$P_s$ and $Q_t$ being polynomials in $r$, and the probabilities being $y_0$, $y_1$, $y_2$, $\cdots$ of occurrences in corresponding cells. If we are confronted with experimental (or even mathematically defined) data, then we can think of utilizing a set of moments (data moments will certainly exist). Now (1) is not carefully defined, for it requires $y_0$, $y_1$, $y_2$, $\cdots$, and these must be non-negative and sum to unity. The linearization of (1) with multiplication by a power of $r$ (including $r^0$) leads to a required set of equations to determine the parameters in $P_s(r)$ and $Q_t(r)$ in terms of $y_0$ in general. We then have an equation for $y_0$, for determining $y_0$ assuming there is a solution.

The corresponding case of a doubly infinite set of probabilities ($y_0$, $y_{\pm 1}$, $y_{\pm 2}$, ...) is treated in a similar fashion and is sometimes referred to as the Type IV case. We show here that potential solutions are easily set up, given a set of moments and assuming that $y_0 \neq 0$; transfers of the origin may take care of this case.

We found the studies of Ord quite illuminating and reference may be made to his [5], [6], and [8] papers. There is also his book [7] on frequency distributions with many references to generalizations.

Of course it would be amiss to omit the name of Karl Pearson [9] to whom the basic notion is usually attributed.

## 2. BASIC FORMULAE FOR THE SEMI-INFINITE CASE

### 2.1. Development of Formulae

Let the model be

$$y_r = \left(1 + \frac{\alpha - r}{C_0 + C_1 x + C_2 x^2}\right) y_{r-1} = k_r y_{r-1}, \tag{2}$$

$$(r = 1, 2, \cdots; x = r - \mu_1'; \mu_1' = E(r))$$

subject to

$$y_0 + k_1 y_0 + k_2 k_1 y_0 + \cdots = 1.$$

It is assumed that $k_s \geq 0$, $s = 1, 2, \cdots$. The form in (2) is similar to that used by Ord [5], [6], [8]. Our interest is fitting (2) to statistical data defined as frequencies $n_0$, $n_1$, $\cdots$; $n_s$ being the frequency in the $s$th cell. The approach is computer oriented; it is readily generalized.

The four parameters can be determined by moments using

$$\sum_{r=1}^{\infty}(C_0 + C_1x + C_2x^2)(y_r - y_{r-1})x^s = \sum_{r=1}^{\infty} x^s(\alpha - r)y_{r-1}, \quad (s = 0, 1, 2, 3)$$

The right hand component may be defined in terms of $(\alpha - \mu_1' - x)$. Basic elements are;

$$\begin{aligned}(i)\ \sum_{r=1}^{\infty} x^s(y_r - y_{r-1}) &= \sum_{r=0}^{\infty} x^s y_r - (-\mu_1')^s y_0 - \sum_{r=1}^{\infty}(r - 1 - \mu_1' + \mu_1')^s y_{r-1} \\ &= \mu_s - (-\mu_1')^s y_0 - \left\{\mu_s + \binom{s}{1}\mu_1'\mu_{s-1} + \cdots + \binom{s}{s}\mu_1'^s\right\} \\ &= -(-\mu_1')^s y_0 - \nu_s\end{aligned} \qquad (3)$$

in terms of the mean and central moments, where

$$\nu_s = E(x+1)^s - E(x)^s.$$

For example, $\nu_0 = 0$, $\nu_1 = 1$, $\nu_2 = 1$, $\nu_3 = 3\mu_2 + 1$.

$$(ii)\ \sum_{r=1}^{\infty} x^s y_{r-1} = \sum_{r=1}^{\infty}(\overline{x-1} + 1)^s y_{r-1} = \mu_s + \binom{s}{1}\mu_{s-1} + \cdots = \lambda_s. \qquad (4)$$

For example, $\lambda_0 = 1$, $\lambda_1 = 1$, $\lambda_2 = \mu_2 + 1$, $\lambda_3 = \mu_3 + 3\mu_2 + 1$.

The equations now appear as :

$$C_0(\nu_s + k_s y_0) + C_1(\nu_{s+1} + k_{s+1}y_0) + C_2(\nu_{s+2} + k_{s+2}y_0) + C_3\lambda_s = \lambda_{s+1} \qquad (5)$$

where $k_s = (-\mu_1')^s$ , $C_3 = \alpha - \mu_1'$ and $s = 0, 1, 2, 3$. By elementary operations on (5), we find

$$C_0 y_0 + (1 - y_0\mu_1')C_1 + (1 + y_0\mu_1'^2)C_2 + C_3 = 1 \qquad (6)$$

and the remaining three equations, in matrix form

$$\underline{M}\,[C_1, C_2, C_3]' = [h_1, h_2, h_3]' - C_0\,[1, \nu_1 + \mu_1', \nu_2 + \mu_1']' \qquad (7)$$

where

$$\underline{M} = \begin{bmatrix} \nu_1 + \mu_1' & \nu_2 + \mu_1'\nu_1 & \lambda_1 + \mu_1'\lambda_0 \\ \nu_2 + \mu_1'\nu_1 & \nu_3 + \mu_1'\nu_2 & \lambda_2 + \mu_1'\lambda_1 \\ \nu_3 + \mu_1'\nu_2 & \nu_4 + \mu_1'\nu_3 & \lambda_3 + \mu_1'\lambda_2 \end{bmatrix},$$

and

$$h_1 = \lambda_2 + \mu_1'\lambda_1, h_2 = \lambda_3 + \mu_1'\lambda_2, h_3 = \lambda_4 + \mu_1'\lambda_3.$$

From (7), $[C_1, C_2, C_3]'$ is used in (6) to determines $C_0$, and a return to (7) to determine $C_1, C_2$, and $C_3$. Then $y_0$ and subsequent probabilities follow from (2).

### 2.2. Examples

#### 2.2.1. Poisson distribution

Poisson distributions are those with $y_r = e^{-\theta}\theta^r/r!$ For this $C_0 = \theta$, $C_1 = 1$, $C_2 = 0$, and $C_3 = 0$. Once $y_0$ is evaluated accurately, then subsequent $y_r$ turn out to be all exact Poisson probabilities. For example,

| | $y_r$ | | | | |
|---|---|---|---|---|---|
| $r$ | 0 | 1 | 2 | 3 | 4 |
| $\theta = 0.25$ | 0.7788 | 0.1947 | 0.0243 | 0.0020 | 0.0001 |
| $\theta = 0.5$ | 0.6065 | 0.3033 | 0.0758 | 0.0126 | 0.0016 |
| $\theta = 0.75$ | 0.4724 | 0.3543 | 0.1329 | 0.0332 | 0.0062 |

#### 2.2.2. Stuttering Poisson Distribution

Stuttering Poisson distributions are those with probability generation function $e^{-a-b-c+at+bt^2+ct^3}$; we restrict attention here to $a = 1$, $b = 1/2$, $c = 1/3$. The moments are $\mu_1' = 3$, $\mu_2 = 6$, $\mu_3 = 14$, and $\mu_4 = 144$. The solutions are

$$\begin{cases} C_0 = (-2.476 + 14.35y_0)/\Delta \\ C_1 = (-0.6993 + 4.421y_0)/\Delta \\ C_2 = (0.01748 - 0.1206y_0)/\Delta \\ C_3 = (0.3217 - 1.904y_0)/\Delta \\ \Delta = -0.3601 + 2.177y_0 \end{cases}$$

Using 75 terms in (2) gives $y_0 = 0.15527298$. True probabilities are generated by

$$P_{r+1} = \frac{1}{r+1}\{aP_r + 2bP_{r-1} + 3cP_{r-2}\}. \quad (r = 0, 1, 2, \cdots; P_r = 0 \;\; r < 0)$$

| $r$ | Discrete Model | True Value |
|---|---|---|
| 0 | 0.1553 | 0.1599 |
| 1 | 0.1678 | 0.1599 |
| 2 | 0.1650 | 0.1599 |
| 3 | 0.1476 | 0.1599 |
| 4 | 0.1204 | 0.1199 |
| 5 | 0.0899 | 0.0879 |
| 6 | 0.0619 | 0.0613 |
| 7 | 0.0396 | 0.0384 |
| 8 | 0.0238 | 0.0235 |
| 9 | 0.0135 | 0.0137 |
| 10 | 0.0074 | 0.0075 |
| 11 | 0.0039 | 0.0041 |
| 12 | 0.0020 | 0.0021 |

### 2.2.3. Binomial distribution

Binomial distributions are those with probability $\binom{n}{r} p^r(1-p)^{n-r}$; here we examine the case $n$=10. Here $C_0 = npq$, $C_1 = q$, $C_2 = 0$, $C_3 = p$, and frequencies

| | $y_r$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $p = 0.25$ | 0.0563 | 0.1877 | 0.2816 | 0.2503 | 0.1460 | 0.0584 | 0.0162 | 0.0031 | 0.0004 | 0.0000 | 0.000C |
| $p = 0.5$ | 0.0010 | 0.0093 | 0.0440 | 0.1172 | 0.2051 | 0.2461 | 0.2051 | 0.1172 | 0.0440 | 0.0093 | 0.0010 |
| $p = 0.75$ | 0.0000 | 0.0000 | 0.0004 | 0.0031 | 0.0162 | 0.0584 | 0.1460 | 0.2503 | 0.2816 | 0.1877 | 0.0563 |

The recursion is $y_r = \{p(n-r+1)/(qr)\}\, y_{r-1}$, and the derived probabilities are correct.

