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COMPLETE FLUID EQUATIONS FOR LOW-n SINGULAR MODES IN AXISYMMETRIC TOROIDAL PLASMAS

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A system of equations governing the singular region of low-n MHD instabilities in a collisional toroidal plasma with magnetic shear has been derived from Braginskii's magnetized fluid equations.[1] These equations are an extension of the equations of Glasser, Greene, and Johnson[2] (GGJ), incorporating all important fluid processes. They can be used to describe the narrow layer in the neighborhood of a singular surface for tearing and interchange modes. Formal ordering assumptions describing the behavior of the perturbations in the singular region are introduced into Braginskii's equations. The ordering is chosen to allow for the maximum range of physical effects while retaining the principal terms in GGJ and ensuring the validity of the fluid equations. Time dependence is expressed in terms of a complex growth rate by restricting consideration to modes which vary rapidly on the time scale of variation of the equilibrium. One spatial dimension is treated as ignorable because of axisymmetry, retaining a single Fourier component. While the pressure must be constant along a field line in order to exclude fast sound waves, temperature and density are not individually constant. A second-order ODE along the field line with periodic coefficients is derived for the temperature. A periodic solution for temperature must be obtained in order to complete the determination of the variation along the field line. A second spatial dimension can then be eliminated by averaging over this known behavior. The third spatial dimension, the distance across the singular layer, survives as the independent variable of a 12th-order system of ODEs. Laplace transformation, using specially constructed contours in the complex plane, is used to reduce the order from 12 to 4. Asymptotic limits of the solutions are matched to ideal MHD behavior in the region far from the singular region, yielding a dispersion relation which can be solved to obtain growth rates and stability criteria. Most of the new effects are related primarily to ion behavior. Transverse ion thermal conductivity and viscosity prop up the singular layer width and prevent it from falling below the ion gyration radius. Parallel ion viscosity in the presence of toroidal variation of the magnetic field strength along the field line imposes a drag on the perturbed parallel velocity. Gyroviscosity in the presence of the same toroidal variation breaks the reflectional symmetry of the singular region equations about the singular surface. In addition, the equations incorporate anisotropic resistivity, diamagnetic rotation, and related effects.

Singular modes are among the most prominent in determining the limits on the operations of tokamaks and similar toroidal plasmas. Low-n resistive tearing modes cause disruptions in tokamaks and limit achievable current density. Low-n resistive interchange modes cause anomalous transport in spheromaks and reversed field pinches. High-n resistive ballooning modes cause anomalous transport in high-beta tokamaks and limit achievable β. All these modes are characterized by global ideal MHD regions matched to narrow singular regions where nonideal effects are dominant. The singular region acts like the grid of a triode, where many small effects can influence large flows of energy. Existing theories include very limited dynamics in the singular region. The goal of this work is to develop a complete linear theory of the singular region, including all important dynamical effects. The present phase of the work treats the more collision fluid regime. A later phase will treat the less collisional gyrokinetic regime. This paper concerns the derivation and form of the fluid equations for the singular region of low-n modes. Later work will treat high-n ballooning modes. In addition, the ordering in the present work must be amended before it is applicable to the neighborhood of the field reversal surface of the RFP.

We begin with Braginskii's equations for a magnetized fluid, the equations for the total fluid mass,

$$\frac{\partial N}{\partial t} + \nabla\cdot(N\mathbf{v}) = 0,$$

momentum

$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\Big) = \frac{1}{c}\mathbf{J}\times\mathbf{B} - \nabla p - \nabla\cdot\pi,$$

the temperature for species $j = e, i$,

$$\frac{3}{2}N\Big(\frac{\partial T_j}{\partial t} + \mathbf{v_j}\cdot\nabla T_j\Big) + NT_j\nabla\cdot\mathbf{v_j} + \nabla\cdot\mathbf{q}_j + \pi_j : \nabla\mathbf{v}_j = Q_j,$$

