

1979 02 1989

PEP Note--303  
May 1979

## EMITTANCE IN A FODO-CELL LATTICE

R.H. Helm and H. Wiedemann

Stanford Linear Accelerator Center  
Stanford University, Stanford, California 94305

Several PEP Notes (1), (2) give formulas and curves for the variation of beam parameters in a FODO lattice as functions of focal length. However, because of approximations used in Refs.(1) and (2), the results for emittance are not accurate when the focusing is too strong. We give here the emittance calculation correct for any phase advance per cell between 0 and  $\pi$ .

We consider the case of a symmetrical FODO cell with equal focusing and defocusing quadrupoles in thin-lens approximation, and with the space between the lenses completely filled by rectangular bending magnets. The horizontal matrix per half-cell is

$$M = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \rho \sin \theta & 2\rho \sin^2 \frac{\theta}{2} \\ 0 & 1 & 2 \tan \frac{\theta}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - k\rho \sin \theta & \rho \sin \theta & 2\rho \sin^2 \frac{\theta}{2} \\ -k^2 \rho \sin \theta & 1 + k\rho \sin \theta & 2 \tan \frac{\theta}{2} (1 + \frac{1}{2} k\rho \sin \theta) \\ 0 & 0 & 1 \end{bmatrix}$$

where  $k = 1/f$  is the inverse focal length of half a quadrupole,  $\rho$  is the bending radius of a magnet, and  $\theta$  is the bending angle per half-cell. It follows from the transformation matrix that the horizontal functions are

$$\sin \psi = k \rho \sin \theta \quad (1)$$

$$\hat{\beta} = \frac{2 \rho \sin \theta (1 + k \rho \sin \theta)}{\sin 2 \psi} \quad (2)$$

$$\hat{\eta} = \frac{2 \rho (1 - \cos \theta) (1 + \frac{1}{2} k \rho \sin \theta)}{\sin^2 \psi} \quad (3)$$

where  $\psi$  phase advance per half-cell, and  $\hat{\beta}$  and  $\hat{\eta}$  are the betatron and dispersion functions at the focusing lens. The values of  $\check{\beta}$  and  $\check{\eta}$  (at the defocusing lens) are given by changing the sign of  $k$  in Eqs. (2) and (3). Just inside the entrance of the bend magnet,

$$\beta_0 = \hat{\beta} \quad (4)$$

$$\eta_0 = \hat{\eta} \quad (5)$$

$$\alpha_0 = -\frac{1}{2} \beta_0' = (k - \frac{1}{\rho} \tan \frac{\theta}{2}) \hat{\beta} \quad (6)$$

$$\eta_0' = -(k - \frac{1}{\rho} \tan \frac{\theta}{2}) \hat{\eta} \quad (7)$$

By substituting Eqs. (1) through (7) in Eq. (20) of Ref. (3) we find after interminable algebra

$$\langle X \rangle = \frac{2 \rho}{\sin \theta \sin^2 \psi \sin 2 \psi} \left\{ 16 \sin^4 \frac{\theta}{2} - 2 \sin^2 \psi \sin^2 \frac{\theta}{2} (5 - 2 \cos \theta - 3 \frac{\sin \theta}{\theta}) + \sin^4 \psi (2 + \cos \theta - 3 \frac{\sin \theta}{\theta}) \right\} \quad (8a)$$

In small bending-angle approximation,

$$\langle X \rangle \approx \frac{2 \rho \theta^3}{\sin^2 \psi \sin 2 \psi} \left\{ 1 - \frac{3}{4} \sin^2 \psi + \frac{1}{60} \sin^4 \psi \right\} \quad (8b)$$

If the phase advance  $\psi$  is also small,

$$\langle X \rangle \approx \frac{\rho \theta^3}{\psi^3} \quad (8c)$$

If the phase advance per cell,  $2\psi$ , is close to  $\pi$ ,

$$\langle \mathcal{M} \rangle = \frac{8}{15} \frac{\rho e^3}{\pi - 2\psi}$$

Figure 1 shows Eq.(8b) compared to Eq.(24) of Ref.(1). The behavior of the emittance integral, in varying as  $\psi^{-3}$  at low tunes, going through a minimum around  $2\psi = 3\pi/4$  and diverging at  $2\psi = \pi$ , has been found by numerical studies to be typical for a wide range of cell designs. For example, similar behavior is found in cells with thick quadrupoles, with bend magnets which do not fill the space between the quadrupoles, in combined function as well as separated function cells, and in cells with unequal vertical and horizontal tunes.

#### REFERENCES

1. H. Wiedemann, Scaling of FODO-Cell Parameters, PEP-39, Stanford Linear Accelerator Center internal note (1973).
2. J. Rees and B. Richter, Preliminary Design of a 15-GeV Electron-Positron Variable Tune Storage Ring, SPEAR-167/PEP-69, Stanford Linear Accelerator Center internal note (1973).
3. R.H. Helm, M.J. Lee, P.L. Morton, and M. Sands, Evaluation of Synchrotron Radiation Integrals, IEEE Trans. on Nucl. Sci., NS-20, p. 900, (1973).

#### DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

46 5373

WE SEMI-LOGARITHMIC 1 CYCLES X 90 DIVISIONS  
RUMPLE 5 3294 50