### 2.2.4. Ord's Example of Type I Distribution

Using a model noted by Ord [5]

$$y_r = \left(1 + \frac{C_2 - x}{C_0 + C_1 x}\right) y_{r-1}$$

$$(r = 1, 2, 3, \cdots, \infty;\ x = r - \mu_1';\ C_2 = \alpha - \mu_1')$$

and moments $\mu_1'$, $\mu_2$, $\mu_3$ gives the equations

$$\begin{cases} C_0 y_0 + C_1(1 - \mu_1' y_0) + C_2 = 1 \\ C_0 + C_1(1 + \mu_1') + C_2(1 + \mu_1') = \mu_2 + \mu_1' + 1 \\ C_0(1 + \mu_1') + C_1(3\mu_2 + 1 + \mu_1') + C_2(1 + \mu_2 + \mu_1') = 1 + 3\mu_2 + \mu_3 + \mu_2\mu_1' + \mu_1' \end{cases}$$

and solutions, when $\mu_3 = 0$

$$\begin{cases} C_0 = \mu_2 \left\{2 + \mu_1' y_0 \left[1 - (1 + \mu_1')^2/\mu_2\right]\right\}/\Delta \\ C_1 = \left\{1 + y_0 \left[\mu_2 - (1 + \mu_1')^2\right]\right\}/\Delta \\ C_2 = \left\{1 - y_0 \left[3\mu_2 - \mu_1'^2 - 2\mu_1' + 1 + \mu_1'^2(1 + \mu_1')/\mu_2\right]\right\}/\Delta \\ \Delta = 2 - y_0 \left[2 + \mu_1' + \mu_1'^2(1 + \mu_1')/\mu_2\right]. \end{cases}$$

As an example, take the triangular distribution

$$y_r = \frac{r+1}{(n+1)^2}, \quad (r = 0, 1, \cdots, n), \quad y_r = \frac{2n - r + 1}{(n+1)^2}. \quad (r = n+1, n+2, \cdots, 2n)$$

with moments

$$\mu_1' = n, \qquad \mu_2 = n(n+2)/6, \qquad \mu_3 = 0.$$

For $n = 4$,

| $r$ | True Value | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0.0400 | 0.0432 | 0.0425 | 0.0495 |
| 1 | 0.0800 | 0.0774 | 0.0766 | 0.0742 |
| 2 | 0.1200 | 0.1211 | 0.1204 | 0.1096 |
| 3 | 0.1600 | 0.1632 | 0.1626 | 0.1535 |
| 4 | 0.2000 | 0.1861 | 0.1852 | 0.1910 |
| 5 | 0.1600 | 0.1750 | 0.1739 | 0.1910 |
| 6 | 0.1200 | 0.1311 | 0.1300 | 0.1363 |
| 7 | 0.0800 | 0.0742 | 0.0734 | 0.0644 |
| 8 | 0.0400 | 0.0289 | 0.0287 | 0.0215 |
| 9 | - | - | 0.0064 | 0.0062 |
| 10 | - | - | 0.0004 | 0.0018 |
| 11 | - | - | -0.0000 | 0.0006 |
| 12 | - | - | 0.0000 | 0.0002 |
| 13 | - | - | - | 0.0001 |
| 14 | - | - | - | 0.0000 |
| $\mu_1'$ | 4.0 | 3.9649 | 3.9999 | 3.9999 |
| $\mu_2$ | 4.0 | 3.8616 | 4.0001 | 3.9977 |
| $\sqrt{\beta_1}$ | 0 | -0.0661 | 0.0003 | -0.0094 |
| $\beta_2$ | 1.77 | 2.3460 | 2.4493 | 2.8069 |

The basic series for $y_0$ was taken to 9, 13, and 21 terms in computing the probabilities in columns 1, 2, and 3. For the parameters,

$$\begin{cases} C_0 = (32 - 336y_0)/\Delta \\ C_1 = (4 - 84y_0)/\Delta \\ C_2 = (4 - 36y_0)/\Delta \\ \Delta = 8 - 104y_0. \end{cases}$$

## 3. BASIC FORMULAE FOR THE BOUNDED CASE

### 3.1. Development of Formulae

Let the model be

$$y_r = \left(1 + \frac{C_3 - x}{C_0 + C_1 x + C_2 x^2}\right) y_{r-1} = k_r y_{r-1},$$

$$(r = 1, 2, \cdots N; x = r - \mu_1'; \mu_1' = E(r))$$

subject to

$$y_0 + k_1 y_0 + k_2 k_1 y_0 + \cdots + k_N k_{N-1} \cdots k_3 k_2 k_1 y_0 = 1,$$

and

$$y_N = k_N - k_1 y_0.$$

It is assumed that $k_s \geq 0$, $s = 1, 2, \cdots, N$ . We use

$$\sum_{r=1}^{N} y_r x^s = \mu_s - (-\mu_1')^s y_0$$

$$\sum_{r=1}^{N} y_{r-1} x^s = \lambda_s - T^s y_N \quad (T = N - \mu_1' + 1, \lambda_s = E(x+1)^s)$$

$$\sum_{r=1}^{N} (y_r - y_{r-1}) x^s = -\nu_s - (-\mu_1')^s y_0 + T^s y_N \quad (\nu_s = E(x+1)^s - E(x^s))$$

The six parameters can be determined by moments using

$$(y_r - y_{r-1})(C_0 + C_1 x + C_2 x^2) = (C_3 - x) y_{r-1}$$

$$\begin{aligned} C_0(-y_0 + y_N) + C_1(-\nu_1 + \mu_1' y_0 + T y_N) &+ C_2(-\nu_2 - \mu_1'^2 y_0 + T^2 y_N) \\ &- C_3(\lambda_0 - y_N) = -(\lambda_1 - T y_N) \end{aligned}$$

$$\begin{aligned} C_0(-\nu_1 + \mu_1' y_0 + T y_N) + C_1(-\nu_2 - \mu_1'^2 y_0 &+ T^2 y_N) + C_2(-\nu_3 + \mu_1'^3 y_0 + T^3 y_N) \\ &- C_3(\lambda_1 - T y_N) = -(\lambda_2 - T^2 y_N) \end{aligned}$$

$$\begin{aligned} C_0(-\nu_2 - \mu_1'^2 y_0 + T^2 y_N) + C_1(-\nu_3 + \mu_1'^3 y_0 &+ T^3 y_N) + C_2(-\nu_4 - \mu_1'^4 y_0 + T^4 y_N) \\ &- C_3(\lambda_2 - T^2 y_N) = -(\lambda_3 - T^3 y_N) \end{aligned}$$

$$\begin{aligned} C_0(-\nu_3 + \mu_1'^3 y_0 + T^3 y_N) + C_1(-\nu_4 - \mu_1'^4 y_0 &+ T^4 y_N) + C_2(-\nu_5 + \mu_1'^5 y_0 + T^5 y_N) \\ &- C_3(\lambda_3 - T^3 y_N) = -(\lambda_4 - T^4 y_N). \end{aligned}$$

From the last three equations, set up $C_0$, $C_1$, $C_2$ in terms of $C_3$, insert these derived forms in the first equation and thus find $C_0$, $C_1$, $C_2$, and $C_3$ in terms of $y_0$ and $y_N$. Then $y_0 + k_1 y_0 + \cdots = 1$ or $y_0 = 1/(1 + k_1 + k_2 k_1 + \cdots + k_N k_{N-1} \cdots k_1)$ and $k_N k_{N-1} k_{N-2} \cdots k_1 y_0 = y_N$.

