LONG-TIME BEHAVIOR OF A NUCLEAR REACTOR*

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A fundamental problem of reactor physics is the determination of the long-time behavior of the neutron population in a nuclear reactor. In particular, one is interested in the question whether the total neutron density has a purely exponential behavior as $t \to \infty$. We formulate this problem as an abstract Cauchy problem, show that the solution is given by a semigroup, and investigate the asymptotic behavior of the semigroup.

1. INTRODUCTION

A fundamental problem of reactor physics is the determination of the asymptotic behavior of a nuclear reactor for large tines. Inside a reactor (a highly heterogeneous composite structure of many different materials) neutrons are generated by fission processes. The neutrons move about freely (i.e., rectilinearly and with constant velocity) until they interact with a nucleus of the reactor material; in the course of an interaction a neutron may disappear entirely (absorption), it may change its velocity (scattering), or it may trigger a fission process, as a result of which one or more new neutrons appear. The relevant space and time scales are such that interactions can be viewed as localized and instantaneous events. The equation that describes the rate of change of the neutron density inside the reactor is a linear transport equation; the dependent variable is the neutron velocity distribution function (f). If Ω denotes the rector domain (a bounded open convex subset of $\mathbb{R}^3$), and S is the neutron velocity range (a ball or spherical shell centered at the origin in $\mathbb{R}^3$), then $f(x,\xi,t)dxd\xi$ represents the (expected) number of neutrons in a volume element dx centered at a point $x \in \Omega$ whose velocities lie in a velocity element $d\xi$ centered at the velocity $\xi \in S$ at time t. The linear transport equation is a balance equation for f over the element $dxd\xi$ about (x,ξ),

$$(1.1) \qquad \frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\cdot \xi f(x,\xi,t) - h(x,\xi)f(x,\xi,t) + \int_S k(x,\xi\leftarrow\xi')f(x,\xi',t)d\xi', \quad x \in \Omega,\ \ \xi \in S,\ \ t > 0 .$$

The first term on the right is the (spatial) divergence of the neutron flux, which represents the effect of the free streaming; the second term represents the loss due to interactions at x, $h(x,\xi)d\xi$ being the collision frequency for neutrons with the velocity in the range $d\xi$ about ξ at the point x; the third term represents the gain due to interactions at x, $k(x,\xi\leftarrow\xi')d\xi$ being the (expected) number of neutrons emerging with a velocity in the range $d\xi$ about ξ after an

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interaction of a neutron with the velocity ξ' with a nucleus of the reactor material at x. With Eq. 1.1 are prescribed an initial condition,

$$(1.2)\qquad f(x,\xi,0) = f_0(x,\xi), \qquad (x,\xi) \in \Omega\times S \ ,$$

and a boundary condition on $\partial\Omega$. The boundary condition expresses the fact that no neutrons enter the reactor from outside ("zero incoming flux"); it may be formulated as

$$(1.3)\qquad f(x,\xi,t) = 0 \ , \qquad x \in \partial\Omega \ , \quad \xi \in S_x \ , \quad t \geq 0 \ ,$$

where $S_x = \{\xi \in S: x + t\xi \in \Omega \text{ for some } t > 0\}$, $x \in \partial\Omega$.

The quantity of interest is the total neutron density inside the reactor, i.e., the integral $\int_\Omega \int_S f(x,\xi,t)dxd\xi$; in particular, its asymptotic behavior as $t \to \infty$. For practical purposes one wants to know under which conditions on the functions h and k the integral behaves like a pure exponential as $t \to \infty$. We might add that, for many reactor materials, the functions h and k vary rapidly with the neutron velocity: they may display resonances, etcetera. As we shall see, a satisfactory solution to this problem has not yet been given. Partial answers are available, and new results from the theory of strongly continuous semigroups of positive operators in Banach lattices are being applied.

In the next section we give the functional formulation of the reactor problem as an abstract Cauchy problem. In Section 3 we show that this abstract Cauchy problem is solved by a strongly continuous semigroup of positive operators. In the final Section 4 we discuss some results about the asymptotic behavior of the semigroup. Details of the proofs, as well as related results, can be found in our forthcoming monograph [1, Chapter 12].