Ohm's Law,

$$Ne\Big(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\Big) = \mathbf{R} + \frac{1}{c}\mathbf{J}\times\mathbf{B} - \nabla p_e$$

the potential representation of the electromagnetic fields,

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla\times\mathbf{A},$$

Ampère's law without displacement current,

$$\nabla^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{J},$$

and quasi-neutrality,

$$\nabla\cdot\mathbf{J} = 0.$$

The viscosity tensor π can be expressed as a sum of five 4th-rank tensors acting on the velocity gradients,

$$\pi = \sum_{i=0}^{4} \pi_i, \qquad \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = \frac{\mathbf{B}}{B}, \qquad \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij},$$

in terms of the parallel terms

$$\pi_0 = -3\eta_0 \left[\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 - \frac{1}{3}\mathbf{I}\right]\left[\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 - \frac{1}{3}\mathbf{I}\right] : \nabla\mathbf{v},$$

transverse terms

$$\pi_1 = -\eta_1 \Big[(\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_2)(\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_2) + (\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3)(\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3)\Big] : \nabla\mathbf{v},$$

$$\pi_2 = -\eta_2 \Big[(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_1)(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_1) + (\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_1)(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_1)\Big] : \nabla\mathbf{v},$$

and gyroviscous terms

$$\pi_3 = -\eta_3 \Big[(\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3)(\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_2) - (\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_2)(\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3)\Big] : \nabla\mathbf{v},$$

$$\pi_4 = -\eta_4 \Big[(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_1)(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_1) - (\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_1)(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_1)\Big] : \nabla\mathbf{v},$$

The momentum exchange terms can be expressed as a sum of friction and thermal force terms, $\mathbf{R} = \mathbf{R}_u + \mathbf{R}_T$,

$$\mathbf{R}_u = \frac{1}{Ne}\alpha \cdot \mathbf{j} = \frac{1}{Ne}\left[\alpha_{\|}\hat{\mathbf{e}}_1 \cdot \mathbf{j} - \alpha_{\wedge}\hat{\mathbf{e}}_1 \times \mathbf{j} + \alpha_{\perp}\left(\mathbf{I} - \hat{\mathbf{e}}_1\hat{\mathbf{e}}_1\right) \cdot \mathbf{j}\right],$$

$$\mathbf{R}_T = -\beta \cdot \nabla T = -\left[\beta_{\|}\hat{\mathbf{e}}_1 \cdot \nabla T + \beta_{\wedge}\hat{\mathbf{e}}_1 \times \nabla T + \beta_{\perp}\left(\mathbf{I} - \hat{\mathbf{e}}_1\hat{\mathbf{e}}_1\right) \cdot \nabla T\right].$$

The heat flux can be expressed as a sum of thermal conductivity and thermoelectric terms, $\mathbf{q} = \mathbf{q}_u + \mathbf{q}_T$,

$$\mathbf{q}_u = -\frac{T}{Ne}\beta \cdot \mathbf{j} - \frac{T}{Ne}\left[\beta_{\|}\hat{\mathbf{e}}_1 \cdot \mathbf{j} + \beta_{\wedge}\hat{\mathbf{e}}_1 \times \mathbf{j} + \beta_{\perp}\left(\mathbf{I} - \hat{\mathbf{e}}_1\hat{\mathbf{e}}_1\right) \cdot \mathbf{j}\right],$$

$$\mathbf{q}_T = -\chi \cdot \nabla T = -\left[\chi_{\|}\hat{\mathbf{e}}_1 \cdot \nabla T + \chi_{\wedge}\hat{\mathbf{e}}_1 \times \nabla T + \chi_{\perp}\left(\mathbf{I} - \hat{\mathbf{e}}_1\hat{\mathbf{e}}_1\right) \cdot \nabla T\right],$$

The transport coefficients obey scaling laws expressed in terms of the cyclotron frequency $\Omega = eB/mc$ and the collision mean free time $\tau \sim T^{3/2}/N$ for each species. These scaling laws are

$$\alpha_{\|} \sim \frac{Nm}{\tau}, \qquad \alpha_{\wedge} \sim \frac{\alpha_{\|}}{\Omega\tau}, \qquad \alpha_{\perp} \sim \alpha_{\|},$$

$$\beta_{\|} \sim N, \qquad \beta_{\wedge} \sim \frac{\beta_{\|}}{\Omega\tau}, \qquad \beta_{\perp} \sim \frac{\beta_{\|}}{\Omega^2\tau^2},$$

$$\chi_{\|} \sim \frac{NT\tau}{m}, \qquad \chi_{\wedge} \sim \frac{\chi_{\|}}{\Omega\tau}, \qquad \chi_{\perp} \sim \frac{\chi_{\|}}{\Omega^2\tau^2},$$

$$\eta_0 \sim NT\tau, \qquad \eta_1 \sim \eta_2 \sim \frac{\eta_0}{\Omega^2\tau^2} \qquad \eta_3 \sim \eta_4 \sim \frac{\eta_0}{\Omega\tau},$$