If the model is

$$y_r = \left(1 + \frac{C_2 - x}{C_0 + C_1 x}\right) y_{r-1}$$

then the general solutions are, assuming existence

$$\begin{cases} C_0 = (-2\mu_2^2 + y_0 A_1 + y_N A_2 + y_0 y_N A_3)/\Delta \\ C_1 = (-\mu_2 - \mu_3 + y_0 B_1 + y_N B_2 + y_0 y_N B_3)/\Delta \\ C_2 = (-\mu_2 + \mu_3 + y_0 D_1 + y_N D_2 + y_0 y_N D_3)/\Delta \\ \Delta = -2\mu_2 + y_0 E_1 + y_N E_2 + y_0 y_N E_3 \end{cases}$$

where

$$
\begin{aligned}
A_1 &= \mu_1'(\mu_2 + 2\mu_1'\mu_2 + \mu_3 + \mu_1'\mu_3 + \mu_1'^2\mu_2 - \mu_2^2) \\
A_2 &= \mu_1'(\mu_2 - \mu_1'\mu_2 - 2\mu_3 - \mu_1'\mu_3 - \mu_1'^2\mu_2 + \mu_2^2) - \mu_3 + 3\mu_2^2 + \mu_2 \\
&+ m(-\mu_2 + 2\mu_3 + 2\mu_1'\mu_2 - \mu_2^2 + 2\mu_1'\mu_3 + 3\mu_1'^2\mu_2) \\
&- m^2(\mu_2 + \mu_3 + 3\mu_1'\mu_2) + m^3\mu_2 \\
A_3 &= -\mu_1'\left\{m(m-1)^2 + m[-3(\mu_2 + \mu_1' + \mu_1'^2 + \mu_1'\mu_2) - \mu_3 - \mu_1'^3]\right. \\
&+ \left. m^2(4\mu_1' + 2\mu_2 + 2\mu_1'^2) - m^3\mu_1'\right\} \\
B_1 &= \mu_2 + 2\mu_1'\mu_2 + \mu_1'^2\mu_2 + \mu_1'\mu_3 - \mu_2^2 + \mu_3 \\
B_2 &= m(m-1)^2 - \mu_1'(1 + 2\mu_1' + \mu_2 + \mu_1'^2 + \mu_1'\mu_2 + \mu_3) + \mu_2(1 + \mu_2) \\
&+ m(4\mu_1' + \mu_2 + 3\mu_1'^2 + \mu_3 + 2\mu_1'\mu_2) - m^2(3\mu_1' + \mu_2) \\
B_3 &= -m(m-1)^2 - m(3\mu_1'(1 + \mu_2 + \mu_1') + \mu_1'^3 + 3\mu_2 + \mu_3) \\
&+ m^2(4\mu_1' + 2\mu_1'^2 + 2\mu_2) - m^3\mu_1' \\
D_1 &= \mu_1'(\mu_1' + \mu_1'^2 - 2\mu_2 - \mu_1'\mu_2 - 2\mu_3) + \mu_2 + 3\mu_2^2 - \mu_3 \\
D_2 &= -\mu_2[1 + \mu_1'(1 - \mu_1') + 3\mu_2] + \mu_3 + 2\mu_1'\mu_3 + m(\mu_2 - 2\mu_3 - 2\mu_1'\mu_2) + m^2\mu_2 \\
D_3 &= m\mu_1'[1 + 3(\mu_2 + \mu_1' + \mu_1'^2)] + m^2(-2\mu_1' - 4\mu_1'^2 + 2\mu_1'\mu_2 + \mu_3) + m^3(\mu_1' - \mu_2) \\
E_1 &= \mu_1'(\mu_1' + \mu_1'^2 + \mu_2) + 2\mu_2 \\
E_2 &= m(m-1)^2 - \mu_1'[\mu_2 + (1 + \mu_1')^2] + \mu_2 + m[\mu_1'(4 + 3\mu_1') + \mu_2] - 3m^2\mu_1' \\
E_3 &= -m(m-1)^2 - 3m[\mu_1'(1 + \mu_1') + \mu_2] + m^2[\mu_1'(4 + \mu_1') + \mu_2] - m^3\mu_1'
\end{aligned}
$$

and $m = N + 1$.

### 3.2. Examples

#### 3.2.1. Triangular Distribution

For the discrete triangular distribution of 2.2.4. with $N = 8$, $\mu_1' = 4$, $\mu_2 = 4$, and $\mu_3 = 0$ using the fundamental equations

$$
\begin{cases}
C_0(-y_0 + y_8) + C_1(-1 + 4y_0 + 5y_8) + C_2(y_8 - 1) = -1 + 5y_8 \\
C_0(-1 + 9y_0) + C_1(4 - 36y_0) + 4C_2 = 0 \\
C_0(4 - 36y_0) + (-8 + 144y_0) = 12
\end{cases}
$$

we find solutions

$$
\begin{cases}
C_0 = [8 - 84y_0 + y_8(-88 + 612y_0)]/\Delta \\
C_1 = [1 - 21y_0 + y_8(-5 + 153y_0)]/\Delta \\
C_2 = [1 - 9y_0 + y_8(-17 + 153y_0)]/\Delta \\
\Delta = 2 - 26y_0 + y_8(-26 + 306y_0) \\
k_r = 1 + (C_2 - x)/(C_0 + C_1 x). \quad (x = r - 4, r = 1, 2, \cdots, 8).
\end{cases}
$$

If we assume the true probabilities in the parameters, then

$$C_0 = 5.125, \quad C_1 = 0.5, \quad C_2 = 0.5, \quad \Delta = 0.4096.$$

We use our scheme for $y_0$ and $y_N$ ($N = 8$) iteratively and we find the 15th iteratives to be $y_0 = 0.0394944261$, and $y_8 = 0.0394944261$ with the recursive parameters

$$C_0 = 5.102939, \quad C_1 = 0.500000, \quad C_2 = 0.500000, \quad \Delta = 0.423591678.$$

The evaluated probabilities (symmetric to 6 significant figures) are,

| $r$ | 0 and 8 | 1 and 7 | 2 and 6 | 3 and 5 | 4 |
|---|---|---|---|---|---|
| $y_r$ | 0.039494 | 0.07786 | 0.125302 | 0.166136 | 0.182414 |
| True | 0.040000 | 0.08000 | 0.120000 | 0.160000 | 0.200000 |

and computed moments

$$\mu_1' = 4.000000, \quad \mu_2 = 4.000000, \quad \mu_3 = -0.000001.$$

From numerical evidence it appears that $k_5k_4 = k_6k_3 = k_7k_2 = k_8k_1 = k_9k_0$. From the formulas for $C_0$, $C_1$, $C_2$ we find this leads to $C_1 = 1/2$, $C_2 = 1/2$ and $y_0 = y_8$. Hence

$$k_r = \frac{2C_0 + 5 - r}{2C_0 + r - 4} \quad (r = 0, 1, \cdots, 8)$$

and

$$C_0 = \frac{8 - 172y_0 + 612y_0^2}{2 - 52y_0 + 306y_0^2}.$$

The equation for $y_0$ is

$$y_0(2 + k_1 + 2k_1k_2 + 2k_1k_2k_3 + k_1k_2k_3k_4) = 1.$$

Using it iteratively with $y_0 = 0.04$ initially, the 14th iterate is $y_0 = 0.039494425$.

### 3.2.2. Non-Symmetric Distribution

Consider another example with $\mu_1' = 2.2$, $\mu_2 = 1.56$ and $\mu_3 = -0.144$, which has probabilities

$$y_0 = 1/10, \quad y_1 = 2/10, \quad y_2 = 3/10, \quad y_3 = 2/10, \quad y_4 = 2/10.$$

The model is

$$y_r = \left(1 + \frac{C_2 - x}{C_0 + C_1 x}\right) y_{r-1} \quad (r = 1, 2, 3, 4;\ x = r - 2.2)$$

and given the moments, solutions are

$$\begin{cases} C_0 = 4.8672 - 28.776y_0 - y_4(20.16 - 105.6y_0)/\Delta \\ C_1 = 1.4160 - 13.080y_0 - y_4(4.80 - 48.0y_0)/\Delta \\ C_2 = 1.7040 - 10.712y_0 - y_4(8.40 - 51.2y_0)/\Delta \\ \Delta = 3.1200 - 22.040y_0 - y_4(15.0 - 104.0y_0). \end{cases}$$

After 107 iterations, we have $y_0 = 0.09276122$, and $y_4 = 0.19184435$, with computed moments, 2.2001, 1.5596, and -0.1498.

