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CONF-780768--2

LA-UR -78-2951

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SUBMITTED TO: Springer-Verlag
(For publication in proceedings of conference: "The Permutation Group and its Applications in Physics and Chemistry" University of Bielefeld, Germany, July 3 - July 12, 1978.)

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DEVELOPMENT ADMINISTRATION
CONTRACT W-7405-ENG. 36

REPRESENTATIONS OF THE SYMMETRIC GROUP AS SPECIAL CASES OF THE BOSON POLYNOMIALS IN U(n)

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ABSTRACT

The set of all real, orthogonal irreps of S_n are realized explicitly and non-recursively by specializing the boson polynomials carrying irreps of the unitary group. This realization makes use of a 'calculus of patterns', which is discussed. 37 references

*Paper presented by L.C. Biedenharn. Work supported, in part, by the National Science Foundation.

**Work performed under the auspices of the USERDA.

I. *Introduction*

The purpose of the present paper is to show how some recent investigations in the unitary group -- motivated by applications to quantum physics -- can be specialized to yield interesting -- and we hope, useful -- results for the symmetric group. Our main result is to obtain an explicit, non-recursive, set of real, orthogonal irreps for S_n, whose realization by means of the pattern calculus (as explained below) is (we believe) new.

Let us indicate, very briefly, why the unitary group figures so prominently in quantum physics. The *state*, ψ, of a quantal system is a *ray* in Hilbert space of unit length; *observables* are self-adjoint operators, O, mapping the Hilbert space into itself. A *symmetry* is a mapping of states into states, and operators into operators such that the probability $|\langle\phi|O|\psi\rangle|$ is preserved. The fundamental theorem (Wigner-Artin) -- essentially the fundamental theorem of projective geometry -- now states: any symmetry can be implemented by a semi-linear unitary transformation. It follows that the unitary group is of basic interest in quantum physics.

Let us remark also that the study of the symmetric group by broadening the investigation to the unitary group is itself a familiar technique; it was used extensively by Weyl, and is one of the principal themes in G. de B. Robinson's monograph on S_n.

It is necessary to explain now precisely what is meant by a "boson", and by a "boson operator". These terms are physicist's jargon for concepts known to mathematicians as the Weyl algebra, (or as it is also called the generators of the Heisenberg group). The boson, a, and its conjugate, $\bar{a}$, are elements of an algebra (Weyl algebra) satisfying the commutation rule: $[\bar{a},a]=1$, where 1 is the unit operator. More generally, we consider n bosons: a_i, $i=1,2,\ldots n$ and their conjugates: $\bar{a}_i$, $i=1,\ldots n$ obeying the rules:

$$[a_i,a_j] = [\bar{a}_i,\bar{a}_j] = 0; \; [\bar{a}_i,a_j] = \delta_{ij} \quad .$$

[The name "boson" contrasts with the physicist's term "fermion", which replaces commutation in the rules above by anti-commutation.]

Boson polynomials are simply polynomials (over $\mathbb{C}$) with the bosons $\{a_i\}$ as indeterminates. There is a natural scalar product associated to the boson polynomials by the commutation rule, if we define the abstract vector $|0\rangle$ to be annihilated by all conjugate bosons:

$\bar{a}_i|0\rangle \equiv 0$. Then to the boson monomial $(a_i)^k$, we associate the Hilbert space vector: $(a_i)^k|0\rangle \equiv |\psi\rangle$, and the scalar product: $\langle\psi|\psi\rangle = \langle 0|(\bar{a}_i)^k(a_i)^k|0\rangle = k!$.

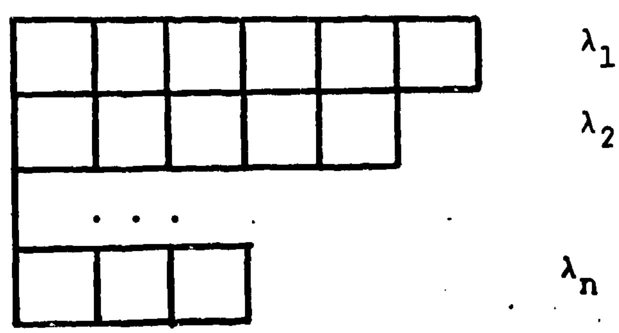
We remark that the technique of boson operator construction can be phrased in the language of the umbral calculus.

II. *Young tableaux, Weyl tableaux, and Gel'fand patterns*

One of the first problems that one confronts in discussing the irreducible representations of the unitary group U(n) is that of devising a comprehensible notation. This problem was solved in an elegant way by Gel'fand and Zetlin[1] by utilizing the Weyl branching law[2] for U(n). In order to explain this notation in familiar terms, it is convenient to appeal to the concept of standard Young tableaux of the symmetric group S_n since the relationship of these tableaux (to the irreducible representations of S_n) is well known to the participants of this conference.

The first concept required is that of a *Young frame*: a Young frame $Y_{[\lambda]}$ of *shape* $[\lambda]=[\lambda_1\lambda_2\cdots\lambda_n]$, where the λ_i are non-negative integers satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, is a diagram consisting of λ_1 boxes (nodes) in row 1, λ_2 boxes in row 2,..., λ_n boxes in row n, arranged as illustrated in Fig. 1.

Fig. 1



(1)

A *Weyl tableau* is a Young frame in which the boxes have been "filled in" with integers selected from 1, 2,...,n. A Weyl tableau is *standard* if the sequence of integers appearing in each row of $Y_{[\lambda]}$ is *non-decreasing* as read from left to right and the sequence of integers appearing in each column is *strictly increasing* as read from top to bottom. The *weight* or *content* (W) of a Weyl tableau $Y_{[\lambda]}$ is defined to be the row vector $(W)=(w_1,w_2,\ldots,w_n)$, where w_k equals the number of times integer k appears in the pattern. If $\lambda_1+\lambda_2+\ldots+\lambda_n=N$, then also $w_1+w_2+\ldots+w_n=N$. We shall call $[\lambda]$ a *partition* of N into n parts, or more often, a partition when N is unspecified. We generally count the 0's in determining the parts of a partition. For example, the partitions of 4 into 3 parts are [4 0 0], [3 1 0], and [2 2 0]. When the number of parts is understood, one frequently omits the zeroes (writing [4], [3 1], and [2 2] in the examples).