We perturb about an axisymmetric equilibrium, with all equilibrium scalars independent of the toroidal polar angle ϕ and with no fluid velocity and no strong anisotropy, satisfying the equations

$$\mathbf{J} = \frac{c}{4\pi}\nabla \times \mathbf{B}, \qquad \nabla \cdot \mathbf{J} = \nabla \cdot \mathbf{B} = 0,$$

$$\frac{1}{c}\mathbf{J} \times \mathbf{B} = \nabla P, \qquad \sigma \equiv \frac{\mathbf{J}\cdot\mathbf{B}}{B^2}, \qquad \mathbf{J} = \sigma\mathbf{B} + c\frac{\mathbf{B}\times\nabla P}{B^2},$$

$$\mathbf{B} = RB_T\nabla\phi + \frac{1}{2\pi}\nabla\phi \times \nabla\chi$$

$$\nabla\cdot\mathbf{J} = \mathbf{B}\cdot\nabla\left[\sigma + \frac{2\pi cRB_T}{B^2}\frac{dP}{d\chi}\right] = 0,$$

$$\mathbf{B}\cdot\nabla N = \mathbf{B}\cdot\nabla T = \mathbf{B}\cdot\nabla P = \mathbf{v} = 0,$$

$$\mathbf{B}\cdot\nabla(RB_T) = \mathbf{B}\cdot\nabla\chi = \mathbf{B}\cdot\nabla\Phi = 0.$$

Perturbed quantities in the singular region are described in terms of a coordinate system closely related to and derived from Hamada coordinates. The Hamada system uses the coordinate volume V enclosed within a magnetic flux surface to label the surface, and angular coordinates θ and ζ, which increase by 1 in going around the torus the short way and the long way, respectively, to label a point on the surface. Such coordinates can be constructed for which the Jacobian $J \equiv (\nabla V\cdot\nabla\theta\times\nabla\zeta)^{-1} = 1$. The poloidal magnetic flux is denoted $\chi(V)$ and the toroidal magnetic flux is denoted $\psi(V)$. The magnetic field has the representation

$$\mathbf{B} = \nabla V \times (\psi'\nabla\theta - \chi'\nabla\zeta)$$

The magnetic field components can then be expressed as $\mathbf{B}\cdot\nabla\theta = \chi'$, $\mathbf{B}\cdot\nabla\zeta = \psi'$, and $\mathbf{B}\cdot\nabla V = 0$, so that for any quantity f, the derivative along the field line is given by

$$\mathbf{B}\cdot\nabla f = \left(\chi'\frac{\partial}{\partial\theta} + \psi'\frac{\partial}{\partial\zeta}\right)f(V,\theta,\zeta)$$

The safety factor is defined as $q(V) = \psi'/\chi' = d\psi/d\chi$. For any quantity f depending only on V, $f'(V)$ denotes df/dV. Equilibrium scalars are independent of ζ. The scalar quantity $\mathbf{B} \cdot \nabla V \times \nabla\theta$ can be expressed as $2\pi R B_T$.

Modified Hamada coordinates introduce further simplification in the neighborhood of a rational surface, where the safety factor $q_0 \equiv q(V_0) = m/n$ is a rational number. We define the new coordinate $\beta \equiv \zeta - q_0\theta$ and make the formal coordinate transformation $V, \theta, \zeta \rightarrow V, \theta, \beta$, which preserves the unit Jacobian. Any function which is independent of ϕ or ζ is also independent of β. Two of the magnetic field components in this coordinate system remain the same, $\mathbf{B} \cdot \nabla\theta = \chi'$, $\mathbf{B} \cdot \nabla V = 0$, but the third one becomes

$$\mathbf{B} \cdot \nabla\beta = \chi'(q - q_0) \approx \chi' q' x$$

where $x \equiv V - V_0$ is the small distance from the rational surface and the approximation is obtained from the first term in a Taylor expansion, assuming the shear q' does not vanish at the rational surface. For any function f, we can write

$$\mathbf{B} \cdot \nabla f = \chi' \Big(\frac{\partial}{\partial\theta} + q' x \frac{\partial}{\partial\beta} \Big) f(V, \theta, \beta),$$

$$\Big\langle f(V, \theta, \beta) \Big\rangle \equiv \int_0^1 f(V, \theta, \beta) d\theta$$

