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PARTICLE-CONFINEMENT CRITERIA FOR AXISYMMETRIC FIELD-REVERSED MAGNETIC CONFIGURATIONS

by

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I. INTRODUCTION

Particle-confinement criteria for a fusion system is fundamentally important because it is the basis for many essential calculations, e.g., evaluation of particle-confinement time, particle distribution functions, possible loss-cone-like instability, etc. In previous studies, Wang and Miley[1] obtained the confinement criteria for a spherical-Hill's-vortex magnetic configuration; Lovelace et al.[2] obtained similar results when considering ion-ring equilibria where they considered ions with canonical angular momentum $P_\theta < 0$ (defining the z-component of the external magnetic field $B_{ex,z} > 0$). In this report, the general conditions for particle confinement are derived for an axisymmetric system and specialized to a general class of vortex-type field-reversed magnetic configurations. The criteria for closed-field and open-field confinements are obtained, including both the lower and upper bounds of P_θ/q where q is the charge of the particle. The constraint on P_θ/q represents a necessary condition for particle confinement. In certain limits, the commonly used criterion for Hamiltonian, $H < -\omega_o P_\theta$, where $\omega_o \equiv qB_o/mc$, is deduced from a more general form as a special case. In addition, a new criterion, $-B_o R_w^2/2c < P_\theta/q < 0$, where R_w = wall radius, and B_o = vacuum field, is found necessary to be imposed, which reduces the confined region in (E,P_θ) space. These results can be applied to calculations for Field-Reversed Mirrors and Field-Reversed Theta Pinches.

II. DERIVATION OF CONFINEMENT CRITERIA

The Hamiltonian for a particle with mass m and charge q in an axisymmetric magnetic field $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$, $\underset{\sim}{A} = A_\theta(r,z)\hat{\theta}$, and electric field $\underset{\sim}{E} = -\nabla V$,

$V = V(r,z)$, is given by

$$H = \frac{P_r^2}{2m} + \frac{P_z^2}{2m} + \frac{(P_\theta - \frac{q}{c}\psi)^2}{2mr^2} + qV, \tag{1}$$

where P_z, P_r, P_θ are the canonical momenta

$$P_z = m\dot{z}, \quad P_r = m\dot{r}, \quad P_\theta = mr^2\dot{\theta} + \frac{q}{c}\psi, \tag{2}$$

and

$$\psi = rA_\theta. \tag{3}$$

As was noted earlier,[3] since P_θ is a constant of motion, Eq. (1) can be viewed as

$$H = \begin{pmatrix} \text{effective kinetic} \\ \text{energy in } (r,z) \\ \text{motion, } T \end{pmatrix} + \begin{pmatrix} \text{effective potential} \\ \text{energy in } (r,z) \\ \text{motion, } U \end{pmatrix} \tag{4}$$

where

$$T(r,z) = \frac{P_r^2}{2m} + \frac{P_z^2}{2m}, \tag{5}$$

$$U(r,z) = \frac{(P_\theta - \frac{q}{c}\psi)^2}{2mr^2} + qV. \tag{6}$$

The consideration of particle confinement is, therefore, equivalent to considering the (r,z) motion of a particle with total energy H moving in a potential U(r,z).

The equipotential contours in (r,z) plane (or equipotential surfaces in (r, θ, z) configuration space) are determined by

$$U(r,z) = \frac{(P_\theta - \frac{q}{c}\psi)^2}{2mr^2} + qV = constant. \tag{7}$$

A different value of the constant gives a different contour. Therefore, for a particle with total energy H and canonical angular momentur P_θ to be confined in certain spatial region R, the necessary and sufficient condition is that the equipotential contour U(r,z) = H must locate within region R. In other words, if U_{crit} is the maximum permissible potential barrier, i.e., every point of the contour $U(r,z) = U_{crit}$ is located either inside R or on the boundary of R, then the necessary and sufficient condition is $H < U_{crit}$.

