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EMITTANCE IN A FODO-CELL LATTICE

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Several PEP Notes (1), (2) give formulas and curves for the variation of beam parameters in a FODO lattice as functions of focal length. However, because of approximations used in Refs.(1) and (2), the results for emittance are not accurate when the focusing is too strong. We give here the emittance calculation correct for any phase advance per cell between 0 and π.

We consider the case of a symmetrical FODO cell with equal focusing and defocusing quadrupoles in thin-lens approximation, and with the space between the lenses completely filled by rectangular bending magnets. The horizontal matrix per half-cell is

$$M = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \rho\sin\theta & 2\rho\sin^2\frac{\theta}{2} \\ 0 & 1 & 2\tan\frac{\theta}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-k\rho\sin\theta & \rho\sin\theta & 2\rho\sin^2\frac{\theta}{2} \\ -k^2\rho\sin\theta & 1+k\rho\sin\theta & 2\tan\frac{\theta}{2}\,(1+\frac{1}{2}k\rho\sin\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

where $k = 1/f$ is the inverse focal length of half a quadrupole, ρ is the bending radius of a magnet, and θ is the bending angle per half-cell. It follows from the transformation matrix that the horizontal functions are

$$\sin\psi = k\,\rho\,\sin\theta \tag{1}$$

$$\hat{\beta} = \frac{2\,\rho\,\sin\theta\,(1 + k\,\rho\,\sin\theta)}{\sin 2\psi} \tag{2}$$

$$\hat{\eta} = \frac{2\,\rho\,(1 - \cos\theta)(1 + \tfrac{1}{2}\,k\,\rho\,\sin\theta)}{\sin^2\psi} \tag{3}$$

where ψ phase advance per half-cell, and $\hat{\beta}$ and $\hat{\eta}$ are the betatron and dispersion functions at the focusing lens. The values of $\check{\beta}$ and $\check{\eta}$ (at the defocusing lens) are given by changing the sign of k in Eqs.(2) and (3). Just inside the entrance of the bend magnet,

$$\beta_o = \hat{\beta} \tag{4}$$

$$\eta_o = \hat{\eta} \tag{5}$$

$$\alpha_o = -\tfrac{1}{2}\,\beta_o' = \left(k - \frac{1}{\rho}\tan\frac{\theta}{2}\right)\hat{\beta} \tag{6}$$

$$\eta_o' = -\left(k - \frac{1}{\rho}\tan\frac{\theta}{2}\right)\hat{\eta} \tag{7}$$

By substituting Eqs.(1) through (7) in Eq.(20) of Ref.(3) we find after interminable algebra

$$\langle\mathcal{H}\rangle = \frac{2\,\rho}{\sin\theta\,\sin^2\psi\,\sin 2\psi}\left\{16\sin^4\frac{\theta}{2} - 2\sin^2\psi\,\sin^2\frac{\theta}{2}\left(5 - 2\cos\theta - 3\,\frac{\sin\theta}{\theta}\right) + \sin^4\psi\left(2 + \cos\theta - 3\,\frac{\sin\theta}{\theta}\right)\right\} \tag{8a}$$

In small bending-angle approximation,

$$\langle\mathcal{H}\rangle \simeq \frac{2\rho\theta^3}{\sin^2\psi\;\sin 2\psi}\left\{1 - \frac{3}{4}\sin^2\psi + \frac{1}{60}\sin^4\psi\right\} \tag{8b}$$

If the phase advance ψ is also small,

$$\langle\mathcal{H}\rangle \simeq \frac{\rho\theta^3}{\psi^3} \tag{8c}$$

If the phase advance per cell, 2ψ, is close to π,

$$\langle\mathcal{H}\rangle \simeq \frac{8}{15}\,\frac{\rho\theta^3}{\pi - 2\psi}$$

Figure 1 shows Eq.(8b) compared to Eq.(24) of Ref.(1). The behavior of the emittance integral, in varying as ψ^{-3} at low tunes, going through a minimum around $2\psi = 3\pi/4$ and diverging at $2\psi = \pi$, has been found by numerical studies to be typical for a wide range of cell designs. For example, similar behavior is found in cells with thick quadrupoles, with bend magnets which do not fill the space between the quadrupoles, in combined function as well as separated function cells, and in cells with unequal vertical and horizontal tunes.