Example. The standard Weyl patterns corresponding to the Young frame  are:

$$\begin{array}{ccc}
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} & \\
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \\
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} & \\
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} & &
\end{array} \qquad (2)$$

Young's[3] interest was in invariant theory, utilizing the symmetric group, and he considered frames with n nodes filled in with integers 1 to n. To our knowledge Weyl[4] was the first to use Young frames filled in with repeated integers. We therefore refer to these latter tableaux as *Weyl tableaux*, reserving the term *Young tableaux* for the more restricted case.

Gel'fand patterns. An elegant geometrical notation for codifying the constraints imposed on the entries of a Young pattern is provided by a Gel'fand pattern which we now define.

A Gel'fand pattern is a triangular array of n rows of integers, there being one entry in the first row, two entries in the second row, ..., and n entries in the nth row. The entries in each row 2, 3,..., n-1, are arranged so as to fall between the entries in the row above and below, as illustrated below:

$$\begin{pmatrix} [m] \\ (m) \end{pmatrix} = \begin{pmatrix} m_{1n} & & m_{2n} & & \cdots & & & & m_{nn} \\ & \ddots & & & & & & \iddots & \\ & & m_{13} & & m_{23} & & m_{33} & & \\ & & & m_{12} & & m_{22} & & & \\ & & & & m_{11} & & & & \end{pmatrix} \qquad (3)$$

The integral entries m_{ij}, $i \leq j = 1,2,\ldots,n$, in this array are required to satisfy the following rules:

(*i*) $m_{1n} \geq m_{2n} \geq \ldots \geq m_{nn}$; (4)

(*ii*) for each specified partition $[m_{1n} \ldots m_{nn}]$, the entries in the remaining rows $j=n-1, n-2, \ldots, 1$ may be any integers which satisfy the "betweenness conditions"

$$m_{1j+1} \geq m_{1j} \geq m_{2j+1} \geq m_{2j} \geq m_{3j+1} \geq m_{3j} \geq \ldots \geq m_{j-1j} \geq m_{jj} \geq m_{j+1j+1}. \tag{5}$$

These betweenness conditions are, in fact, just the Weyl branching law for the chain of unitary subgroups given by:

$$U(n) \supset U(n-1) \supset \ldots \supset U(2) \supset U(1) \quad . \tag{6}$$

Example. For $n=3$, and $[m_{13}m_{23}m_{33}] = [2\ 1\ 0]$, there are eight Gel'fand patterns as displayed below:

$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{pmatrix} \quad \begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{pmatrix}$$

$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{pmatrix} \quad \begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{pmatrix} \quad \begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{pmatrix} \tag{7}$$

$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{pmatrix} \quad \begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{pmatrix}$$

$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{pmatrix}$$

Mapping between Gel'fand patterns and Standard Weyl tableaux. There is a one-to-one correspondence between the set of Gel'fand patterns (m) having nth row $[m_{1n}m_{2n}\ldots m_{nn}]$ (with $m_{nn} \geq 0$) and the set of standard Weyl tableaux of this shape.

The mapping between Gel'fand patterns and standard Weyl tableaux is described as follows: The shape of the frame is $[m_{1n}m_{2n}\ldots m_{nn}]$, and the rows of the frame are filled in according to the following rules

$$\begin{array}{llll}
\text{row 1:} & m_{11}\ 1\text{'s},\ m_{12}-m_{11}\ 2\text{'s}, & m_{13}-m_{23}\ 3\text{'s}\ \ldots, & m_{1n}-m_{1n-1}\ n\text{'s} \\
\text{row 2:} & m_{22}\ 2\text{'s}, & m_{23}-m_{22}\ 3\text{'s},\ldots, & m_{2n}-m_{2n-1}\ n\text{'s} \\
\vdots & & & \\
\text{row } j\text{:} & m_{jj}\ j\text{'s}, & m_{jj+1}-m_{jj}\ (j-1)\text{'s},\ldots, & m_{jn}-m_{jn-1}\ n\text{'s} \\
\vdots & & & \\
\text{row } n\text{:} & & & m_{nn}\ n\text{'s}\ .
\end{array} \tag{8}$$

Using the rule (8), we see that the set of Gel'fand patterns (7) is mapped to the set of Weyl tableaux (2). Conversely, from each standard Weyl tableau (2), we construct in an obvious way the Gel'fand pattern in the set (7).

The *weight* or *content* of a Gel'fand pattern (m), is the row vector $(W) = (w_1 w_2 \ldots w_n)$, where w_j is defined to be the sum of the entries in row j of (m) minus the sum of the entries in row j-1 ($w_1 \equiv m_{11}$):

$$w_j = \sum_{i=1}^{j} m_{ij} - \sum_{i=1}^{j-1} m_{ij-1}\ . \tag{9}$$

Clearly, this definition of weight coincides with that given earlier for a standard Weyl tableau.

The constraint in a standard Weyl tableau that each row (column) should comprise a set of nondecreasing (strictly increasing) nonnegative integers is realized in a Gel'fand pattern by the 'geometrical' rule that the integers (m_{ij}) satisfy the betweenness conditions.

III. *Carrier spaces of the representations of the symmetric group.*

Two important pattern results for the symmetric group S_n are: (a) The set of irreps of S_n is in one-to-one correspondence with the set of partitions ([λ]) of n into n parts; (b) the set of basis vectors of a carrier space of irrep [λ] of S_n is in one-to-one correspondence with the set of standard Young tableaux of shape [λ] having weight (W) = (1,1,...,1). [The number of basis vectors (the number of standard patterns) is then the dimension of the irrep.]