In order to retain all terms which are of comparable importance, in the fluid regime, we construct an ordering which satisfies four criteria: 1. it is consistent with conditions for the validity of Braginskii's fluid equations; it is consistent with retaining all the physics in GGJ; it is as consistent as possible with conditions in a realistic magnetic confinement system, subject to limitations imposed by the first two criteria; it allows the inclusion of a maximum number of physical effects. The following set of conditions satisfies all of these conditions.

$$n \sim q \sim 1, \qquad \frac{r_{L,i}}{z} \sim \frac{\lambda_i}{R} \sim \epsilon \ll 1$$

$$\beta \sim \frac{a}{R} \sim \frac{B_P}{B} \sim \frac{z}{a} \sim \epsilon^2, \qquad \frac{r_{L,i}}{a} \sim \epsilon^3$$

$$\frac{\nu_i}{\Omega_i} \sim \frac{z}{R} \sim \frac{m_e}{m_i} \sim \epsilon^4, \qquad \frac{r_{L,i}}{R} \sim \epsilon^5$$

$$\frac{\omega}{\Omega_i} \sim \frac{\omega^*}{\Omega_i} \sim \epsilon^6,$$

with n the toroidal mode number, q the safety factor, r_L, i the ion Larmor radius, z the resistive layer thickness, a a characteristic poloidal distance, R the major radius, β the ratio of fluid pressure to field energy density, B the total field strength, B_P the poloidal field strength, m_e and m_i the electron and ion masses, λ_i the ion mean free path, ν_i the ion collision frequency, ω the mode frequency, ω^* the diamagnetic rotation frequency, and Ω_i the ion cyclotron frequency.

To lowest order, the momentum equation can be solved for the transverse components of the current to yield

$$\mathbf{j}^{(0)} = j_{\parallel}^{(0)}\hat{\mathbf{e}}_1 + \frac{c}{B^2}\mathbf{B} \times \nabla p^{(0)}.$$

and the parallel component of the equation of motion yields

$$\frac{\partial p^{(0)}}{\partial \theta} = 0.$$

The quasineutrality equation yields the lowest-order result

$$\frac{\partial}{\partial \theta}\Big[j_{\parallel}^{(0)} + \frac{2\pi c R B_T}{\chi' B^2}\frac{\partial p^{(0)}}{\partial V}\Big] = 0.$$

This can be used to express the parallel current in terms of a quantity which is constant on a field line and known variation along the field line,

$$j_{\parallel}^{(0)}B = \Big\langle \mathbf{j}\cdot\mathbf{B}\Big\rangle^{(0)}\frac{B^2}{< B^2 >} + \frac{2\pi c R B_T}{\chi'}\frac{\partial p^{(0)}}{\partial V}\Big(\frac{B^2}{< B^2 >} - 1\Big).$$

Similarly, the transverse components of Ohm's law can be solved for the transverse velocity,

$$\mathbf{v}^{(0)} = v_{\parallel}^{(0)}\hat{\mathbf{e}}_1 + \frac{c}{B^2}\mathbf{B} \times \nabla\hat{\varphi}^{(0)},$$

where we have defined the quantity

$$\hat{\varphi}^{(0)} \equiv \varphi^{(0)} + \frac{p_i^{(0)}}{Ne}$$

The parallel component of Ohm's law yields the lowest-order result,

$$\frac{\partial \hat{\varphi}^{(0)}}{\partial \theta} = 0,$$

and the lowest-order pressure equation yields incompressibility to lowest order,

$$\frac{\partial}{\partial \theta}\Big[v_{\parallel}^{(0)} + \frac{2\pi c R B_T}{\chi' B^2}\frac{\partial \hat{\varphi}^{(0)}}{\partial V}\Big] = 0$$

This can be used to express the parallel velocity in terms of a quantity which is constant on a field line and known variation along the field line,

$$v_{\parallel}^{(0)}B = \Big\langle \mathbf{v}\cdot\mathbf{B}\Big\rangle^{(0)}\frac{B^2}{< B^2 >} + \frac{2\pi c R B_T}{\chi'}\frac{\partial \hat{\varphi}^{(0)}}{\partial V}\Big(\frac{B^2}{< B^2 >} - 1\Big)$$