### 3.2.3. Truncated Poisson Distribution

The truncated Poisson distribution may be considered as a doubly bounded distribution

$$P_r = k_N^* \frac{\theta^r}{r!}, \quad (r = 0, 1, 2, \cdots, N, \theta > 0)$$

and $k_N^* = (1 + \theta/1! + \cdots + \theta^N/N!)^{-1}$, $P_0 = k_N^*$, $P_N = k_N^* \theta^N/N!$. Recurrence is

$$y_r = \left(1 + \frac{C_2 - x}{C_0 + C_1 x}\right) y_{r-1} \quad (r = 1, 2, \cdots, N; x = r - \mu_1')$$

where $C_0 = \mu_1'$, $C_1 = 1$, $C_2 = \theta - \mu_1'$, and $\mu_1' = \theta(1 - k_N^* \theta^N/N!)$.

For another example, we take $P_s = k^* \theta^{s+2}/(s+2)!$, $(s = 0, 1, 2, 3, 4)$, with $\theta = 1$. The moments are $\mu_1' = 0.390716$, $\mu_2 = 0.446954$, $\mu_3 = 0.549920$, and $\mu_4 = 1.303602$. We find $C_0 = 2.390716$, $C_1 = 1$, and $C_2 = -1.390716$, and computed $y_s$ identical to the true probabilities $P_s$.

## 4. THE DOUBLY INFINITE CASE

### 4.1. Formulae

Suppose the probability $y_r$ now includes the negative integers and $r = 0, \pm 1, \pm 2, \cdots$. Then

$$\sum_{r=-\infty}^{\infty} x^s (y_r - y_{r-1}) = -\nu_s$$

$$\sum_{r=-\infty}^{\infty} x^s y_{r-1} = \lambda_s.$$

In other words the new equations for $(C_0, C_1, C_2, C_3)$ are those in (5) with $y_0$ taken to be zero (this does not mean that the actual $y_0$ is zero). We have after elementary operations,

$$\begin{cases} C_1 + C_2 + C_3 = 1, \\ C_0 + 3\mu_2 C_2 = \mu_2, \\ 3\mu_2 C_1 + (4\mu_3 + 3\mu_2)C_2 + \mu_2 C_3 = \mu_3 + 2\mu_2, \\ 3\mu_2 C_0 + (\mu_3 + 3\mu_2)C_1 + (5\mu_4 + 6\mu_3 + 4\mu_2)C_2 + (2\mu_2 + \mu_3)C_3 = \mu_4 + 3\mu_3 + 3\mu_2. \end{cases}$$

leading to the solution (if valid)

$$\left\{\begin{array}{l} C_0 = \mu_2(4\beta_2 - 3\beta_1 - 1/\mu_2)/\Delta, \\ C_1 + \alpha = \mu_1' + (8\beta_2 - 9\beta_1 - 12 + 1/\mu_2)/\Delta, \\ C_2 = (2\beta_2 - 3\beta_1 - 6 + 1/\mu_2)/\Delta, \\ \alpha = \mu_1' + 1/2 - (\mu_3/\mu_2)(\beta_2 + 3 - 1/\mu_2)/\Delta, \\ \Delta = 10\beta_2 - 12\beta_1 - 18 + 2/\mu_2, \end{array}\right. \tag{8}$$

with $\sqrt{\beta_1} = \mu_3/\mu_2^{3/2}$ and $\beta_2 = \mu_4/\mu_2^2$. We have then

$$y_0 = \left\{1 + (k_1 + k_0^{-1}) + (k_2k_1 + k_{-1}^{-1}k_0^{-1}) + (k_3k_2k_1 + k_{-2}^{-1}k_{-1}^{-1}k_0^{-1}) + \cdots\right\}^{-1} \tag{9}$$

involving in some cases a slowly converging series.

## 4.2. Examples

### 4.2.1. Pearson Type IV Moments

Consider the case, $\mu_1' = 1$, $\mu_2 = 2$, $\sqrt{\beta_1} = 1.5$, $\beta_2 = 12$. Using (8)

$$C_0 = 1.072368, \quad C_1 = 0.750120, \quad C_2 = 0.154605, \quad C_3 = 0.095274,$$

and using 75 terms of (9), the returned computed moments are

$$\mu_1' = 0.999998, \quad \mu_2 = 1.999805, \quad \sqrt{\beta_1} = 1.4930, \quad \beta_2 = 11.3460.$$

### 4.2.2. Discrete Pseudo-Normal Distribution

Next consider the discrete pseudo-normal distribution, using the moments for the continuous case, $\mu_1' = 0$, $\mu_2 = 1$, $\sqrt{\beta_1} = 0$, $\beta_2 = 3$. Here $y_0 = 0.3974$ ( using 100 or so terms of (9)) and $1/\sqrt{2\pi} = 0.3989$. Moreover $y_1/y_0 = 0.6112$ and $e^{-1/2} = 0.6065$. Using the probabilities from the recursion, the first four moment parameters check to at least five significant digits and

$$y_r = \left\{\frac{(r-4)^2 + 2}{(r+3)^2 + 2}\right\} y_{r-1}. \quad (r = 0, \pm 1, \cdots)$$

For a general normal case, with $\mu_1' = \mu$, $\mu_2 = \sigma^2$, $\sqrt{\beta_1} = 0$, $\beta_2 = 3$, we find

$$y_r = \left\{\frac{x^2 - (6\sigma^2 + 2)x + 12\sigma^4 + 5\sigma^2 + 1}{x^2 + 6\sigma^2 x + (12\sigma^2 - 1)\sigma^2}\right\} y_{r-1}. \quad (r = 0, \pm 1, \cdots; \ x = r - \mu_1').$$

Now define

$$y_r = ke^{-r^2/2}, \qquad (r = 0, \pm 1, \cdots).$$

Then direct calculation finds $\mu_2 = 0.999999788$, $\mu_4 = 3.00007069$. Using these moments, we retrieve $y_0$, $y_{\pm 1}$, etc. and the recurrence

$$y_r = \left( \frac{11.9980855 - 7.999362604r + r^2}{10.9987 + 5.999362608r + r^2} \right) y_{r-1} \quad (r = 0,\ \pm 1,\ \cdots),$$

with $y_0$ differing slightly from 0.3974 found by taking $\mu_2 = 1$, $\mu_4 = 3$ ($\mu_1' = 0$, $\mu_3 = 0$).

### 4.2.3. Pearson Type VII Moments

Suppose a distribution has moments

$$\mu_1' = 0, \quad \mu_2 = 3, \quad \mu_3 = 0, \quad \mu_4 = 75.$$

The Type IV model yields the results:

$$y_r = \left( \frac{12 - 4r + r^2}{9 + 2r + r^2} \right) y_{r-1}. \quad (r = 0,\ \pm 1,\ \pm 2,\ \cdots)$$

The first few values of the probabilities are (there being symmetry)

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $y$ | 0.2751 | 0.2063 | 0.09709 | 0.03641 | 0.01324 | 0.00512 | 0.00215 |

Note that $y_1/y_0 = 0.75$, $y_2/y_1 = 0.4706$, $y_3/y_2 = 0.375$. The returned moments are

$$\mu_1' = 0, \quad \mu_2 = 2.9997, \quad \mu_3 = -0.0008, \quad \mu_4 = 70.2831,$$

using $y_{-75}$ to $y_{75}$ in the computations. Note that $k_{1-r} = k_r^{-1}$.

The fit is poor when the distribution used is $y_0 = p$, $y_{\pm r} = pq^{|r|}/2$ ($r \neq 0$), with $p = q = 1/2$, and $y_r/y_{r-1} = 1/2$, $r \geq 2$. The example serves as a reminder that some structures will fail in this Type IV and other similar models.

### 4.2.4. Bessel Distribution

Bessel Distribution (see Johnson and Kotz, [3]; Skellam, [10]) can arise as the distribution of the difference between two independent Poisson variables (means $\theta_1$ and $\theta_2$) and probability generating function (P.G.F.); $exp(\theta_1 t + \theta_2/t - \theta_1 - \theta_2)$ with recurrence $\theta_2 P_{s+2} = \theta_1 P_s - (s+1)P_{s+1}$, then

$$Pr(r_1 - r_2 = t) = e^{-\theta_1 - \theta_2}(\theta_1/\theta_2)^{t/2} I_t(2\sqrt{\theta_1\theta_2}) \quad (\theta_1, \theta_2 > 0)$$

in terms of the Bessel function $I_t(\cdot)$.