This latter result may, of course, also be expressed in terms of Gel'fand patterns. For example, the irreps of S_3 are enumerated by the partitions of 3 into 3 parts [3 0 0], [2 1 0], and [1 1 1]. The standard Young tableaux of weight (1,1,1) having these shapes, respect-

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}\ ; \quad \begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}\ , \quad \begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}\ ; \quad \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array} \quad . \qquad (10)$$

Thus, the irreps [3 0 0], [2 1 0], and [1 1 1] are of dimensions 1, 2, and 1, respectively. These same results are enumerated by the Gel'fand patterns

$$\begin{pmatrix} 3 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{pmatrix} ; \begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{pmatrix} , \begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{pmatrix} ; \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{pmatrix} . \qquad (11)$$

The standard Young tableaux for S_n are often enumerated by another indexing scheme -- the *Yamanouchi symbol* [5]

$$(y) = (y_1, y_2, \ldots, y_n) \quad . \qquad (12)$$

Here y_{n-j+1} is the positive integer equal to the row in which j appears in a given standard Young tableau of shape $[\lambda]$ and weight (1,1,...,1). y_{n-j+1} is also the position (counting from the left) in which 1 occurs in the set of differences

$$m_{1j}-m_{1j-1},\ m_{2j}-m_{2j-1}, \ldots, m_{j-1j}-m_{j-1j-1},\ m_{jj} \qquad (13)$$

formed from the entries in the corresponding Gel'fand pattern $[m_{1n} \ldots m_{nn}] = [\lambda_1 \ldots \lambda_n]$ having weight (1,1,...,1). For example, the Yamanouchi symbols for the Young tableaux (10) [and Gel'fand patterns (11), are, respectively,

$$(1,1,1); \quad (2,1,1), \quad (1,2,1); \quad (3,2,1) \quad .$$

IV. *Carrier spaces of the representations of the rotation group.*

The important pattern results for the rotation group [SU(2)] are: (a) The set of irreps of the rotation group is in one-to-one correspondence with the set of partitions [2j 0], j = 0,1/2,1,...; (b) the set of basis vectors of the carrier space of irrep [2j 0] is in one-to-one correspondence with the set of Gel'fand patterns having the partition [2j 0]:

$$\begin{pmatrix} 2j & & 0 \\ & & \end{pmatrix} , \ m = -j,-j+1,\ldots,j \ . \qquad (14)$$

[Observe that the betweenness rule embodies in a natural way the fact that the projection quantum number m runs over the values: $m=-j,\ldots,j$.]

The Weyl tableau corresponding to the Gel'fand pattern (14) is the one-rowed pattern

$$\underbrace{\boxed{1}\boxed{1}\boxed{1}\boxed{\ldots}\boxed{1}}_{j+m}\underbrace{\boxed{2}\boxed{2}\boxed{\ldots}\boxed{2}}_{j-m} \quad . \tag{15}$$

The notation above for SU(2) is a special case of U(2) for which we now give an explicit construction of the basis vectors in terms of boson operators.

The standard Weyl tableau of two rows corresponding to the Gel'fand pattern

$$\begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \quad , \text{ where } m_{12} \geq m_{11} \geq m_{22} \quad , \tag{16}$$

is

$$\begin{array}{l} \overbrace{\boxed{1}\boxed{1}\boxed{\ldots}\boxed{1}}^{m_{22}}\overbrace{\boxed{1}\boxed{1}\boxed{\ldots}\boxed{1}}^{m_{11}-m_{22}}\overbrace{\boxed{2}\boxed{2}\boxed{\ldots}\boxed{2}}^{m_{12}-m_{11}} \\ \boxed{2}\boxed{2}\boxed{\ldots}\boxed{2} \end{array} \tag{17}$$

A mapping from Weyl tableaux to bosons is given by

$$\begin{aligned} 1 &\to \begin{pmatrix} 1 & & 0 \\ & 1 & \end{pmatrix} \to a_1^1 \quad , \\ 2 &\to \begin{pmatrix} 1 & & 0 \\ & 0 & \end{pmatrix} \to a_2^1 \quad , \\ {}^{1}_{2} &\to \begin{pmatrix} 1 & & 1 \\ & 1 & \end{pmatrix} \to \det \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix} \equiv a_{12}^{12} \quad . \end{aligned} \tag{18}$$

[The Weyl tableau $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$ corresponds to antisymmetrized bosons made up of two independent bosons a_i^1 and a_i^2 $(i=1,2,)$.]

Using the correspondence (18), we obtain the following boson state vector, corresponding to the Gel'fand pattern (16) and the Weyl tableau (17):

$$\left| \begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix} \right\rangle = M^{-1/2} \cdot (a_{12}^{12})^{m_{22}} (a_1^1)^{m_{11}-m_{22}} (a_2^1)^{m_{12}-m_{11}} |0\rangle \quad . \tag{19}$$

where the normalization factor is given by:

$$M = \frac{(m_{12}+1)!\,(m_{11}-m_{22})!\,(m_{12}-m_{11})!\,(m_{22})!}{(m_{12}-m_{22}+1)!} \quad . \tag{20}$$

The angular momentum labels for the states (19) are

$$j = \frac{m_{12}-m_{22}}{2}\,, \quad m = m_{11} - \frac{m_{12}+m_{22}}{2} \quad . \tag{21}$$

[The $2m_{22}$ anti-symmetric (paired) bosons are inert as far as angular momentum is concerned, that is, a^{12}_{12} is invariant under unitary unimodular transformations.]

V. *Double tableaux and the rotation matrices.*

A closer inspection of the basis vectors (19) reveals that the Weyl tableau (17) has been used to assign the *subscripts* to the bosons. One sees, in fact, that the superscript assignment originates from the Weyl tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|}\multicolumn{4}{c}{\longleftarrow m_{22} \longrightarrow} & \multicolumn{4}{c}{\longleftarrow m_{12}-m_{22} \longrightarrow}\\ \hline 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ \hline 2 & 2 & \ldots & 2 & \multicolumn{4}{c}{} \\ \cline{1-4} \end{array} \tag{22}$$

corresponding to the maximal Gel'fand pattern

$$\begin{pmatrix} m_{12} & & m_{22} \\ & m_{12} & \end{pmatrix} \quad . \tag{23}$$

A more descriptive notation for the state vector (19) uses a *double Weyl tableau* or a *double Gel'fand pattern:*

$$\left| \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & \ldots & 1 & 1 & \ldots & 1 & 2 & \ldots & 2 \\ \hline 2 & \ldots & 2 & \multicolumn{6}{c}{} \\ \cline{1-3} \end{array} \right| \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 \\ \hline 2 & \ldots & 2 & \multicolumn{6}{c}{} \\ \cline{1-3} \end{array} \Bigg\rangle \tag{24}$$

$$= \left| \begin{pmatrix} & m_{12} & \\ m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix} \right\rangle \equiv M^{-1/2}(a^{12}_{12})^{m_{22}}(a^{1}_{1})^{m_{11}-m_{22}}(a^{1}_{2})^{m_{12}-m_{11}}|0\rangle$$

where we observe that

(*i*) the Young frames have the *same shape;*

(*ii*) by convention the second Gel'fand pattern (23) is inverted over the first one (16) in order to depict explicitly the shared labels $[m_{12}\ m_{22}]$ giving the common shape of the Young frame;

(*iii*) the mapping from the double Weyl tableau to bosons is obtained by pairing off the columns occurring in the *same positions* in the two Weyl tableaux

$$\left\{ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \,,\, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \right\} \rightarrow a^{12}_{12} \quad \left\{ \boxed{i} \,,\, \boxed{j} \right\} \rightarrow a^{j}_{i} \,, \quad i,j = 1,2. \tag{25}$$

[In the patterns in (24) the column pair

$$\left\{ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \,,\, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \right\}$$

occurs m_{22} times; the column pair $\{\boxed{1}\,,\boxed{1}\}$ occurs $m_{11}-m_{22}$ times, and the column pair $\{\boxed{1},\ \boxed{2}\}$ occurs $m_{12}-m_{11}$ times.]