REFERENCES

1. H. Wiedemann, Scaling of FODO-Cell Parameters, PEP-39, Stanford Linear Accelerator Center internal note (1973).
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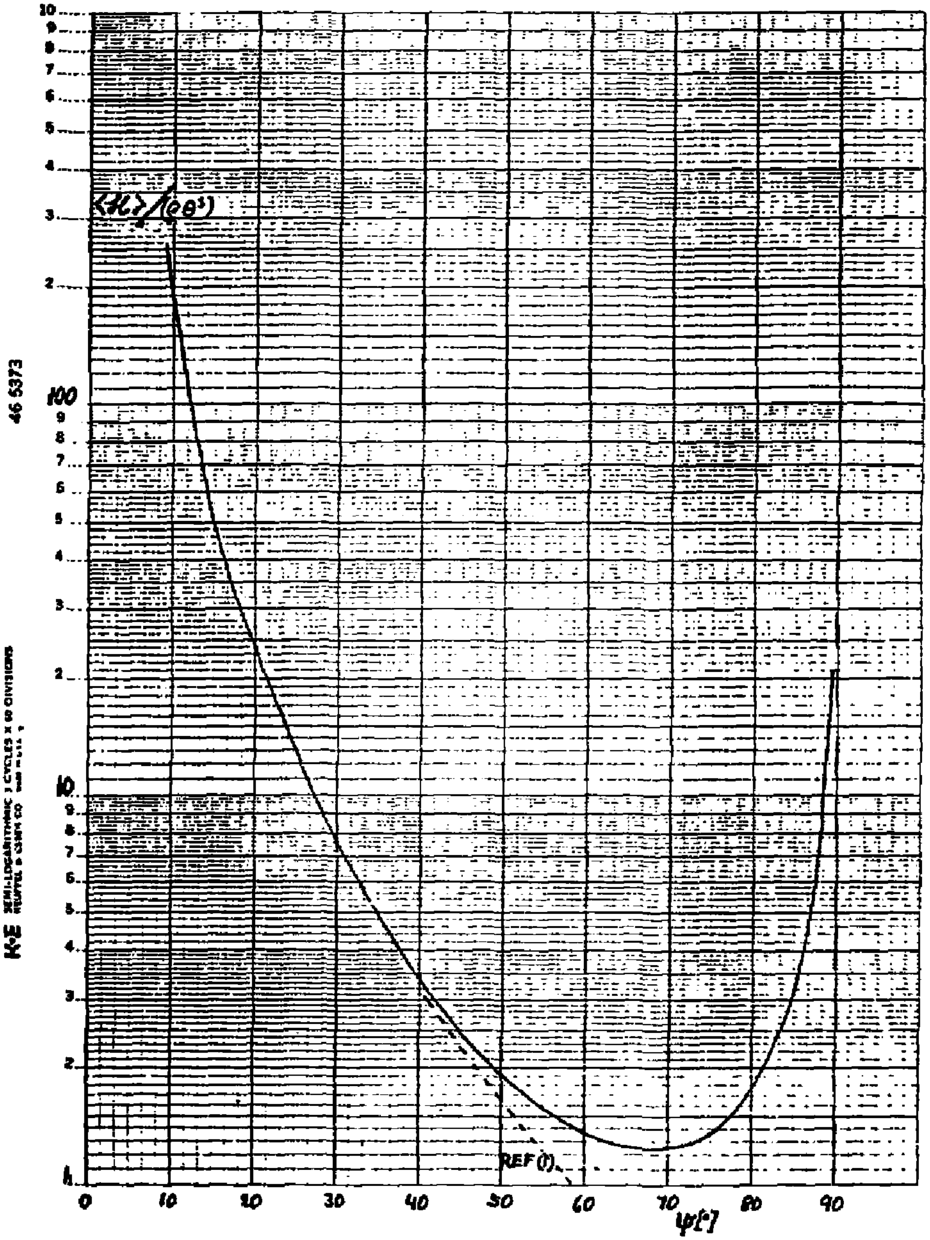

⟨H⟩/(ϱθ³)
REF (1)
ψ[°]
0
10
20
30
40
50
60
70
80
90
100
10
1
46 5373
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