The cumulant generating function is

$$\theta_1 e^{\alpha} + \theta_2 e^{-\alpha} - \theta_1 - \theta_2$$

so that

$$\begin{cases} \kappa_1 = \theta_1 - \theta_2, \\ \kappa_2 = \theta_1 + \theta_2, \\ \kappa_3 = \theta_1 - \theta_2, \\ \kappa_4 = \theta_1 + \theta_2. \end{cases}$$

and if $\theta_1 = \theta_2 = \theta$, then $\mu'_1 = 0$, $\mu_2 = 2\theta$, $\mu_3 = 0$, and $\mu_4 = 2\theta + 12\theta^2$. Moreover, in this case

$$y_0 = e^{-2\theta} I_0(2\theta).$$

If $\theta = 1$, then the doubly infinite model

$$y_r = \left(1 + \frac{C_3 - r}{C_0 + C_1 r + C_2 r^2}\right) y_{r-1}, \qquad (r = 0, \pm 1, \pm 2, \cdots)$$

with $C_0 = 1.5$, $C_1 = 5/12$, $C_2 = 1/112$, $C_3 = 1/2$ and

$$y_r = \left(\frac{24 - 7r + r^2}{18 + 5r + r^2}\right) y_{r-1},$$

and from $y_0 = (1 + 2k_1 + 2k_1k_2 + \cdots)^{-1}$, we find $y_0 = 0.294427$ (true is 0.308508).

| $r$ | Pearson Discrete Distribution | Bessel Distribution |
|---|---|---|
| 0 | 0.294427 | 0.308508 |
| 1 | 0.220820 | 0.215269 |
| 2 | 0.096609 | 0.093239 |
| 3 | 0.027603 | 0.028791 |
| 4 | 0.006134 | 0.006865 |
| 5 | 0.001263 | 0.001330 |
| 6 | 0.000271 | 0.000217 |
| 7 | 0.000064 | 0.000030 |
| 8 | 0.000017 | 0.000004 |
| 9 | 0.000005 | |
| 10 | 0.000002 | |
| 11 | 0.000001 | |

### 4.2.5. Bessel Distribution with Bias

We have $\theta_1 = 4$, $\theta_2 = 1$, then, $\mu'_1 = 3$, $\mu_2 = 5$, $\mu_3 = 3$, and $\mu_4 = 80$, Basic model is

$$y_r = \left(1 + \frac{C_3 - x}{C_0 + C_1 x + C_2 x^2}\right) y_{r-1} \quad (r = 0, \pm 1, \pm 2, \cdots, x = r - 3)$$

and solutions are $C_0 = 4.574468$, $C_1 = 0.7375589$, $C_2 = 0.028369$, and $C_3 = 0.234043$. Pearson estimates $y_0$ is 0.076536 and true is 0.076152.

## 5. A GENERALIZATION

### 5.1. Formulae

Suppose in the recursive scheme in (2) that the numerator is of degree $t(\geq 1)$ whereas the denominator is of degree $r > t$, both in the variable $x = r - \mu_1'$. Then $2r + t - 1$ moments will be required, the model being

$$y_r = \left\{1 + \frac{\sum_{\lambda=0}^{t} \hat{C}_\lambda x^\lambda}{\sum_{\lambda=0}^{r} C_\lambda x^\lambda}\right\} y_{r-1} \quad (r = 1, 2, \cdots; y_0 \neq 0)$$

where $\hat{C}_\lambda$ and $C_\lambda$ are rational fractions with numerator and denominator of degree 1 and linear in $y_0$.

The equations become

$$\begin{aligned} C_0(\nu_s + k_s y_0) &+ C_1(\nu_{s+1} + k_{s+1} y_0) + \cdots + C_r(\nu_{r+s} + k_{r+s} y_0) \\ &+ \hat{C}_0 \lambda_s + \hat{C}_1 \lambda_{s+1} + \cdots + \hat{C}_t \lambda_{s+t} = 0. \\ &\quad (\hat{C}_1 = -1;\ s = 0, 1, \cdots,\ r + t) \end{aligned} \tag{10}$$

We consider the case for which the probabilities are zero on the negative real axis.

The $r + t + 1$ equations are linear in the unknown $y_0$ ($\neq 0$). By multiplying the $s$th equation ($s = 0, 1, \cdots, r + t$) by $\mu_1'$ and adding to the $(s + 1)$th equation, there will be one equation only involving $y_0$ explicitly, namely when $s = 0$. The remaining $r + t$ equations will involve $r + t + 1$ unknowns. By a simple matrix inversion, we can find $r + t$ of the parameters $C$ and $\hat{C}$ in terms of an excluded parameter. A return to (10) with $s = 0$ determines the $r + t + 1$ unknowns.

The type IV case equations follow by taking $y_0 = 0$ in (10) but not in the equation analogous to (9).

### 5.2. The Hansmann Distribution

Hansmann [2] considered the Pearson Type symmetric family defined by

$$\frac{1}{y}\frac{dy}{dx} = \frac{-x}{C_0 + C_2 x^2 + C_4 x^4} \quad (-\infty < x < \infty) \tag{11}$$

and gave explicit solutions for seven different forms. A discrete form of this would be $y_r = k_r y_{r-1}$ with

$$y_r = \left(1 + \frac{\alpha - r}{C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4}\right) y_{r-1} \tag{12}$$

$$(r = \pm 1, \pm 2, \cdots; x = r - \mu_1', C_5 = \alpha - \mu_1')$$

in the doubly infinite case; recall that here $y_{-1} = y_0/k_0$, $y_{-2} = y_0/(k_0 k_{-1})$ etc. In (11) there is no contradiction when positive and negative values of the argument are considered, whereas an exact discrete form would clearly be invalid; however it could be considered when $y_r$ refers to the positive axis only. In the doubly infinite case, $C_1$

and $C_3$ in (12) play an important role.

There are six parameters in (12) determined from

$$\begin{aligned} C_0 y_0 + C_1(\nu_1 - y_0\mu_1') &+ C_2(\nu_2 + y_0\mu_1'^2) + C_3(\nu_3 - y_0\mu_1'^3) \\ &+ C_4(\nu_4 + y_0\mu_1') + C_5\lambda_0 = \lambda_1 \end{aligned} \tag{13}$$

and

$$\begin{aligned} C_0(\nu_s &+ \mu_1'\nu_{s-1}) + C_1(\nu_{s+1} + \mu_1'\nu_s) + C_2(\nu_{s+2} + \mu_1'\nu_{s+1}) \\ &+ C_3(\nu_{s+3} + \mu_1'\nu_{s+2}) + C_4(\nu_{s+4} + \mu_1'\nu_{s+3}) + C_5(\lambda_s + \mu_1'\lambda_{s-1}) = \lambda_{s+1} + \mu_1'\lambda_s. \end{aligned} \tag{14}$$

$$(s = 1, 2, 3, 4, 5)$$

These 5 equations (14) in 6 unknowns are used in (13) to determine the 6 unknowns.

For the double infinite case, use the modified parameters given in section **4**.

### 5.3. Further Comments on the Models

We can not say what further forms of the fundamental recurrence w'l turn out to be useful for practical situations. Much will depend on sample sizes and the range of $r$ in the probability $y_r$; another important factor is the response of the model to sampling errors in higher sample moments (note, for example, that for a quartic denominator and cubic numerator ten moments are required). If we are not dealing with experimental data but with theoretical structures for which a set of moments are available then a model may prove of some use although perhaps difficult to gain insight from.

There are several choices available for the numerator if we extend models like (12). Consider the case of a cubic. Then

$$k_r = \frac{\hat{C}_0 + \hat{C}_1 x + \hat{C}_2 x^2 + \hat{C}_3 x^3}{C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4}$$

and one parameter is available for disposal. The basic form in (1) and its extension in (12) would suggest using $\hat{C}_1 = -1$. Then for the eight unknowns we have in the semi-infinite case

$$\begin{aligned} C_0 y_0 &+ C_1(\nu_1 - \mu_1' y_0) + C_2(\nu_2 + \mu_1'^2 y_0) + C_3(\nu_3 - \mu_1'^3 y_0) \\ &+ C_4(\nu_4 + \mu_1'^4 y_0) + \hat{C}_0\lambda_0 + \hat{C}_1\lambda_1 + \hat{C}_2\lambda_2 + \hat{C}_3\lambda_3 = 0 \end{aligned}$$

along with the seven equations (eight unknowns)

$$\begin{aligned} C_0(\nu_s &+ \mu_1'\nu_{s-1}) + C_1(\nu_{s+1} + \mu_1'\nu_s) + C_2(\nu_{s+2} + \mu_1'\nu_{s+1}) + \\ &+ C_3(\nu_{s+3} + \mu_1'\nu_{s+2}) + C_4(\nu_{s+4} + \mu_1'\nu_{s+3}) + \hat{C}_0(\lambda_s + \mu_1'\lambda_{s-1}) \\ &+ \hat{C}_1(\lambda_{s+1} + \mu_1'\lambda_s) + \hat{C}_2(\lambda_{s+2} + \mu_1'\lambda_{s+1}) + \hat{C}_3(\lambda_{s+3} + \mu_1'\lambda_{s+2}) = 0. \end{aligned}$$

$$(s = 1, 2, \cdots, 7;\ \hat{C}_1 = -1, \lambda_s = E(x+1)_s,\ \nu_s = E\{(x+1)^s - x^s\})$$