The parallel component of Ampere's yields an equation for the parallel component of the vector potential,

$$\frac{\partial^2}{\partial V^2}\Big\langle \mathbf{A}\cdot\mathbf{B}\Big\rangle^{(0)} = -\frac{4\pi}{c}\Big[\frac{< B^2/|\nabla V|^2 >}{< B^2 >}\Big\langle \mathbf{j}\cdot\mathbf{B}\Big\rangle^{(0)} + \frac{2\pi c R B_T}{\chi'}\Big(\frac{< B^2/|\nabla V|^2 >}{< B^2 >} - \Big\langle\frac{1}{|\nabla V|^2}\Big\rangle\Big)\frac{\partial p^{(0)}}{\partial V}\Big].$$

The above relations, governing the behavior of various quantities along a field line, are derived from lowest order in our ordering. At next order, we obtain inhomogenous equations for the variation of first-order quantities along the field lines, expressed in terms of first-order operators on zeroth-order quantities. The lowest-order relations can be used to annihilate the first-order operators and yield constraints on the lowest-order quantities. The parallel component of Ohm's law yields an equation for the electrostatic potential,

$$\chi' q' x \frac{\partial \hat{\varphi}^{(0)}}{\partial \beta} + \frac{1}{c}\frac{\partial}{\partial t}\Big\langle \mathbf{A}\cdot\mathbf{B}\Big\rangle^{(0)} - \Big(\frac{\Phi'}{\chi'} - \frac{1}{Ne}\frac{P'}{\chi'}\Big)\frac{\partial}{\partial \beta}\Big\langle \mathbf{A}\cdot\mathbf{B}\Big\rangle^{(0)} + \frac{\alpha_{\|}}{N^2 e^2}\Big\langle \mathbf{j}\cdot\mathbf{B}\Big\rangle^{(0)} = 0,$$

the parallel component of the momentum equation yields and equation for the perturbed pressure,

$$\begin{aligned}
&\chi' q' x \frac{\partial p^{(0)}}{\partial \beta} + \rho\frac{\partial}{\partial t}\Big\langle \mathbf{v}\cdot\mathbf{B}\Big\rangle^{(0)} - \frac{P'}{\chi'}\frac{\partial}{\partial \beta}\Big\langle \mathbf{A}\cdot\mathbf{B}\Big\rangle^{(0)} \\
&+ 3\eta_0 \frac{<(\hat{\mathbf{e}}_1\cdot\nabla B)^2>}{<B^2>}\Big(\Big\langle \mathbf{v}\cdot\mathbf{B}\Big\rangle^{(0)} + \frac{2\pi c R B_T}{\chi'}\frac{\partial \hat{\varphi}^{(0)}}{\partial V}\Big) \\
&- \eta_2 B^2 \frac{\partial^2}{\partial V^2}\Big[\frac{<|\nabla V|^2>}{<B^2>}\Big\langle \mathbf{v}\cdot\mathbf{B}\Big\rangle^{(0)} + \frac{2\pi c R B_T}{\chi'}\Big(\frac{<|\nabla V|^2>}{<B^2>} - \Big\langle\frac{|\nabla V|^2}{B^2}\Big\rangle\Big)\frac{\partial \hat{\varphi}^{(0)}}{\partial V}\Big] \\
&- \eta_3 B\Big\langle \frac{|\nabla V|^2}{B^3}\mathbf{B}\cdot\nabla B\Big\rangle c\frac{\partial^2 \hat{\varphi}^{(0)}}{\partial V^2} = 0,
\end{aligned}$$