2. FUNCTIONAL FORMULATION

Let Ω be a bounded, open, convex subset of $\mathbb{R}^3$, and let S be a ball of finite radius centered at the origin in $\mathbb{R}^3$. In this section we shall show that the initial-boundary value problem 1.1-3 leads to an abstract initial value problem for the function $f: [0,\infty) \to L^1(\Omega\times S)$. (The choice of an L^1-space is a natural one in the present context, as f is nonnegative and its L^1-norm gives the total number of neutrons inside the reactor.)

We begin with the definition of the collisionless transport operator (-T), which corresponds to the first term in the right member of Eq. 1.1. Two technical difficulties arise: one because the expression $(\partial/\partial x)\cdot\xi f$ is singular at $\xi=0$, the other because the boundary condition 1.3 involves only part of the range of the variable ξ. Let $C^\infty_{B,0}(\Omega\times S)$ be the space of all functions f that satisfy the conditions (i) supp $f \subset \Omega\times S_{\alpha\beta}$ for some $\beta \geq \alpha > 0$, where $S_{\alpha\beta} = \{\xi \in \mathbb{R}^3: \alpha \leq |\xi| \leq \beta\}$; and (ii) f admits a (B,ε)-extension to $\Omega_\varepsilon\times S$ for some $\varepsilon > 0$; here, Ω_ε is a ε-neighborhood of Ω, and a (B,ε)-extension is a function $f_\varepsilon \in C^\infty(\Omega_\varepsilon\times S)$ whose restriction to $\Omega\times S$ coincides with f and which vanishes on each incoming ray up to a point inside Ω (i.e., for each $(x,\xi) \in \Omega\times S$, let $\tau = \tau(x,\xi)$ denote the unique nonnegative number such that $x-\tau\xi \in \partial\Omega$; then there exists a $\eta \in (0,\tau)$ such that $f_\varepsilon(x-s\xi,\xi) = 0$ for all $s > \eta$.) Let T_0 be defined in $C^\infty_{B,0}(\Omega\times S)$ by the expression

$$(2.1)\qquad T_0 f(x,\xi) = \frac{\partial}{\partial x} \cdot \xi f(x,\xi) \ , \qquad (x,\xi) \in \Omega\times S \ , \quad f \in C^\infty_{B,0}(\Omega\times S) \ .$$

Then $\lambda I+T_0(\lambda\in\mathbb{C})$ is a bijective map of $C^{\infty}_{B,0}(\Omega\times S)$ onto itself. If $\mathrm{Re}\lambda > 0$, then $(\lambda I+T_0)^{-1}$ can be extended by continuity to a bounded linear operator R_λ in $L^1(\Omega\times S)$, where

$$(2.2) \qquad R_\lambda g(x,\xi) = \int_0^\tau e^{-\lambda s} g(x-s\xi,\xi)ds , \qquad g \in L^1(\Omega\times S) ,$$

for almost all $(x,\xi) \in \Omega\times S$. This operator R_λ is injective; its inverse is the closure of $\lambda I+T_0$, so if we define T by

$$(2.3) \qquad T = R_\lambda^{-1} - \lambda I ,$$

then T is uniquely defined and T is the closure of T_0.

The second and third term in the right member of Eq. 1.1 give rise to bounded linear operators in $L^1(\Omega\times S)$, provided $h \in L^\infty(\Omega\times S)$ and $h_p \in L^\infty(\Omega\times S)$, where $h_p(x,\xi') = \int_S k(x,\xi\to\xi')d\xi$ for $(x,\xi') \in \Omega\times S$. We shall assume that these conditions are met, and define the operators A_1 and A_2 in $L^1(\Omega\times S)$ by the expressions

$$(2.4) \qquad A_1 f(x,\xi) = h(x,\xi)f(x,\xi) , \qquad (x,\xi) \in \Omega\times S ,$$

$$(2.5) \qquad A_2 f(x,\xi) = \int_S k(x,\xi\leftarrow\xi')f(x,\xi')d\xi' , \qquad (x,\xi) \in \Omega\times S ,$$

for any $f \in L^1(\Omega\times S)$. Then $\|A_1\| = \|h\|_\infty$ and $\|A_2\| = \|h_p\|_\infty$.