## 5.4. Examples

### 5.4.1. Discrete Pseudo-Normal Distribution

Take $\mu_1' = 0$, $\mu_2 = 1$, $\mu_3 = 0$, $\mu_4 = 3$, $\mu_5 = 0$ and $\mu_6 = 15$. The model is

$$y_r = \left(1 + \frac{(\hat{C}_0) - x + \hat{C}_2 x^2}{C_0 + C_1 x + C_2 x^2}\right) y_{r-1}.$$

$$(r = 0, \pm 1, \pm 2, \cdot, \hat{C}_0 = \alpha - \mu_1')$$

Evaluated values are

$$\begin{cases} C_0 = 0.785715 = 11/14 \\ C_1 = 0.428572 = 6/14 \\ C_2 = 0.071429 = 1/14 \\ \hat{C}_0 = 0.5 \\ \hat{C}_2 = 0 \ . \end{cases}$$

This is identical to the earlier case 3.2.2 with $\hat{C}_2$ taken to be zero.

### 5.4.2. Discrete Pseudo-Normal with the First 8 Moments

The equations in matrix form are:

$$\begin{bmatrix} 0 & 1 & 1 & 4 & 7 & 1 \\ 1 & 1 & 4 & 7 & 26 & 1 \\ 1 & 4 & 7 & 26 & 61 & 2 \\ 4 & 7 & 26 & 61 & 232 & 4 \\ 7 & 26 & 61 & 232 & 659 & 10 \\ 26 & 61 & 232 & 659 & 2610 & 26 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ \hat{C}_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 10 \\ 26 \\ 76 \end{bmatrix}$$

$$k_r = \left(1 + \frac{\hat{C}_0 - r}{\sum_{s=0}^{4} C_s r^s}\right). \qquad (r = 0, \pm 1, \pm 2, \cdots)$$

The solution is

$$k_r = \phi(-R)/\phi(R), \qquad (R = r - 1/2)$$

and

$$\phi(R) = 7430.0625 + 3710R + 605.5R^2 - 7R^4,$$

with a negative coefficient of $R^4$. The probabilities are

| $r$ | 0 | $\pm 1$ | $\pm 2$ | $\pm 3$ | $\pm 4$ | $\pm 5$ |
|---|---|---|---|---|---|---|
| $y_r$ | 0.4006 | 0.2432 | 0.05419 | 0.004454 | 0.0001375 | $0.517 \times 10^{-6}$ |

with sum 1.0046; $y_6$ is negative.

A curiosity is the comparison of this case with the approximation $y_r \approx e^{-\frac{1}{2}r^2}/\sqrt{2\pi}$, for which $y_r/y_{r-1} = e^{\frac{1}{2}-r}$.

| Ratio $y_r/y_{r-1}$ | | |
|---|---|---|
| $r$ | Discrete Model | Normal Approx. |
| 1 | 0.606825 | 0.606531 |
| 2 | 0.222874 | 0.223130 |
| 3 | 0.082410 | 0.082085 |
| 4 | 0.030319 | 0.030197 |
| 5 | 0.003759 | 0.011109 |
| -0.5 | 2.718164 | 2.718282 (e) |

### 5.4.3. Discrete Pseudo-Normal with the First 10 Moments

We take $R = r - 1/2$ and

$$\begin{aligned} p_s &= \sum_{-\infty}^{\infty}(y_r - y_{r-1})R^s = 0 \quad if\ s\ is\ even \\ &= \sum_{-\infty}^{\infty} y_r\left[(r-1/2)^s - (r+1/2)^s\right], \\ q_s &= \sum_{-\infty}^{\infty} y_{r-1}R^s = \sum_{-\infty}^{\infty}(r+1/2)^s y_r. \end{aligned}$$

Thus $p_{2s+1} = -2q_{2s+1}$. New "enriched model" is

$$y_r - y_{r-1} = y_{r-1}\left(\frac{-R - 2A_3^*R^3}{A_0^* + A_1^*R + A_2^*R^2 + A_3^*R^3 + A_4^*R^4}\right)$$

and

$$k_r = \frac{A_0^* + (A_1^* - 1)R + A_2^*R^2 - A_3^*R^3 + A_4^*R^4}{A_0^* + A_1^*R + A_2^*R^2 + A_3^*R^3 + A_4^*R^4} = \Phi(R).$$

$$(R = r - 1/2; r = 0, \pm 1, \pm 2, \cdots)$$

Then $\Phi(R)\Phi(-R) = 1$ provided $A_1^* = 1 - A_1^*$ or $A_1^* = 1/2$.

The equations in matrix form with four parameters are:

$$\begin{bmatrix} p_1 & p_3 & 2q_4 & p_5 \\ p_3 & p_5 & 2q_6 & p_7 \\ p_5 & p_7 & 2q_8 & p_9 \\ p_7 & p_9 & 2q_{10} & p_{11} \end{bmatrix}\begin{bmatrix} A_0^* \\ A_2^* \\ A_3^* \\ A_4^* \end{bmatrix} = \begin{bmatrix} -q_2 \\ -q_4 \\ -q_6 \\ -q_8 \end{bmatrix}.$$

The solution is

$$\begin{cases} A_0^* = 31025/(8\cdot 3878) \\ A_1^* = 1939/3878 \\ A_2^* = 411/3878 \\ A_3^* = 44/3878 \\ A_4^* = 2/3878. \end{cases}$$

### 5.4.4. Discrete Pseudo-Normal with the First 16 Moments

The "enriched model" is

$$k_r = \frac{A_0 - A_1R + A_2R^2 - A_3R^3 + A_4R^4 - A_5R^5 + A_6R^6}{A_0 + A_1R + A_2R^2 + A_3R^3 + A_4R^4 + A_5R^5 + A_6R^6}.$$

$$(R = r - 1/2; r = 0, \pm1, \pm2, \cdots)$$

The equations in matrix form with six parameters are:

$$\begin{bmatrix} p_1 & p_3 & 2q_4 & p_5 & 2q_6 & p_7 \\ p_3 & p_5 & 2q_6 & p_7 & 2q_8 & p_9 \\ p_5 & p_7 & 2q_8 & p_9 & 2q_{10} & p_{11} \\ p_7 & p_9 & 2q_{10} & p_{11} & 2q_{12} & p_{13} \\ p_9 & p_{11} & 2q_{12} & p_{13} & 2q_{14} & p_{15} \\ p_{11} & p_{15} & 2q_{14} & p_{15} & 2q_{16} & p_{17} \end{bmatrix} \begin{bmatrix} A_0 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix} = \begin{bmatrix} -q_2 \\ -q_4 \\ -q_6 \\ -q_8 \\ -q_{10} \\ -q_{12} \end{bmatrix}.$$

The solution is

$$\begin{cases} A_0 = 1.00000002049 \\ A_1 = 0.5 \\ A_2 = 0.113276449857 \\ A_3 = 0.014971578751 \\ A_4 = 0.001224410918 \\ A_5 = 0.000059007755 \\ A_6 = 0.000001311283. \end{cases}$$

The probabilities are

| $r$ | with 10 moments | with 16 moments | $e^{-r^2/2}/\sqrt{2\pi}$ |
|---|---|---|---|
| 0 | 0.39894030 | 0.39894228 | 0.39894228 |
| $\pm1$ | 0.24197226 | 0.24197072 | 0.24197072 |
| $\pm2$ | 0.05399025 | 0.05399097 | 0.05399097 |
| $\pm3$ | 0.00443204 | 0.00443185 | 0.00443185 |
| $\pm4$ | 0.00013381 | 0.00013383 | 0.00013383 |
| $\pm5$ | 0.000001486 | 0.000001487 | 0.000001487 |
| $\sum$ | 1.0000 | 1.0000 | 1.0000 |

Note that $y_{-1/2}/y_{-3/2}$ for the 4, 10, 16 moments model has the values 2.697, 2.718278, and 2.718281840 respectively as approximants to $e$.