Say, the system we are considering is bounded by $0 \leq r < R_w$ and $L_1 < z < L_2$. The conditions for absolute confinement, i.e., for particles to be confined in the system, therefore, are that (i) there must be points (r_i, z_i), i = 1, 2, 3, 4, on the equipotential contour U(r,z) = constant such that

$$\left.\begin{array}{l} \left.\frac{dz}{dr}\right|_{(r_1,z_1)} = \left.\frac{dz}{dr}\right|_{(r_2,z_2)} = \left.\frac{dr}{dz}\right|_{(r_3,z_3)} = \left.\frac{dr}{dz}\right|_{(r_4,z_4)} = 0 \,, \\ 0 \leq r \leq R_w \,, \quad L_1 \leq z \leq L_2 \,, \quad i = 1,2,3,4. \end{array}\right\} \tag{8}$$

see Fig. 1, and (ii) if we define

$$U_{crit} \equiv \max\left\{ U(r_i, z_i) \,\middle|\, (r_i, z_i) \text{ satisfies Eq. (8)} \right\} \tag{9}$$

then

$$H < U_{crit}. \tag{10}$$

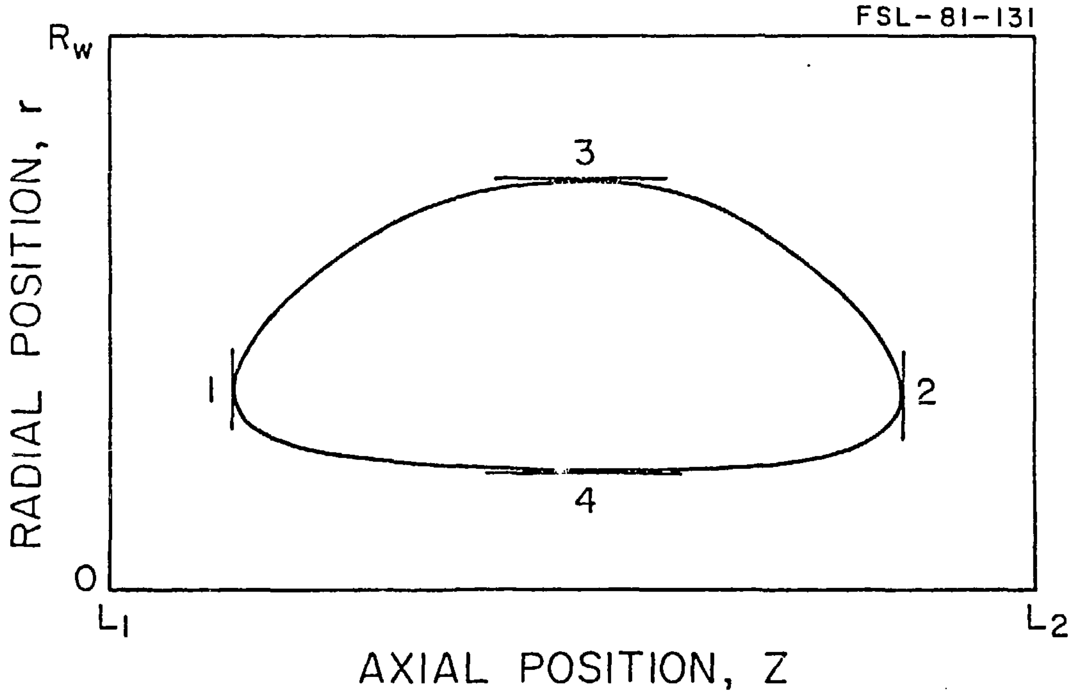

Figure 1. Points (r_i, z_i), $i = 1, 2, 3, 4$, on the equipotential contour $U(r,z) = \text{constant}$ represent the necessary condition, Eq. (8).