The initial-boundary value problem 1.1-3 thus gives rise to the following abstract Cauchy problem in $L^1(\Omega\times S)$:

$$(2.6) \qquad f'(t) = (-T-A_1+A_2)f(t) , \quad t > 0 ; \quad f(0) = f_0 .$$

3. SOLUTION OF THE ABSTRACT CAUCHY PROBLEM

We consider the transport operator $-T-A_1+A_2$ as a perturbation of the streaming operator $-(T+A_1)$ by the bounded operator A_2. The spectrum $\sigma(-(T+A_1))$ is determined by the behavior of $h(x,\xi)$ for small values of $|\xi|$. Let the nonnegative constant λ^* be defined by

$$(3.1) \qquad \lambda^* = \inf\{ \lim_{|\xi|\to 0} h(x,\xi): \ x \in \Omega\} .$$

Assume that

(A) There exists a positive constant c such that $h(x,\xi) \geq \lambda^*-c|\xi|$ for all $x \in \Omega$ and all $\xi \in S$ with $\xi \neq 0$.

Then the right half-plane $\{\lambda \in \mathbb{C}: \mathrm{Re}\lambda > -\lambda^*\}$ belongs to the resolvent set $\rho(-(T+A_1))$; moreover, the resolvent $Q_\lambda = (\lambda I+T+A_1)^{-1}$ satisfies the estimate $\|Q_\lambda^n\| \leq M(\mathrm{Re}\lambda+\lambda^*)^{-n}$ for n=1,2,..., with $M = \exp(c\cdot\mathrm{diam}\Omega)$. It folows from the Hille-Yosida theorem [2, Section IX.1] that $-(T+A_1)$ is the infinitesimal generator of a strongly continuous semigroup $W_1 = [W_1(t): t \geq 0]$ in $L^1(\Omega\times S)$. The expression for $W_1(t)$ is readily found,

$$(3.1)\qquad W_1(t)g(x,\xi) = \exp\left[-\int_0^t h(x-s\xi,\xi)ds\right]g(x-s\xi,\xi)\ , \qquad (x,\xi) \in \Omega\times S\ ,$$

for any $g \in L^1(\Omega\times S)$. The semigroup consists of positive operators. As the underlying space is an L^1-space, the type of the semigroup coincides with the spectral bound of the generator [3, Section 3.3]. The latter is at most equal to $-\lambda^*$; it is exactly equal to $-\lambda^*$ if we assume, in addition, that

(B) For each $\varepsilon > 0$ there exists a ball $B_0 = \{|x-x^0| \leq \rho\}$ wholly contained in Ω and a constant $\eta > 0$ such that $h(x,\xi) < \lambda^*+\varepsilon$ for all $x \in B_0$ and $\xi \in S$ with $|\xi| \leq \eta$.

In fact, if (A) and (B) hold, then $\sigma(-(T+A_1))$ fills the entire half-space $\{\lambda \in \mathbb{C}: \mathrm{Re}\lambda \leq -\lambda^*\}$.

We now add the bounded perturbation A_2 to $-(T+A_1)$. According to the theorem of Hille and Phillips [2, Section IX.2.1], the resulting operator $-(T+A_1)+A_2$ is the infinitesimal generator of a strongly continuous semigroup $W = [W(t): t \geq 0]$ in $L^1(\Omega\times S)$. This semigroup provides the solution of the abstract Cauchy problem.

THEOREM 1. <u>If $f_0 \in \mathrm{dom}T$, then the solution of Eq. 2.6 is uniquely determined and given by</u>

$$(3.2)\qquad f(t) = W(t)f_0\ , \qquad t \geq 0\ .$$

The semigroup W cannot be determined explicitly. However, W can be found from Duhamel's integral equation

$$(3.3)\qquad W(t) = W_1(t) + \int_0^t W_1(t-s)A_2W(s)ds\ , \qquad t \geq 0\ ,$$

by iteration; the result is the following Dyson-hillips expansion:

$$(3.4)\qquad W(t) = \sum_{n=0}^{\infty} W_1^{(n)}(t)\ , \qquad t \geq 0\ ,$$

where $W_1^{(0)}(t) = W_1(t)$, $W_1^{(n)}(t) = W_1(t) + \int_0^t W_1(t-s)A_2W_1^{(n-1)}(s)ds$ for $n=1,2,\ldots$. The series 3.4 converges in the operator norm topology, see [2, Section IX.2.1]. Because A_2 is a positive operator, the semigroup W consists again of positive operators.