The significance of rewriting Eq. (19) in the form of Eq. (24) is that one now recognizes that the latter result generalizes: *The Weyl tableau in the second position* (the upper Gel'fand pattern) *may be taken to be any standard tableau corresponding to the shape* $[m_{12}\ m_{22}]$. The mapping (25) then assigns a definite state vector (boson polynomial) to each pair of standard Weyl tableaux of the same shape.

The method outlined above for associating boson polynomials to double standard tableaux is the natural extension of Eq. (24) and is of interest in its own right [cf. Doubilet, Rota and Stein[8]], but it leads to nonorthogonal boson state vectors, except for the special case (24) [cf. Eq. (35) below]. We therefore develop an alternative method, used primarily by physicists, which utilizes repeated application of a lowering operator,

$$E^{21} = \sum_{i=1}^{2} a^{2}_{i}\,\bar{a}^{1}_{i} \quad , \tag{26}$$

to the vector (24), thereby generating orthonormal boson state vectors. These orthonormal vectors may be expressed in an elegant combinatoric form:

$$\left| \begin{pmatrix} & m'_{11} & \\ m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix} \right\rangle = M^{-\frac{1}{2}}\ B \begin{pmatrix} & m'_{11} & \\ m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix} (A)\,|\,0\,\rangle \quad , \tag{27}$$

where

$$B \begin{pmatrix} & m'_{11} & \\ m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix} (A) = [w_1!\, w_2!\, w'_1!\, w'_2!]^{\frac{1}{2}}\, (a^{12}_{12})^{m_{22}} \times$$

$$\sum_{(\alpha)} \frac{(a^1_1)^{\alpha^1_1}\, (a^1_2)^{\alpha^1_2}\, (a^2_1)^{\alpha^2_1}\, (a^2_2)^{\alpha^2_2}}{(\alpha^1_1)!\ (\alpha^1_2)!\ (\alpha^2_1)!\ (\alpha^2_2)!} \ , \tag{28}$$

in which (W) and (W') are, respectively, weights of the Gel'fand patterns

$$\begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} m_{12} & & m_{22} \\ & m'_{11} & \end{pmatrix} \tag{29}$$

and the summation is over all nonnegative integers α_i^j such that the matrix α has the fixed row and column sums given by (W) and (W'), that is,

$$\alpha = \begin{array}{|cc|l} \alpha_1^1 & \alpha_1^2 & w_1 \\ \alpha_2^1 & \alpha_2^2 & w_2 \\ \hline w_1' & w_2' & \end{array} \tag{30}$$

$(\alpha_1^1 + \alpha_1^2 = w_1,\ \alpha_2^1 + \alpha_2^2 = w_2,\ \alpha_1^1 + \alpha_2^1 = w_1',\ \alpha_1^2 + \alpha_2^2 = w_2'.)$

Observe that while the double Gel'fand patterns in Eq. (27) are in one-to-one correspondence with the double standard Weyl tableaux, we no longer have a simple rule for reading off the general form (28).

We will not give the details here of the derivation of Eqs. (27) and (28), but let us note several important properties of the *double Gel'fand pattern polynomials* (28):

(i) The set of double Gel'fand pattern polynomials of weight (W,W') is a (linearly independent) basis of the vector space spanned by all monomials in the bosons (a_i^j) which contain w_i occurrences of the subscript i and w_j' occurrences of the superscript j.

(ii) The set of double Gel'fand pattern polynomials corresponding to all partitions (m) of the nonnegative integer N is a basis of the vector space of homogeneous polynomials of degree N in the bosons (a_i^j).

(iii) The matrix $D^{(m)}(A)$ having element in row m_{11} $(m_{11}=m_{12},\ldots,m_{22})$ and column m'_{11} $(m'_{11}=m_{12},\ldots,m_{22})$ given by the boson polynomials (28) is a unitary irreducible representation of the group U(2) when the matrix A is replaced by a unitary 2x2 matrix.

(iv) If we replace the bosons a_i^j in Eq. (28) by the elements u_i^j of a 2x2 unitary unimodular matrix U, we obtain the (unitary) irreducible representations of SU(2) (rotation matrices):

$$\mathcal{D}^j \ \text{[illegible]}(U) = \text{[illegible]} \begin{bmatrix} \text{[illegible]} \end{bmatrix}^{j+m'} (U) \tag{31}$$

VI. *The general boson polynomials of* U(n)

Let us turn now to the description of the U(n) boson polynomials stating some of their important properties. There is a vast literature on this subject (cf. Ref. 7-25 and references therein). Our presentation is based on results which may be found in Refs. 6, 9, 10, 11, 14, 17, 18, and 23, to which we refer for further details and proofs. We first sketch the relationship of the U(n) boson polynomials to double standard tableaux. Consider the double standard Weyl tableau of shape $[\lambda] = [\lambda_1\lambda_2\ldots\lambda_n]$:

$$\left(\begin{array}{l|l} \begin{array}{|c|c|c|c|} \hline i_{11} & i_{12} & \cdots & i_{12_1} \\ \hline i_{21} & i_{22} & \cdots & i_{2\lambda_2} \\ \hline \vdots & & & \\ \hline i_{n1} & i_{n2} & \cdots & i_{n\lambda_n} \\ \hline \end{array} & \begin{array}{|c|c|c|c|} \hline j_{11} & j_{12} & \cdots & j_{1\lambda_1} \\ \hline j_{21} & j_{22} & \cdots & j_{2\lambda_2} \\ \hline \vdots & & & \\ \hline j_{n1} & j_{n2} & \cdots & j_{n\lambda_n} \\ \hline \end{array} \end{array}\right) \qquad (32)$$

Alternatively, this double standard tableau may be denoted by the double Gel'fand pattern

$$\begin{pmatrix} (m') \\ [m] \\ (m) \end{pmatrix} \qquad (33)$$

where the left and right tableaux in (32) correspond, respectively, to the upper and lower Gel'fand patterns in (33).