the quasineutrality equation yields an equation for the parallel current,

$$\begin{aligned}
&\frac{\chi' q' x}{<B^2>}\frac{\partial}{\partial \beta}\Big\langle \mathbf{j}\cdot\mathbf{B}\Big\rangle^{(0)} - \rho c^2\Big[\Big\langle\frac{|\nabla V|^2}{B^2}\Big\rangle + \Big(\frac{2\pi R B_T}{\chi'}\Big)^2\Big(\Big\langle\frac{1}{B^2}\Big\rangle - \frac{1}{<B^2>}\Big)\Big]\frac{\partial^3 \hat{\varphi}^{(0)}}{\partial t \partial V^2} \\
&- \Big\{\frac{\chi''}{\chi'^2} + \frac{2\pi R B_T q'}{<B^2>} + 4\pi\frac{P'}{\chi'}\Big[\Big\langle\frac{1}{B^2}\Big\rangle \\
&\quad + \Big(\frac{2\pi R B_T}{\chi'}\Big)^2\Big(\Big\langle\frac{1}{B^2|\nabla V|^2}\Big\rangle - \frac{<1/|\nabla V|^2>^2}{<B^2/|\nabla V|^2>}\Big)\Big]\Big\} c\frac{\partial p^{(0)}}{\partial \beta} \\
&- \frac{2\pi R B_T}{\chi'}\frac{P'}{\chi'}\Big(\frac{<1/|\nabla V|^2>}{<B^2/|\nabla V|^2>} - \frac{1}{<B^2>}\Big)\frac{\partial^2}{\partial V \partial \beta}\Big\langle \mathbf{A}\cdot\mathbf{B}\Big\rangle^{(0)} \\
&- 3\eta_0 \frac{2\pi c R B_T}{\chi'}\frac{<(\hat{\mathbf{e}}_1\cdot\nabla B)^2>}{<B^2>^2}\frac{\partial}{\partial V}\Big(\Big\langle \mathbf{v}\cdot\mathbf{B}\Big\rangle^{(0)} + \frac{2\pi c R B_T}{\chi'}\frac{\partial \hat{\varphi}^{(0)}}{\partial V}\Big) \\
&+ \Big[\eta_1 B^2\Big\langle\frac{|\nabla V|^4}{B^4}\Big\rangle + \eta_2 B^2\Big(\frac{2\pi R B_T}{\chi'}\Big)^2\Big\langle|\nabla V|^2\Big(\frac{1}{B^2} - \frac{1}{<B^2>}\Big)^2\Big\rangle\Big] c^2\frac{\partial^4 \hat{\varphi}^{(0)}}{\partial V^4} \\
&+ \frac{\eta_2 B^2}{<B^2>}\frac{2\pi c R B_T}{\chi'}\Big(\frac{<|\nabla V|^2>}{<B^2>} - \Big\langle\frac{|\nabla V|^2}{B^2}\Big\rangle\Big)\frac{\partial^3}{\partial V^3}\Big\langle \mathbf{v}\cdot\mathbf{B}\Big\rangle^{(0)} \\
&+ \frac{\eta_3 B}{<B^2>}\Big\langle\frac{|\nabla V|^2}{B^3}\mathbf{B}\cdot\nabla B\Big\rangle\frac{2\pi c R B_T}{\chi'} c\frac{\partial^3 \hat{\varphi}^{(0)}}{\partial V^3} = 0.
\end{aligned}$$

and the pressure equation yields an equation for the parallel velocity,

$$
\frac{\chi' q' x}{<B^2>}\frac{\partial}{\partial\beta}\Big\langle \mathbf{v}\cdot\mathbf{B}\Big\rangle^{(0)} - \Big(\frac{\chi''}{\chi'^2} + \frac{2\pi R B_T q'}{<B^2>}\Big) c \frac{\partial\hat{\varphi}^{(0)}}{\partial\beta}
$$

$$
- \frac{c^2}{N^2 e^2}\Big[\alpha_\perp \Big\langle \frac{|\nabla V|^2}{B^2}\Big\rangle + \alpha_\parallel \Big(\frac{2\pi R B_T}{\chi'}\Big)^2 \Big(\Big\langle \frac{1}{B^2}\Big\rangle - \frac{1}{<B^2>}\Big)\Big] \frac{\partial^2 p^{(0)}}{\partial V^2}
$$

$$
+ \frac{2\pi R B_T}{\chi'}\Big(\frac{<1/|\nabla V|^2>}{<B^2/|\nabla V|^2>} - \frac{1}{<B^2>}\Big)\frac{\partial^2}{\partial V \partial t}\Big\langle \mathbf{A}\cdot\mathbf{B}\Big\rangle^{(0)}
$$

$$
+ 4\pi\Big[\Big\langle\frac{1}{B^2}\Big\rangle + \Big(\frac{2\pi R B_T}{\chi'}\Big)^2\Big(\Big\langle\frac{1}{B^2|\nabla V|^2}\Big\rangle - \frac{<1/|\nabla V|^2>^2}{<B^2/|\nabla V|^2>}\Big)\Big]\frac{\partial p^{(0)}}{\partial t}
$$