### 5.4.5. Discrete Bessel Distribution with the First 10 Moments

We consider the case with $\theta_1 = \theta_2 = 1/2$. The first 10 moments are 0, 1, 0, 4, 0, 31, 0, 379, 0, and 6556. The basic model with four moments is

$$y_r = \left[\frac{(2R-4)^2 + 11}{(2R+4)^2 + 11}\right] y_{r-1} \qquad (R = r - 1/2)$$

and "enriched model with 10 moments is

$$y_r = \left(\frac{A_0 - A_1R + A_2R^2 - A_3R^3 + A_4R^4}{A_0 + A_1R + A_2R^2 + A_3R^3 + A_4R^4}\right) y_{r-1}.$$

The solution is

$$A_0 = 1145/(8 \cdot 236), \quad A_1 = 118/236, \quad A_2 = 115/236, \quad A_3 = 24/236, \quad A_4 = 2/236.$$

| | Basic Model | 10 moments | 16 moments | $e^{-1}I_r(1)$ |
|---|---|---|---|---|
| $r$ | $y_r$ | $y_r$ | $y_r$ | $y_r$ |
| 0 | 0.422312 | 0.455065 | 0.464351 | 0.465760 |
| ±1 | 0.234617 | 0.213919 | 0.208621 | 0.207910 |
| ±2 | 0.046923 | 0.049170 | 0.049910 | 0.049939 |
| ±3 | 0.006120 | 0.008246 | 0.008177 | 0.008155 |
| ±4 | 0.000927 | 0.001026 | 0.001007 | 0.001007 |
| ±5 | 0.000185 | 0.000098 | 0.000100 | 0.000100 |
| ±6 | 0.000047 | 0.000008 | 0.000008 | 0.000008 |
| ±7 | 0.000014 | 0.000001 | 0.000001 | 0.000001 |
| ±8 | 0.000005 | | | |
| ±9 | 0.000002 | | | |
| ±10 | 0.000001 | | | |
| $\sum y_r$ | 1.000000 | 1.000000 | 1.000000 | |

## 6. THE ALGEBRAIC STRUCTURE FOR THE DOUBLY BOUNDED CASE

### 6.1. First Order Equations

From the fundamental moment parameter in (3) we deduce the following:

$$\begin{aligned}
\sum_{r=1}^{N}(y_r - y_{r-1})(x^s - Tx^{s-1}) &= -l_s \quad (s = 1, 2, \ldots) \\
&= -\left[\hat{l}_s + (-1)^s \mu_1'^{s-1}(N+1)y_0\right] \qquad (15) \\
\sum_{r=1}^{N} y_{r-1}(x^s - Tx^{s-1}) &= n_s \\
&= \lambda_s - T\lambda_{s-1} \qquad (16) \\
\sum_{r=1}^{N}(y_r - y_{r-1})(x^s + \mu_1' x^{s-1}) &= -k_s \\
&= -\left[\hat{k}_s - T^{s-1}(N+1)y_N\right] \qquad (17) \\
\sum_{r=1}^{N} y_{r-1}(x^s + \mu_1' x^{s-1}) &= m_s
\end{aligned}$$

$$= \quad \hat{m}_s - T^{s-1}(N+1)y_N \tag{18}$$

where

$$\begin{cases} \hat{l}_s = \nu_s - T\nu_{s-1} \\ \hat{k}_s = \nu_s + \mu_1'\nu_{s-1} \\ \hat{m}_s = \lambda_s + \mu_1'\lambda_{s-1} \end{cases}$$

**6.2. Second Order Equations**

$$\begin{aligned} \sum_{r=1}^{N}(y_r - y_{r-1}0(x+\mu_1')(x-T)x^{s-2} &= \sum_{r=1}^{N}(y_r - y_{r-1})\left[x^2 + (\mu_1' - T)x - \mu_1'T\right]x^{s-2} \\ &= -\nu_s + (T-\mu_1')\nu_{s-1} + \mu_1'T\nu_{s-2} \\ &= -L_s \end{aligned} \tag{19}$$

$$\begin{aligned} \sum_{r=1}^{N} y_{r-1}(x+\mu_1')(x-T)x^{s-2} &= \sum_{r=1}^{N} y_{r-1}x^{s-2}\left[x^2 + (\mu_1' - T)x - \mu_1'T\right] \\ &= M_s \qquad (s = 2,3,\cdots) \\ &= \lambda_s + (\mu_1' - T)\lambda_{s-1} - \mu_1'T\lambda_{s-2} \end{aligned} \tag{20}$$

It will be seen from (15-16) that the moment-parameter involved eliminates $y_N$ as an explicit component; similarly the moment operator in (17-18) eliminates $y_0$. Again the "quadratic" moment parameter in (19-20) eliminates $y_0$ and $y_N$ as far as they appear explicitly. These formulas show that, in the general discrete model, the parameters of the multiplier $k_s$ will be ratios of polynomials each involving a constant, and terms in $y_0$, $y_N$ and $y_0y_N$ only. We illustrate using the basic model for which

$$k_r = 1 + \frac{\hat{C}_0 - x}{C_0 + C_1x + C_2x^2}. \qquad (r = 1, 2, \cdots, N) \tag{21}$$

Using (15-20) the equations now become

$$\begin{cases} [\hat{l}_1 - (N+1)y_0]C_0 + [\hat{l}_2 + \mu_1'(N+1)y_0]C_1 + [\hat{l}_3 - \mu_1'^2(N+1)y_0]C_2 + n_1\hat{C}_0 = n_2 \\ \left[\hat{k}_1 - (N+1)y_N\right]C_0 + \left[\hat{k}_2 - T(N+1)y_N\right]C_1 + \left[\hat{k}_3 - T(N+1)y_N\right]C_2 \\ \qquad\qquad + [\hat{m}_1 - (N+1)y_N]\hat{C}_0 = \hat{m}_2 - T(N+1)y_N \\ L_2C_0 + L_3C_1 + L_4C_2 + M_2\hat{C}_0 = M_3 \\ L_3C_0 + L_4C_1 + L_5C_2 + M_3\hat{C}_0 = M_4 \end{cases} \tag{22}$$

(Note that the powers of $\mu_1'$ in the first equation of (22) alternate in sign, whereas those of $T$ in the second equation do not.)

Th last two equations do not contain $y_0$ or $y_N$; $y_0$ is linear in the first equation, and $y_N$ linear in the second equation. Again it will be seen that the first three columns of the underlying matrix refer to the denominator of $k_r$. Clearly the determinant of the system $\Delta$ involves only a constant, $y_0$, $y_N$ and $y_0y_N$; similarly for the numerators of $C_0$,

$C_1$, $C_2$ and $\hat{C}_0$. Algebraic solutions of (22) and similar generalizations should present no problem using a computer language such as Maple; one would assume the existence of moments, and non-singular matrices. Numerically one would use $y_0^*$ and $y_N^*$ as seeds in (22) and attempt iterative solutions of $\sum_1^N y_r = 1$, and $y_N = y_N \cdot y_{N-1} \cdots \cdot y_0$.

To study generalization of (21), suppose the denominator is $C_0 + C_1 x + \cdots C_\lambda x^\lambda$, and the numerator $\hat{C}_0 + \hat{C}_1 x + \cdots + \hat{C}_u x^u$. We elect which parameter shall be unity; suppose it is $\hat{C}_u$ for simplicity; then $\lambda + u + 1$ equations are needed. The linearized model is then operated on by (15) and (16); then (17) and (18) and finally (19) and (20). The first $\lambda + 1$ columns of the matrix will contain $(\hat{l}, \hat{k}, L)$, and the remaining $u$ columns will contain $(n, m, M)$.

## 7. THE EXPONENTIAL AND THE NORMAL

### 7.1. The Continued Fraction

Recalling that for the discrete normal we assume there is the approximation $y_r = e^{-r^2/2}/\sqrt{2\pi}$, so that $k_r = y_r/y_{r-1} = e^{1/2-r} = e^R$. But there is a continued fraction (c.f.) for $k_r$, namely

$$e^R = \frac{1}{1-}\,\frac{R}{1+}\,\frac{R}{2-}\,\frac{R}{3+}\,\frac{R}{2-}\,\frac{R}{5+}\cdots \qquad (|R| < \infty) \tag{23}$$

with convergents $\chi_s(R)/\omega_s(R)$, where

| $s$ | $\chi_s(R)$ | $\omega_s(R)$ |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |
| 2 | 1 | $1 - R$ |
| 3 | $2 + R$ | $2 - R$ |
| 4 | $6 + 2R$ | $6 - 4R + R^2$ |
| 5 | $12 + 6R + R^2$ | $12 - 6R + R^2$ |
| 6 | $60 + 24R + 3R^2$ | $60 - 36R + 9R^2 - R^3$ |
| 9 | $1680 + 840R + 180R^2 + 20R^3 + R^4$ | $1680 - 840R + 180R^2 - 20R^3 + R^4$ |

It will be seen that there is a subset of convergents of the form $[\phi(R)/\phi(-R)]$, $\phi(\cdot)$ being a polynomial. In fact the subset is $\chi_{4s+1}(R)/\omega_{4s+1}(R)$. And these consist of polynomials which are positive for all real $R$; they are thus compatible with the desired structure of $k_r$.