With each pair of columns in corresponding positions in the left and right patterns of the double standard Weyl pattern (32), we now associate a determinantal boson by the rule

$$\left\{\begin{matrix} i_{1k} & j_{1k} \\ i_{2k} & j_{2k} \\ \vdots & \vdots \\ i_{\lambda'_k k} & j_{\lambda'_k k} \end{matrix}\right\} \rightarrow a_{i_{1k}\ldots i_{\lambda'_k k}}^{j_{1k}\ldots j_{\lambda'_k k}} , \qquad (34)$$

where

$$a_{i_1\ldots i_k}^{j_1\ldots j_k} \equiv \det\begin{pmatrix} a_{i_1}^{j_1} & \cdots & a_{i_1}^{j_k} \\ \vdots & & \vdots \\ j_1 & & j_k \end{pmatrix}$$

A boson polynomial corresponding to a double standard Weyl tableau is defined as the product of the determinantal bosons (34) taken over all columns $1,2,\ldots,\lambda_1$ of the frame. Using the double Gel'fand patterns to denote the polynomials, we have:

$$P\begin{pmatrix}(m')\\ [m]\\ (m)\end{pmatrix}(A) \equiv \prod_{k=1}^{\lambda_1} a_{i_{1k}\cdots i_{\lambda'_k k}}^{j_{1k}\cdots j_{\lambda'_k k}} . \qquad (35)$$

We note two special cases of Eq. (35)

$$P\begin{pmatrix}(m')\\ [m]\\ (\max)\end{pmatrix}(A) = \prod_{k=1}^{n}\left(a_{12\ldots k}^{12\ldots k}\right)^{m_{kn}-m_{k+1n}} , \qquad (36)$$

$$P\begin{pmatrix}(\max)\\ [m]\\ (\text{semi-max})\end{pmatrix}(A) = \prod_{k=1}^{n-1}\left(a_{12\ldots k}^{12\ldots k}\right)^{m_{kn-1}-m_{k+1n}} \times \prod_{k=1}^{n}\left(a_{12\ldots k-1n}^{12\ldots k-1k}\right)^{m_{kn}-m_{kn-1}} , \qquad (37)$$

where $m_{ij} \equiv 0$ for $i > j$, $a_{12\ldots k-1n}^{12\ldots k-1k} \equiv a_n^k$ for $k=1$, and special pattern notations have been introduced:

$$\begin{pmatrix}[m]\\ (\max)\end{pmatrix} \equiv \begin{bmatrix} m_{1n} & m_{2n} & \cdots & & m_{n-1n} & m_{nn} \\ & \ddots & m_{1n} & \ddots & m_{2n} & \\ & & & m_{1n} & & \end{bmatrix} , \qquad (38)$$

$$\begin{pmatrix}[m]\\ (\text{semi-max})\\ (\max)\end{pmatrix} \equiv \begin{bmatrix} m_{1n} & m_{2n} & \cdots & m_{n-1n} & & m_{nn} \\ & m_{1n-1} & m_{2n-1} & \ldots & m_{n-1n-1} & \\ & & (\max) & & & \end{bmatrix} . \qquad (39)$$

The weight (W,W') or content of the double standard tableau (32) (and of the double Gel'fand pattern (33) is defined to be [cf. Eqs. (1) and (9)]

$$(W,W') = (w_1,\ldots,w_n,\ w'_1,\ldots,w'_n) , \qquad (40)$$

where (W) and (W') are, respectively, the weights of the left and right standard tableaux (upper and lower Gel'fand patterns).

As noted earlier the boson state vectors corresponding to the polynomials (35) are not, in general, orthogonal [cf. Eq. (44)-(46) below], and the main emphasis in physics has been on the construction of *orthonormal basis vectors* denoted in the double Gel'fand pattern notation by

$$\left| \begin{pmatrix} (m') \\ [m] \\ (m) \end{pmatrix} \right\rangle \equiv [M([m])]^{-\frac{1}{2}} \, B \begin{pmatrix} (m') \\ m \\ (m) \end{pmatrix} (A) | \, 0 \rangle \qquad (41)$$

where

$$M([m]) = H^{[m]} = \prod_{i=1}^{n} p_{ij}! / \prod_{i<j} (p_{in} - p_{jn}) \qquad (42)$$

in which

$$p_{in} \equiv m_{in} + n - i \, , \quad (p_{in} \text{ is called a "partial hook"}). \qquad (43)$$

The boson polynomials

$$B \begin{pmatrix} (m') \\ [m] \\ (m) \end{pmatrix} (A) \qquad (44)$$

occurring in Eq. (41) and the double tableau polynomials

$$P \begin{pmatrix} (m') \\ [m] \\ (m) \end{pmatrix} (A) \qquad (45)$$

span the same vector spaces. However, only for the patterns

$$\begin{pmatrix} (max) \\ [m] \\ (max) \end{pmatrix} , \quad \begin{pmatrix} (max) \\ [m] \\ (semi\text{-}max) \end{pmatrix} , \quad \begin{pmatrix} (semi\text{-}max) \\ [m] \\ (max) \end{pmatrix} \qquad (46)$$

do the polynomials agree (up to a normalization factor).

We will now state the form of the boson polynomials (44) referring to Refs. 11, 13, 17, and 23 for a discussion of the properties which characterize these orthonormal forms and for the derivations of the results below.