$$
+ \Big[\frac{1}{P}\frac{\partial p^{(0)}}{\partial t} - \frac{3}{4T}\frac{\partial}{\partial t}\Big(T_e^{(0)} + <T_i>^{(0)}\Big)\Big] + \Big[\frac{P'}{P} - \frac{3}{2}\frac{T'}{T}\Big]\frac{c}{\chi'}\frac{\partial\hat{\varphi}^{(0)}}{\partial\beta}
$$

$$
- \Big(\frac{\Phi'}{\chi'} - \frac{1}{Ne}\frac{P'}{\chi'}\Big)\frac{c}{<B^2>}\frac{\partial p^{(0)}}{\partial\beta} - \frac{cN'}{2N^2 e\chi'}\Big[p^{(0)} + N\Big(<T_i>^{(0)} - T_e^{(0)}\Big)\Big] = 0.
$$

Strong electron parallel thermal conductivity forces the lowest-order electron temperature to be constant along a field line, $\chi' \partial T_e^{(0)}/\partial\theta = 0$. At first order, the electron temperature equation relates the electron and ion temperatures through the heat exchange term,

$$
\frac{3}{2}N\Big[\frac{\partial T_e^{(0)}}{\partial t} + \frac{cT'}{\chi'}\frac{\partial\hat{\varphi}^{(0)}}{\partial\beta}\Big] + \frac{1}{2}N\Big[\frac{\partial}{\partial t}\Big(T_e^{(0)} + <T_i>^{(0)}\Big) + \frac{cT'}{\chi'}\frac{\partial\hat{\varphi}^{(0)}}{\partial\beta}\Big]
$$

$$
- \frac{1}{2}\Big[\frac{\partial p^{(0)}}{\partial t} + \frac{cP'}{\chi'}\frac{\partial\hat{\varphi}^{(0)}}{\partial\beta}\Big] - \Big[\frac{3}{2}NT' - TN'\Big]\frac{c}{Ne\chi'}\frac{\partial p^{(0)}}{\partial\beta}
$$

$$
+ \frac{3}{\tau_e}\frac{m_e}{m_i}N\Big[T_e^{(0)} - <T_i>^{(0)}\Big] = 0.
$$

The ion temperature equation is more complicated, containing all components of the ion thermal conductivity,

$$
\frac{\partial}{\partial\theta}\Big[\frac{\chi_{i\parallel}}{B^2}\chi'^2\frac{\partial T_i^{(0)}}{\partial\theta}\Big] + 2\pi c R B T \chi_{i\wedge} B \chi' \frac{\partial}{\partial\theta}\Big(\frac{1}{B^2}\Big)\frac{\partial T_i^{(0)}}{\partial V} + \chi_{i\perp}|\nabla V|^2\frac{\partial^2 T_i^{(0)}}{\partial V^2}
$$

$$
- \frac{3}{2}N\Big[\frac{\partial T_i^{(0)}}{\partial t} + \frac{cT'}{\chi'}\frac{\partial\hat{\varphi}^{(0)}}{\partial\beta}\Big] - \frac{1}{2}N\Big[\frac{\partial}{\partial t}\Big(T_e^{(0)} + T_i^{(0)}\Big) + \frac{cT'}{\chi'}\frac{\partial\hat{\varphi}^{(0)}}{\partial\beta}\Big]
$$

$$
+ \frac{1}{2}\Big[\frac{\partial p^{(0)}}{\partial t} + \frac{cP'}{\chi'}\frac{\partial\hat{\varphi}^{(0)}}{\partial\beta}\Big] + \frac{3}{\tau_e}\frac{m_e}{m_i}N\Big[T_e^{(0)} - T_i^{(0)}\Big] = 0.
$$