It is of some interest to note the definite integral form (not apparently given in the standard textbooks on c.f.s):

$$\begin{aligned}
\chi_{2s}(R) &= \frac{1}{(s-1)!}\int_0^\infty e^{-t}t^s(t+R)^{s-1}dt \\
\omega_{2s}(R) &= \frac{1}{(s-1)!}\int_0^\infty e^{-t}t^{s-1}(t-R)^s dt \\
\chi_{2s+1}(R) &= \frac{1}{s!}\int_0^\infty e^{-t}t^s(t+R)^s dt
\end{aligned}$$

$$\omega_{2s+1}(R) = \frac{1}{s!}\int_0^\infty e^{-t}t^s(t-R)^s dt$$

These can be proved by integration by parts and reference to the c.f. in (23).

### 7.2. Relation to Whittaker Functions

There are asymptotic forms for $R \to \infty$ and $R \to -\infty$. From Whittaker and Watson [11] we first of all consider positive $R$. Now

$$W_{k,m}(z) = \frac{e^{-z/2}z^k}{\Gamma(1/2-k+m)}\int_0^\infty e^{-t}t^{-k-1/2+m}(1+t/z)^{k-1/2+m}dt \quad (\Re(k-1/2+m) \leq 0)$$

for all $z$ except negative reals. Hence

$$\begin{cases} \omega_{2s}(-R) = R^{s-1/2}e^{R/2}W_{1/2,s}(R) \\ \omega_{2s+1}(-R) = R^s e^{R/2}W_{0,s+1/2}(R) \quad (R > 0) \end{cases}$$

When $R > 0$ there are two components in the integrals. Here

$$\begin{aligned} \omega_{2s}(R) &= \frac{1}{(s-1)!}\int_0^\infty e^{-t}t^{s-1}(t-R)^s dt \\ &= \frac{(-1)^s}{(s-1)!}\int_0^R e^{-t}t^{s-1}(R-t)^s + \frac{e^{-R}}{(s-1)!}\int_0^\infty e^{-t}t^s(R+t)^{s-1}dt \end{aligned}$$

The first term relates to the confluent hypergeometric

$$M(a,b;z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}dt, \quad (\Re b > \Re a > 0)$$

and the second to Whittaker's $W$. In fact,

$$\omega_{2s}(R) = (-1)^s R^{2s}\frac{\Gamma(s+1)}{\Gamma(2s+1)}M(s,2s+1;-R) + sR^{s-1/2}e^{-R/2}W_{-1/2,s}(R). \quad (R < 0)$$

Similarly,

$$\omega_{2s+1}(R) = (-1)^s R^{2s+1}\frac{\Gamma(s+1)}{\Gamma(2s+2)}M(s+1,2s+2;-R) + e^{-R/2}R^s W_{0,s+1/2}(R)$$

Whittaker and Watson [11] give the asymptotic form for $|R|$ large, the basic asymptotic being

$$W_{k,m}(z) \sim e^{-z/2}z^\kappa. \qquad (|\arg z| \leq \pi - \alpha < \pi)$$

In our present context it will be seen that

$$\lim_{s\to\infty}\left(\frac{\chi_{4s+1}(-R)}{\omega_{4s+1}(-R)}\right) = e^{-R} = \lim_{s\to\infty}\left(\frac{\omega_{4s+1}(R)}{\omega_{4s+1}(-R)}\right)$$

### 7.3. Application to the Normal

We can now set up the discrete normal not using moments but appropriate convergents of (23). For example, we may use

$$k_r = k_R^{(5)} = \frac{12 - 6R + R^2}{12 + 6R + R^2}$$

and the equation

$$y_0[1 + 2k_1^{(5)} + 2k_2^{(5)}k_1^{(5)} + \cdots] = 1, \tag{24}$$

to determine $y_0$. Note however there is a small error in using $\sum_{-\infty}^{\infty} y_r = 1$; incidentally Aitken [1] gave $\sum_{-\infty}^{\infty} e^{-r^2/2} = 2.506628288$ (nearly) or $\sum_{-\infty}^{\infty} y_r \sim 1.000,000,005$.

An alternative is to assume $y_0 = 1/\sqrt{2\pi}$ and determine the sum from (24). For the former choice, $y_0 = 0.397466120$ and $1/\sqrt{2\pi} \sim 0.398942280$.

Higher order convergents, such as

$$k_r^{4s+1} = \omega_{4s+1}(R)/\omega_{4s+1}(-R) \qquad (s = 2.3.\cdots)$$

lead to using the limiting value $exp(-R)$ so that $y_r/y_0$ is seen as the product

$$\prod_{s=1}^{r} exp[-(2s-1)/2],$$

a very interesting interpretation, providing a link between c.f.s and the discrete normal.

### 7.4. Moments of the Discrete Normal Distribution

We use Hermite polynomials $\{H_s(x)\}$, an orthogonal system with respect to the basic normal density $exp(-x^2/2)/\sqrt{2\pi}$, and for which

$$\begin{cases} H_s(x) = e^{-D_x^2/2}x^s, \\ x^s = e^{D_x^2/2}H_s(x). \quad (D_x \equiv d/dx;\ D_x H_s(x) = sH_{s-1}(x)) \end{cases} \tag{25}$$

Consider then the evaluation of

$$h_{2s}^* = \sum_{r=-\infty}^{\infty} \frac{H_s(r)e^{-r^2/2}}{\sqrt{2\pi}}. \qquad (s = 0, 1, \ldots)$$

Then the Euler summation formula (see Konrad Knopp), [4] shows that

$$h_{2s}^* = \int_{-\infty}^{\infty} \frac{H_{2s}(x)e^{-x^2/2}dx}{\sqrt{2\pi}} + \Re_1,$$

where

$$\Re_1 = \int_0^{\infty} P_3(x)\frac{d^3}{dx^3}\left[\frac{H_{2s}(x)e^{-x^2/2}}{\sqrt{2\pi}}\right]dx,$$

and

$$P_3(x) = \sum_{n=1}^{\infty} \frac{2\sin(2n\pi x)}{(2n\pi)^3}.$$

Integration by parts, with assumptions regarding uniform convergence, leads to

$$k_{2s}^* = \delta_{s,0} + h_{2s}, \tag{26}$$

where

$$h_{2s} = 2(-1)^s \sum_{t=1}^{\infty} (2\pi t)^{2s} e^{-2(\pi t)^2}.$$

For examples,

$$h_0^* = 1 + 2(e^{-2\pi^2} + e^{-8\pi^2} + e^{-18\pi^2} + \cdots),$$
$$h_2^* = -8\pi^2(e^{-2\pi^2} + 2^2 e^{-8\pi^2} + 3^2 e^{-18\pi^2} + \cdots),$$

with regard to $h_0^*$. Dr. Robert Byers using a variable precision package "Derive", gives

$$\sqrt{2\pi} h_0^* = 2.506,628,288,042,905,544,830,67,$$

and

$$\sqrt{2\pi}(h_0^* - 1) = 0.000,000,013,411,905,042,414,91.$$

From (25) and (26) we now have the general formula for the central moments of the discrete normal distribution, namely

$$\mu_{2s} = \sum_{m=0}^{s} \frac{(2s)^{(2m)}}{2^m m!} \left( \frac{\delta_{2s-2m,0} + h_{2s-2m}}{1 + h_0} \right).$$

For examples,

$$\mu_2 = [h_2 + (1 + h_0)]/(1 + h_0),$$
$$\mu_4 = [h_4 + 6h_2 + 3(1 + h_0)]/(1 + h_0).$$

## 8. CONCLUDING REMARKS

We have studied various aspects of the Pearson discrete model in generalized forms, including the doubly infinite case, semi-infinite case and the bounded case. The examples chosen have been directed at gaining insight into the models and their implementation. New interesting properties have been discovered with respect to the normal case; on the one hand using the standard normal moments as against the corrected moments.

Applications to empirical data and estimation problems are not undertaken at this time.

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