We begin with the statement of the simplest polynomials which are those corresponding to a Young frame having 1 row with p boxes so that $[m] = [p0\ldots0] = [p\dot{0}]$:

$$B \begin{pmatrix} (m') \\ [p\,\dot{0}] \\ (m) \end{pmatrix} (A) = \left[\prod_{i=1}^{n} (w_i)! \, (w_i')! \right]^{1/2} \times \sum_{\boxdot} \prod_{i,j=1}^{n} (a_i^j)^{a_i^j} / (a_i^j)! \qquad (47)$$

where [W] and [W'] denote the *weights* of the lower and upper Gel'fand patterns, respectively, and $\boxdot$ denotes the following square matrix of nonnegative integers with constraints on the sums of the entries in

$$\boxed{\alpha} = \begin{array}{|cccc|l} \alpha_1^1 & \alpha_1^2 & \cdots & \alpha_1^n & w_1 \\ \alpha_2^1 & \alpha_2^2 & \cdots & \alpha_2^n & w_2 \\ & \vdots & & & \\ \alpha_n^1 & \alpha_n^2 & \cdots & \alpha_n^n & w_n \\ \hline w_1' & w_2' & & w_n' & \end{array} \tag{48}$$

The symbols $w_i\,(w'_j)$ written to the right of row i (below column j) designate that the entries in row i (column j) are constrained to add to $w_i\,(w_j')$. The sum over $\boxed{\alpha}$ in Eq. (47) is to be taken over all non-negative integers α_i^j (for $i,j = 1,2,\ldots,n$) which satisfy these constraints.

The general result has a form similar to Eq. (47):

$$B\begin{pmatrix}(m')\\ [m]\\ (m)\end{pmatrix}(A) = M^{1/2}([m]) \sum_{\alpha} C\begin{pmatrix}(m')\\ [m]\\ (m)\end{pmatrix}(\alpha) \times \prod_{i,j=1}^{n} (a_i^j)^{\alpha_i^j}/[(\alpha_k^j)!]^{1/2}; \tag{49}$$

where the coefficients C in this result are given by

$$C\begin{pmatrix}(m')\\ [m]\\ (m)\end{pmatrix}(\alpha) = \left\langle \begin{pmatrix}[m]\\ (m')\end{pmatrix} \middle| \left\langle [w \;\begin{matrix}(\Gamma_n)\\ (\alpha_n)\end{matrix}\; \dot{0}] \right\rangle \cdots \left\langle [w \;\begin{matrix}(\Gamma_2)\\ (\alpha_2)\end{matrix}\; \dot{0}] \right\rangle \left\langle [w \;\begin{matrix}(\Gamma_1)\\ (\alpha_1)\end{matrix}\; \dot{0}] \right\rangle \middle| (0) \right\rangle \tag{50}$$

in which $\begin{pmatrix}[w_i & \dot{0}]\\ \alpha_i & \end{pmatrix}$ denotes the Gel'fand pattern

$$\begin{pmatrix}[w_i \;\; \dot{0}]\\ (\alpha_i)\end{pmatrix} = \begin{pmatrix} \alpha_i^1+\alpha_i^2+\ldots+\alpha_i^n & & 0\ldots\ldots 0 \\ & \ddots & \vdots \\ & & 0 \\ \alpha_i^j+\alpha_i^2 & & \\ & \alpha_i^1 & \end{pmatrix} \tag{51}$$

where (Γ_k) is the operator pattern which is uniquely determined by the Δ pattern

$$[\Delta(\Gamma_k)] = [m_{1k} m_{2k} \ldots m_{kk} \dot{0}] - [m_{1k-1} m_{2k-1} \ldots m_{k-1k-1} \dot{0}] \,. \tag{52}$$

We can not go into an explanation here of the general structure of the special case of interest for S_n. It is sufficient here to note that the general coefficients (50) are explicitly known.

We complete this general discussion with several observations on the properties of the boson polynomial (49): *The important properties (i) and (ii) noted earlier (end of Sec. V) apply as stated to the double Gel'fand pattern polynomials*

$$B\begin{pmatrix}(m')\\ [m]\\ (m)\end{pmatrix}(A) \quad .$$

Property (iii) also generalizes to the group U(n), *where the rows and columns of the matrix* $B^{[m]}(A)$ *are now to be enumerated by the* U(n-1) *Gel'fand patterns* ((m),(m')). [Similar statements also apply to the polynomials (35).] *Finally, we have also the transformation property under the combined left and right translations of the boson matrix,*

$$A \to \tilde{U}\,A\,V, \quad U, V \in U(n) \quad , \tag{53}$$

given by

$$B\begin{pmatrix}(m')\\ [m]\\ (m)\end{pmatrix}(\tilde{U}AV) = \sum_{(\mu)(\mu')} D^{[m]}_{(\mu)(m)}(U)\,D^{[m]}_{(\mu')(m')}(V)\,B\begin{pmatrix}(\mu')\\ [m]\\ (m)\end{pmatrix}(A), \tag{54}$$

where ~ *denotes matrix transposition, and*

$$\{D^{[m]}(U)\,|\,U \in U(n)\} \tag{55}$$

is the (unitary) matrix representation of U(n) *obtained by the identification*

$$D^{[m]}_{(m)(m')}(U) = B\begin{pmatrix}(m')\\ [m]\\ (m)\end{pmatrix}(U) \quad . \tag{56}$$

VII. *The Young-Yamanouchi real, proper orthogonal irreducible representations of* S_n.

Let us begin by considering the Cayley n x n permutation representation of S_n. For this one lets P denote a permutation by the rule:

$$P = \begin{pmatrix}1 & 2 & \dots & n\\ i_1 & i_2 & \dots & i_n\end{pmatrix} \quad . \tag{57}$$

Then the correspondence

$$P \to [e_{i_1} e_{i_2} \dots e_{i_n}] \equiv I_P \quad , \tag{58}$$

where e_i denotes a unit column vector with 1 in row i and zeroes elsewhere - is a representation of S_n by nxn matrices.

Since the general boson polynomial admits of an interpretation of the argument A by an nxn indeterminate, it is a well-defined operation to replace A by I_P, in Eq. (49). One obtains

$$B \begin{pmatrix} (m') \\ [m] \\ (m) \end{pmatrix} (I_P)$$

$$= [M([m])]^{1/2} \delta_{w'_1 w_{i_1}} \delta_{w'_2 w_{i_2}} \cdots \delta_{w'_n w_{i_n}} \; C \begin{pmatrix} (m') \\ [m] \\ (m) \end{pmatrix} (a_P) \quad , \qquad (59)$$

where a_P denotes the nxn numerical array

$$(a_P) = [w_{i_1} e_{i_1}, w_{i_2} e_{i_2}, \ldots, w_{i_n} e_{i_n}] \quad . \qquad (60)$$

Let us next specialize to representations having labels [m] which are partitions of n, and at the same time *restrict the two Gel'fand patterns* (m) *and* (m') *such that the weights* $[W]=[W']=[\dot{1}]$. It follows at once from Eq. (59) that these special boson polynomials take the form:

$$B \begin{pmatrix} (m') \\ [m] \\ (m) \end{pmatrix} (I_P) = [M([m])]^{1/2} \; C \begin{pmatrix} (m') \\ [m] \\ (m) \end{pmatrix} (I_P) \quad . \qquad (61)$$

It is useful to give a special notation to these objects; let us define

$$D^{[m]}_{(m),(m')}(P) = B \begin{pmatrix} (m') \\ [m] \\ (m) \end{pmatrix} (I_P) \quad . \qquad (62)$$

Then

$$\{D^{[m]}(P) \,|\, P \in S_n\} \qquad (63)$$

is an irreducible real, orthogonal representation of S_n.