By Eq. (7), on the equipotential contour we have

$$\left(\frac{dz}{dr}\right)^{-1} = \frac{dr}{dz} = -\frac{\partial U/\partial z}{\partial U/\partial r} = -\frac{(P_\theta - \frac{e}{c}\psi)\, r\,\psi_z - mcr^3 V_z}{(P_\theta - \frac{e}{c}\psi)(\frac{c}{e}P_\theta - \psi + r\psi_r) - mcr^3 V_r}, \tag{11}$$

where subindices z and r stand for the corresponding partial derivatives.

The Eqs. (8) - (11) are very general. To get more explicit form for confinement criteria, we need to specify the magnetic and electric fields. In case of no electric field, V = 0. Furthermore, if we consider a class of vortex-type field-reversed magnetic configurations given by

$$\psi = \begin{cases} -\dfrac{3B_o}{4R_s^2}\, r^2 (R_s^2 - r^2 - \zeta^2) & , \quad r^2 + \zeta^2 < R_s^2 , \\ \dfrac{B_o}{2}\, r^2 \left[1 - \dfrac{R_s^3}{(r^2+\zeta^2)^{3/2}} \right] & , \quad r^2 + \zeta^2 > R_s^2 , \end{cases} \tag{12}$$

where $\zeta \equiv z/k$, $k \equiv Z_s/R_s \equiv$ elongation factor, R_s and Z_s are the maximum r- and z- coordinates of the separatrix, $0 \leq R_s, r < R_w$, $-L < Z_s, Z < L$. The configuration described by Eq. (12) is shown in Fig. 2.

For particles confined in open-field region, the condition $\frac{dr}{dz} = 0$ implies

$$r\zeta = 0 , \tag{13}$$

which can always be satisfied by certain (r,ζ) in the open-field region where $r^2 + \zeta^2 > R_s^2$, $0 \leq r \leq R_w$, $0 \leq \zeta \leq \zeta_{max} \equiv L/k$. No constraint is obtained. However, the condition $\frac{dz}{dr} = 0$ implies

$$\frac{P_\theta}{e} + \frac{B_o}{2c}\, r^2 \left\{ 1 + \frac{R_s^3}{(r^2+\zeta^2)^{3/2}} \left[\frac{3r^2}{r^2+\zeta^2} - 1 \right] \right\} = 0 . \tag{14}$$