4. ASYMPTOTIC BEHAVIOR

The type of the semigroup W coincides with the spectral bound of the transport operator. We denote the latter by λ_0,

$$(4.1)\qquad \lambda_0 = \sup\{\mathrm{Re}\lambda: \lambda \in \sigma(-(T+A_1)+A_2)\}\ .$$

It follows from the general theory of strongly continuous semigroups of positive operators that $\lambda_0 \in \sigma(-(T+A_1)+A_2)$, see [3, Section 3.3].

The perturbation A_2 is a partial identity in $L^1(\Omega\times S)$, so it is certainly not compact. However, the operator $A_2W_1(t)A_2$ is an integral operator in $L^1(\Omega\times S)$,

$$(4.2) \qquad A_2W_1(t)A_2f(x,\xi) = \int_\Omega \int_S H_t(x,\xi,x',\xi')f(x',\xi')dx'd\xi' ,$$

where

$$H_t(x,\xi,x',\xi') = \frac{1}{t^3} k\left(x,\xi + \frac{x-x'}{t}\right)k\left(x', \frac{x-x'}{t} + \xi'\right) \times \exp\left(-\int_0^t h\left(x-s \frac{x-x'}{t}, \frac{x-x'}{t}\right)ds\right) , \qquad t > 0 .$$

The representation 4.2 enables us to use compactness arguments.

THEOREM 2. If, for some positive integer n, $(W_1(t)A_2)^n$ is compact for all $t > 0$ and the function $[W_1(t_1)A_2W_1(t_2)A_2 \ldots W_1(t_n)A_2 \colon t_1 > 0, t_2 > 0, \ldots, t_n > 0]$ is continuous in the uniform operator topology, then $\{\lambda \in \mathbb{C}\colon \mathrm{Re}\lambda = -\lambda^*\} \subset \sigma(-(T+A_1)+A_2)$, so $\lambda_0 \geq -\lambda^*$; if $\lambda_0 > -\lambda^*$, then $\sigma(-(T+A_1)+A_2)$ contains finitely many points λ_k $(k=0,\ldots,m)$ in each right half-plane $\mathrm{Re}\lambda > -\lambda^* + \varepsilon$ $(\varepsilon > 0)$, each of these points is an eigenvalue of $-(T+A_1)+A_2$ with finite geometric multiplicity, and

$$(4.3) \qquad W(t) = \sum_{k=0}^{m} e^{\lambda_k t} e^{tD_k} P_k + Z_n(t)(I-P) ,$$

where $\|Z_n(t)\| = o(\exp(-\lambda^*+\varepsilon)t)$ as $t \to \infty$; here P_k and D_k are the projection and nilpotent operator associated with λ_k, and $P = P_0 + \ldots + P_k$.

The representation 4.3 can be sharpened if one can show that the semigroup W is irreducible. In the present context, W is irreducible if there exists a $t_0 > 0$ such that $W(t)$ is positivity improving for each $t \geq t_0$. Indeed, if W is irreducible, then λ_0 is a simple eigenvalue, the projection P_0 is positivity improving, and there exists a $\varepsilon > 0$ such that the real part of any other point of $\sigma(-(T+A_1)+A_2)$ is less than $\lambda_0 - \varepsilon$. Thus,

$$(4.4) \qquad W(t) = e^{\lambda_0 t} P_0 + Z(t)(I-P_0) ,$$

where $Z = [Z(t)\colon t \geq 0]$ is a semigroup in $(I-P_0)L^1(\Omega\times S)$. Although the spectral bound of the generator of Z is strictly less than λ_0, one can only conclude that the type of the semigroup Z is less than or equal to λ_0, as Z does not necessarily consist of positive operators.

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