Consider now the specific form taken by the matrix elements of these irreps. From Eq. (50) we obtain

$$D^{[m]}_{(m),(m')}(P) = [n!/\dim[m]]^{1/2} \times \left\langle \begin{pmatrix} [m] \\ (m) \end{pmatrix} \middle| \left\langle [1 \overset{\gamma_n}{\underset{i_n}{}} \dot{0}] \right\rangle \cdots \left\langle [1 \overset{\gamma_2}{\underset{i_2}{}} \dot{0}] \right\rangle \left\langle [1 \overset{\gamma_1}{\underset{i_1}{}} 0] \right\rangle \middle| \begin{matrix} [\dot{0}] \\ (0) \end{matrix} \right\rangle$$

$$\text{for } P = \begin{pmatrix} 1 & 2 & \ldots & n \\ i_1 & i_2 & \ldots & i_n \end{pmatrix} \quad , \qquad (64)$$

where dim[m] denotes the dimension of the irreducible representation [m] of S_n. We shall now explain in detail the meaning of the quanti-

(i) The symbol

$$\left\langle [1 \begin{smallmatrix}\gamma \\ \\ i\end{smallmatrix} \dot{0}] \right\rangle \qquad ; \qquad (65)$$

denotes a *fundamental Wigner operator* of U(n) [cf. Refs. 9, 17, 18] in which $(\begin{smallmatrix}1 & \dot{0} \\ & i\end{smallmatrix})$ is an abbreviated notation for the n-rowed Gel'fand pattern which has weight [0 ...0 1 0...0] with the 1 appearing in position i; similarly, $(_1{}^{\gamma}{}_{0})$ denotes the inverted Gel'fand pattern which has weight [0...0 1 0...0] with the 1 appearing in position γ. Thus, we have:

$$i, \gamma = 1, 2, \ldots, n \qquad (66)$$

in the symbol (65). For example, for n=3, there are 9 fundamental Wigner operators, a typical example being

$$\left\langle 1 \begin{smallmatrix}2 \\ 1\end{smallmatrix} \dot{0} \right\rangle = \left\langle \begin{smallmatrix} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{smallmatrix} \right\rangle \quad . \qquad (67)$$

[We will see below that, while upper and lower patterns in Eq. (65) run over the same numerical patterns, the role of the two patterns in the definition of a fundamental Wigner operator (65) are qualitatively different.]

(ii) The sequence in integers

$$(\gamma_n, \gamma_{n-1}, \ldots, \gamma_1) \qquad (68)$$

appearing in the upper patterns in Eq. (64) is the Yamanouchi symbol of the Gel'fand pattern

$$\begin{pmatrix}[m] \\ (m')\end{pmatrix} \quad . \qquad (69)$$

[Cf. Eqs. (12) and (13).]

Our remaining task is to define the concept of a fundamental Wigner operator in U(n) and to show how the coefficients in Eq. (64) are calculated.

Let $H^{[m]}$ denote a carrier space for irreducible representation [m] of U(n). Then an orthonormal basis of the space $H^{[m]}$ is:

$$\left\{ |(m)\rangle \quad \begin{matrix}(m) \text{ is a Gel'fand pattern of the} \\ \text{Young frame } Y_{[m]}\end{matrix} \right\} \quad . \qquad (70)$$

The fundamental Wigner operator denoted by

$$\left\langle [1 \begin{smallmatrix}\gamma \\ \\ i\end{smallmatrix} \dot{0}] \right\rangle \qquad (71)$$

[m] [m]+Δ(m)

$\binom{1\ \dot{0}}{\tau}$. [If $m_{\tau n}+1 < m_{\tau+1,n}$, then $H^{[1n]+\Delta(\tau)}$ contains only the zero vector.] The mapping (71) is now defined explicitly by giving its action on each basis vector (70) of $H^{[m]}$:

$$\left\langle [1_i^\tau \dot{0}] \right\rangle \left| \begin{matrix}[m]\\(m)\end{matrix} \right\rangle = \sum_{(m')} \left\langle \begin{matrix}[m]+\Delta(\tau)\\(m')\end{matrix} \right| \left\langle [1_i^\tau \dot{0}] \right\rangle \left| \begin{matrix}[m]\\(m)\end{matrix} \right\rangle \times \left| \begin{matrix}[m]+\Delta(\tau)\\(m')\end{matrix} \right\rangle, \tag{72}$$

where

$$\left\langle \begin{matrix}[m]+\Delta(\tau)\\(m')\end{matrix} \right| \left\langle [1_i^\tau \dot{0}] \right\rangle \left| \begin{matrix}[m]\\(m)\end{matrix} \right\rangle \tag{73}$$

denotes a real number (matrix element) which we now describe.