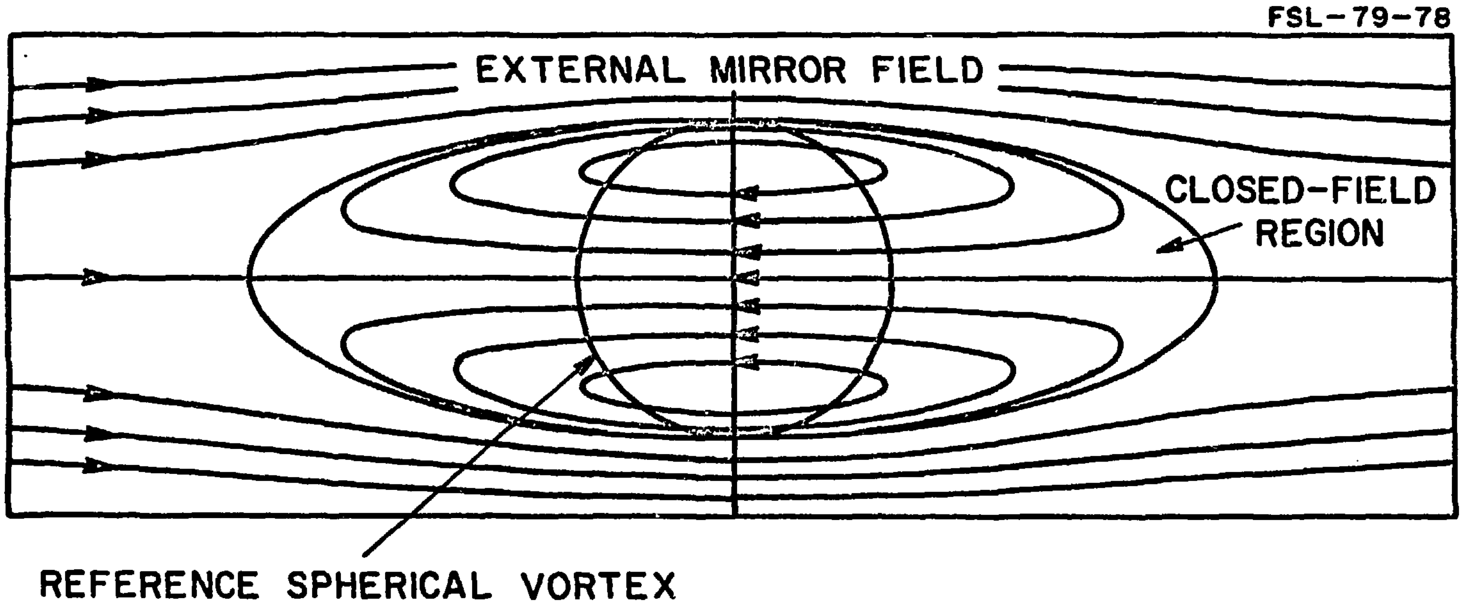

Figure 2. The magnetic configuration described by Eq. (12).

Since Eq. (14) must be satisfied by certain (r,ζ) in the open-field region, it thus requires

$$-\frac{B_o R_w^2}{2c}\left(1+\frac{2R_s^3}{R_w^3}\right) < \frac{P_\theta}{q} < 0 . \tag{15}$$

This is a necessary condition to be satisfied by confined particles.

In order to apply condition (ii), i. e., to calculate U_{crit}, we need to know, depending on R_w and L, whether the radial bound or axial bound is more stringent. If the radial bound is more stringent, i. e., the maximum permissible equipotential surface touches the radial wall of the system first before it touches the axial wall, then

$$\begin{aligned} U_{crit} &= \min\left\{ U(r=R_w, z) \right\} \\ &= U(r,z)\Big|_{\text{at } \frac{dr}{dz}=0,\ r=R_w} . \end{aligned} \tag{16}$$

Using Eq. (13), Eq. (16) becomes

$$\begin{aligned} U_{crit} &= U(R_w, 0) \\ &= \frac{1}{2mR_w^2}\left\{ P_\theta - \frac{qB_o}{2c} R_w^2 \left(1-\frac{R_s^3}{R_w^3}\right)\right\}^2 . \end{aligned} \tag{17}$$

On the other hand, if the axial bound is more stringent, then

$$\begin{aligned} U_{crit} &= \min\left\{ U(r, z=L) \right\} \\ &= U(r,z)\Big|_{\text{at } \frac{dz}{dr}=0,\ z=L} . \end{aligned} \tag{18}$$

Using Eq. (14) Eq. (17) becomes

$$U_{crit} = U(r = r_c,\ z = L)$$

$$= \frac{1}{2mr_c^2}\left\{P_\theta - \frac{qB_0}{2c}\left[r_c^2 - \frac{R_s^3\, r_c^2}{(r_c^2+\zeta_{max}^2)^{3/2}}\right]\right\}^2 , \tag{19}$$

where $\zeta_{max} \equiv LR_s/z_s$, and r_c is the solution of

$$\frac{P_\theta}{q} + \frac{B_0}{2c} r_c^2 \left\{1 + \frac{R_s^3}{(r_c^2+\zeta_{max}^2)^{3/2}}\left[\frac{3r_c^2}{r_c^2+\zeta_{max}^2} - 1\right]\right\} = 0 . \tag{20}$$

As mentioned before, the Eq. (15) is a necessary condition to be satisfied by P_θ/q, of the confined particles. The actual lower and upper bounds of P_θ/q, depending on whether the radial or axial confinement is more stringent, are derived as follows. If the radial confinement is more stringent, i.e., $U_{crit} = U(R_w, 0)$, then Eq. (14) requires

$$-\frac{B_0 R_w^2}{2c}\left(1 + \frac{2R_s^3}{R_w^3}\right) < \frac{P_\theta}{q} < 0 . \tag{15}$$