The mathematical structure of these equations can be understood as follows. To analyze the stability of a particular mode with a particular rational surface, we choose a toroidal Fourier mode $\exp(-2\pi n\beta i)$ and a singular surface with $q_0 = m/n$, and let $\partial/\partial\beta \to -2\pi n i$. We Laplace transform the time dependence and let $\partial/\partial t \to s$, where s is the complex growth rate to be determined. We Laplace transform the variation in $x = V - V_0$ across the singular surface and let $\partial/\partial V \to ik$ and $x \to i\partial/\partial k$. The parallel component of Ampere's Law is now an algebraic equation for $< \mathbf{A} \cdot \mathbf{B} >^{(0)}$ in terms of $< \mathbf{j} \cdot \mathbf{B} >^{(0)}$ and $p^{(0)}$. The electron temperature equation is algebraic. The ion temperature equation is a 2nd-order ordinary differential equation with periodic coefficients in the independent variable θ along a field line on a rational surface. For each value of k, it must be solved to obtain a periodic solution for $T^{(0)}$. Then the moments $< T >^{(0)}$ and $< |\nabla V|^2 T >^{(0)}$ must be evaluated in terms of $p^{(0)}$ and $\hat{\varphi}^{(0)}$. The parallel components of Ohm's Law and the momentum equation, the pressure equation, and the quasineutrality equation now form a 4th-order system of ordinary differential equations with independent variable k and dependent variables $p^{(0)}$, $\hat{\varphi}^{(0)}$, $< \mathbf{j} \cdot \mathbf{B} >^{(0)}$, and $< \mathbf{v} \cdot \mathbf{B} >^{(0)}$.

The solutions to these equations must be matched to the ideal MHD solutions in the outer region, far from the singular surface. A procedure for accomplishing this matching has been developed by GGJ for analytical solutions and by Glasser, Jardin, and Tesauro for numerical solutions. Laplace transform integrals are used with carefully chosen contours in the complex k-plane for which end-point contributions vanish. The integrals are evaluated by a saddle-point approximation at large x to obtain their asymptotic behavior. This procedure must be generalized to the more complicated equations studied here. The matching condition becomes a dispersion relation which may be solved for the complex growth rate s.

The core of this system of equations is identical to that of Glasser, Greene and Johnson. They describe resistive instabilities in general axisymmetric plasma configurations with finite beta and toroidicity, including tearing and interchange modes. The new terms presented here arise from the large number of additional dissipative terms in Braginskii's equations.

Parallel viscous terms, proportional to η_0, introduce a strong drag in toroidal systems which tends to bring the perturbed parallel velocity of the mode to rest relative to the equilibrium and may therefore cause a reduction of the growth rate. They vanish in fictitious 1-dimensional equilibria such as a cylinders and slabs. These terms get large as the collision frequence drops toward the limit of validity of the fluid equations. In the collisionless limit, they constitute the effect referred to by Callen and Shaing as neoclassical MHD. Transverse viscous terms, proportional to η_1 and η_2, introduce a larger velocity diffusion than is present in the simpler resistive theory. This should tend to spread the singular layer thickness and prevent it from shrinking below the ion gyration radius. These terms are commonly neglected on the grounds that they are proportional to the collision frequency and therefore get small in the limit of low collisionality. However, the singular layer thickness also gets small in this limit, and so these terms retain their importance. Gyroviscous terms, proportional to η_3, break the reflectional symmetry about the singular surface and thus mix solutions of different parity. These terms are independent of collision frequency and vanish in equilibria with up-down symmetry.

Diamagnetic effects associated with the Hall and electron pressure gradient terms in Ohm's law produce a Doppler shift in real part of the frequency and may also affect the growth rate. The anisotropy of the resistive terms, proportional the $\alpha_\parallel$ and $\alpha_\perp$, breaks a degeneracy in the large-k behavior of the solutions. It is not clear what if any physical significance this will have. Thermoelectric terms, proportional to $\beta_\parallel$ and $\beta_\perp$, enter the equations for temperature and pressure. It is not clear what physical significance these will have. Anisotropic thermal conductivity contributes to the complicated structure of the temperature equation and causes the temperature to vary along a field line. Transverse thermal conductivity, proportional to $\chi_\perp$, dominates at large values of k, corresponding to large transverse gradients, and causes the temperature to be strongly nonuniform. These terms also contribute to the spreading of the singular layer thickness, as do the transverse viscous terms. Parallel thermal conductivity, proportional to $\chi_\parallel$, dominates at small values of k, corresponding to the region in which the solution matches to ideal MHD, and causes the temperature to be nearly uniform. Reactive cross-field terms proportional to $\chi_\wedge$ further complicate the picture.

Slight modifications of the ordering used here should yield equations describing the singular region of high-n modes in the neighborhood of the field reversal surface of the RFP and long-θ tail of high-n ballooning modes in tokamaks. The remainder of this phase of the work will consist of developing a computer code to numerically solve these equations, match them to the outer region, and systematically study the solutions as $\lambda_i/R \rightarrow 1$. Eventually, there will be a study of the gyrokinetic regime.

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