For the description of the numbers (73), we require a detailed notation for the entries in the rows of a Gel'fand pattern. We introduce the notation $[m]_k = [m_{1k} \dots m_{kk}]$ for the entries in row k, the notation $[1\ \dot{0}]_k$ for the row vector [1 0...0] of length k, and $\Delta_k(\tau_k)$ for the row vector of length k which has 1 in position τ_k $(1 \leq \tau_k \leq k)$ and zeroes elsewhere. In terms of this notation each matrix element (73) may be described in the following manner: Each matrix element (73) is zero unless the Gel'fand pattern

$$\begin{pmatrix}[m]+\Delta(\tau)\\(m')\end{pmatrix}$$

has the form

$$\begin{matrix}[m]_n + \Delta_n(\tau_n) \\ \left(\begin{matrix}[m]_{n-1} + \Delta_{n-1}(\tau_{n-1}) \\ \vdots \\ [m]_i + \Delta_i(\tau_i)\end{matrix}\right) \\ [m]_{i-1} \\ \vdots \\ [m]_1\end{matrix}, \tag{74}$$

where for each prescribed pair, τ and i $(1 \leq \tau \leq n,\ 1 \leq i \leq n)$, the sequence of integers $\tau_n, \tau_{n-1}, \dots, \tau_i$ satisfies

$$\tau_n = \tau \text{ and } 1 \leq \tau_k \leq k \text{ for } k = n-1,\dots,i. \tag{75}$$

Denoting the Gel'fand pattern (74) by the notation

$$\begin{pmatrix}[m]\\(m)\end{pmatrix}_{\tau_n \dots \tau_i}, \tag{76}$$

izes in the following manner:

$$\left\langle \begin{pmatrix}[m]\\(m)\end{pmatrix}_{\tau_n\cdots\tau_i} \middle| \left\langle [1\ \overset{\tau}{\underset{i}{\,}}\ \dot{0}] \right\rangle \middle| \begin{matrix}[m]\\(m)\end{matrix} \right\rangle \tag{77}$$

$$= \prod_{k=i}^{n} \left\langle \begin{matrix}[m]_k+\Delta_k(\tau_k)\\ [m]_{k-1}+\Delta_{k-1}(\tau_{k-1})\end{matrix} \middle| \begin{bmatrix} & \tau_k & \\ [1 & & \dot{0}]_k \\ & \tau_{k-1} & \end{bmatrix} \middle| \begin{matrix}[m]_k\\ [m]_{k-1}\end{matrix} \right\rangle ,$$

in which, by convention, $\tau_{i-1} = i$ and $\Delta_{i-1}(i) = [\dot{0}]_{i-1}$.

Each of the real numbers

$$\left\langle \begin{matrix}[m]_k+\Delta_k(\tau_k)\\ [m]_{k-1}+\Delta_{k-1}(\tau_{k-1})\end{matrix} \middle| \begin{bmatrix} & \tau_k & \\ [1 & & \dot{0}]_k \\ & \tau_{k-1} & \end{bmatrix} \middle| \begin{matrix}[m]_k\\ [m]_{k-1}\end{matrix} \right\rangle \tag{78}$$

in the product (77) is called a *reduced* U(k): U(k-1) *matrix element* and has a very simple interpretation in terms of the *pattern calculus rules* developed in Ref. 7. We state these rules here for the special case required to evaluate the factor (78):

The pattern calculus rules (cf. Ref. 14).

(i) Write out two rows of dots and assign the numerical entries of $\Delta_k(\tau_k)$ and $\Delta_{k-1}(\tau_{k-1})$, as shown:

```
                 (position τk)
0     0     0    1     0     0     0
●     ●     ●··· ●  ···●     ●     ●        row k
   ●     ●    ···   ●  ···  ●     ●         row k-1
   0     0          1       0     0
                 (position τk-1)
```

(ii) Draw an arrow between each point labelled by 1 (tail of arrow) to each point labelled by 0 (head of arrow). Once this arrow-pattern is drawn, remove the 0's and 1's from the diagram.

(iii) In the arrow-pattern assign the partial hook p_{ik} to point i (i-1,2,...,k from left to right) of row k and the partial hook p_{ik-1} to point i (i=1,2,...,k-1) in row k-1 ($p_{ij}\equiv m_{ij}+j-i$).

(iv) Assign a numerical factor to each arrow in the arrow-pattern using the rule

$$p_{tail} - p_{head} + e_{tail}/1 ,$$

where $e_{tail}=1$ if the tail of the arrow is on row k-1 and $e_{tail}=0$ if tail of the arrow is on row k.

N = product of all factors for arrows going *between* rows,

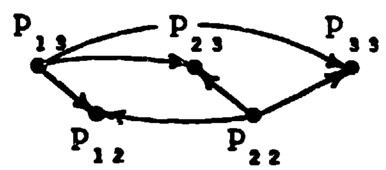
D = product of all factors for arrows going *within* rows.

The reduced U(k):U(k-1) matrix element (78) is then given by

$$S(\tau_{k-1}-\tau_k) \quad \left[\frac{N}{D}\right]^{1/2} \quad , \tag{79}$$

where $S(\tau_{k-1}-\tau_k)$ is +1 for $\tau_{k-1} > \tau_k$ and -1 for $\tau_{k-1}<\tau_k$.

Example. For k=3, $\tau_3=1$, $\tau_2=2$ the arrow-pattern is



and the reduced matrix element (78) has the value given by

$$\left\langle \begin{matrix} m_{13}+1 & m_{23} & m_{33} \\ & m_{12} \quad m_{22}+1 & \end{matrix} \middle| \begin{bmatrix} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \middle| \begin{matrix} m_{13} & m_{23} & m_{33} \\ & m_{12} \quad m_{22} & \end{matrix} \right\rangle$$

$$= - \left[\frac{(p_{13}-p_{12})(p_{22}-p_{33}+1)(p_{22}-p_{33}+1)}{(p_{13}-p_{23})(p_{13}-p_{33})(p_{22}-p_{12}+1)}\right]^{1/2} . \tag{80}$$

The result of applying these rules to Eqs. (78) is:

$$S(\tau_{k-1}-\tau_k) \left[\prod_{\substack{s=1 \\ s\neq\tau_k}}^{k} \frac{(p_{\tau_{k-1}k-1}-p_{sk}+1)}{(p_{\tau_k k}-p_{sk})} \prod_{\substack{t=1 \\ t\neq\tau_{k-1}}}^{k-1} \frac{(p_{\tau_k k}-p_{tk-1})}{(p_{\tau_{k-1}k-1}-p_{tk-1}+1)} \right]^{1/2} . \tag{81}$$

Remarks. Using the above results from the pattern calculus, Eq. (64) *is a completely explicit general result, giving for each* $P \in S_n$, *every element of the irreducible matrix representation* $D^{[m]}(P)$.

Thus we have achieved our stated goal of obtaining the real orthogonal S_n irreps in an explicit, non-recursive, way. The techniques we have used, in particular the pattern calculus, seem to be a natural extension of the ideas underlying the concept of a "hook" (due to Nakayama, and to Frame, et al.[26]) as applied in "hook product" of the Hall-Robinson formula[27].

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