If the axial confinement is more stringent, i.e., $U_{crit} = U(r_c, \zeta_{max})$, then Eq. (14) requires

$$-\frac{B_0 \zeta_{max}^2}{8c}\left(\sqrt{137} - 11\right)\left\{1 + \frac{\sqrt{137} - 13}{(\sqrt{137} - 7)^{5/2}} \cdot \frac{16 R_s^3}{\zeta_{max}^3}\right\} < \frac{P_\theta}{q} < 0 , \tag{21}$$

or approximately,

$$-0.88\,\frac{B_0 \zeta_{max}^2}{c}\left(1 - 0.43\,\frac{R_s^3}{\zeta_{max}^3}\right) < \frac{P_\theta}{q} < 0 . \tag{22}$$

It can be shown that a necessary condition for the axial confinement to be more stringent, i.e., for $U(r_c, \zeta_{max}) < U(R_w, 0)$ is

$$\zeta_{max} < 2.382\, R_w . \tag{23}$$

For particles confined in the closed-field region, $\frac{dr}{dz} = 0$ implies $r\zeta = 0$. Again, this can always be satisfied by certain (r,z) in the closed-field region. The condition $\frac{dz}{dr} = 0$ implies

$$\frac{P_\theta}{g} = -\frac{3 B_o}{4c R_s^2} r^2 (3r^2 + \zeta^2 - R_s^2), \tag{24}$$

which must be satisfied by certain (r, ζ) in the closed-field region where $r^2 + \zeta^2 < R_s^2$, $0 \leq r \leq R_s$, $0 \leq \zeta \leq R_s$. This requires

$$-\frac{3B_o R_s^2}{2c} \leq \frac{P_\theta}{g} \leq \frac{B_o R_s^2}{16c} . \tag{25}$$

And, the maximum permissible potential is given by

$$\begin{aligned} U_{crit} &= \min \left\{ U(r,\zeta) \Big|_{(r,\zeta) \text{ on separatrix}} \right\} \\ &= \min \left\{ \frac{P_\theta^2}{2mr^2} \Big|_{(r,\zeta) \text{ on separatrix}} \right\} \\ &= \frac{P_\theta^2}{2mR_s^2} . \end{aligned} \tag{26}$$

As a limiting case, if $\zeta_{max} >> r_c$, then Eq. (19) reduces to

$$U_{crit} = -\frac{qB_o P_\theta}{4mc(\zeta_{max}^3 - R_s^3)\zeta_{max}^9}\Big\{\zeta_{max}^6 + (\zeta_{max}^3 - R_s^3)$$

$$\cdot\Big[\zeta_{max}^3 - \frac{(qB_o R_s^2 \zeta_{max}^5)^{3/2}}{(qB_o\zeta_{max}^5 - 2cP_\theta(\zeta_{max}^3 - R_s^3))^{3/2}}\Big]\Big\}^2 \quad (27)$$

If, furthermore, $\zeta_{max} >> R_s$, i.e., the system length is much longer than the length of the closed-field region, then Eq. (27) is simplified to

$$U_{crit} = -\frac{qB_o}{mc} P_\theta \quad (28)$$

Note that $r_c < R_w$ and usually $R_s \sim R_w$, therefore, $\zeta_{max} >> R_s$ also implies $\zeta_{max} >> r_c$. Equation (28) is the previous result[1] and used very commonly. In this limiting case, Eq. (14) reduces to

$$\frac{P_\theta}{q} + \frac{B_o}{2c} r^2 = 0$$

which requires

$$-\frac{B_o R_w^2}{2c} < \frac{P_\theta}{q} < 0 \quad (29)$$

Note that the lower bound of P_θ/q in Eq. (29) represents an additional constraint on the confinement criteria, which is not included in previous results.[1,2]

The above criteria derived from Eqs. (8) - (11) are the conditions for a particle having total energy H and canonical angular momentum P_θ to be confined. Since

$$H \equiv \frac{P_r^2}{2m} + \frac{P_z^2}{2m} + \frac{(P_\theta - \frac{q}{c}\psi)^2}{2mr^2} \geq \frac{(P_\theta - \frac{q}{c}\psi)^2}{2mr^2},$$

only part of (H, P_θ) space is energetically assessible by the particles.

Therefore, one more constraint needed to be imposed, when we deal with variables E and P_θ, is

$$H \geq \frac{(P_\theta - \frac{e}{c}\psi)^2}{2mr^2} \quad (30)$$

.

III. SUMMARY AND CONCLUSIONS

(i) The general form of confinement criteria given by Eqs. (8)-(11) can be applied to the calculations for field-reversed mirrors and field-reversed theta pinches. For a general class of vortex-type field-reversed magnetic configurations, Eq. (12), the confinement criteria are summarized as follows. For a particle to be confined within the closed-field region, the total energy H and canonical angular momentum P_θ need to satisfy

$$\begin{cases} -\dfrac{3B_oR_s^2}{2c} \leq \dfrac{P_\theta}{e} \leq \dfrac{B_oR_s^2}{16c}\,, & (25) \\[2ex] H \geq \dfrac{1}{2mr^2}(P_\theta - \dfrac{e}{c}\psi)^2\,, & (30) \\[2ex] H < \dfrac{P_\theta^2}{2mR_s^2}\,. & (31) \end{cases}$$

For α particles to be confined in the open-field region, i. e., absolute confinement, the criteria are

$$\begin{cases} -\dfrac{B_oR_w^2}{2c}\left(1 + \dfrac{2R_s^3}{R_w^3}\right) < \dfrac{P_\theta}{e} < 0\,, \ \text{if}\ U(R_w,0) < U(r_c,\zeta_{max})\,, & (15) \\[2ex] -0.88\,\dfrac{B_oR_w^2}{c}\left(1 - 0.43\dfrac{R_s^3}{\zeta_{max}^3}\right) < \dfrac{P_\theta}{e} < 0\,, \ \text{if}\ U(R_w,0) > U(r_c,\zeta_{max})\,, & (22) \end{cases}$$

$$\begin{cases} H \geq \dfrac{1}{2mr^2}\left(P_\theta - \dfrac{q}{c}\psi\right)^2, & (30) \\ H < U_{crit} = \min\{U(R_w, 0) \text{ and } U(r_c, \zeta_{max})\}, & (32) \end{cases}$$

where $U(R_w, 0)$ and $U(r_c, \zeta_{max})$ are given by Eqs. (17) and (19), respectively. Note that a necessary condition for $U(r_c, \zeta_{max}) < U(R_w, 0)$ is $\zeta_{max} < 2.38\ R_w$.

(ii) As a special case of the general expressions, the criteria for absolute confinement in case of $L >> R_w$ are given by

$$\begin{cases} -\dfrac{B_0 R_w^2}{2c} < \dfrac{P_\theta}{q} < 0, & (29) \\ H \geq \dfrac{1}{2mr^2}\left(P_\theta - \dfrac{q}{c}\psi\right)^2, & (30) \\ H < -\dfrac{qB_0}{mc} P_\theta . & (28) \end{cases}$$

(iii) Typical results for the confined region in (H, P_θ) space are shown in Figs. 3 to 7. Figure 3 is for closed-field confinement. Figures 4 to 7 compare the exact expression with the approximate limiting case for the open-field confinement. Figures 4 and 5 show the cases where the axial open-field confinement is more stringent, while Figs. 6 and 7 are for the cases where the radial open-field confinement being more stringent.

(iv) In practical use of these confinement criteria, as shown by Figs. 3 to 7 and actually can also be understood from the derivation, Eq. (25) and the lower bound in Eq. (15) do not affect the confined region in (H, P_θ) space, because they will be satisfied automatically once the other criteria are satisfied. But the lower bound of P_θ/q in Eqs. (22) and (29) does represent a new criterion necessary to be imposed, which reduces the confined region in (H, P_θ) space. In other words, in case of axial open-field confinements

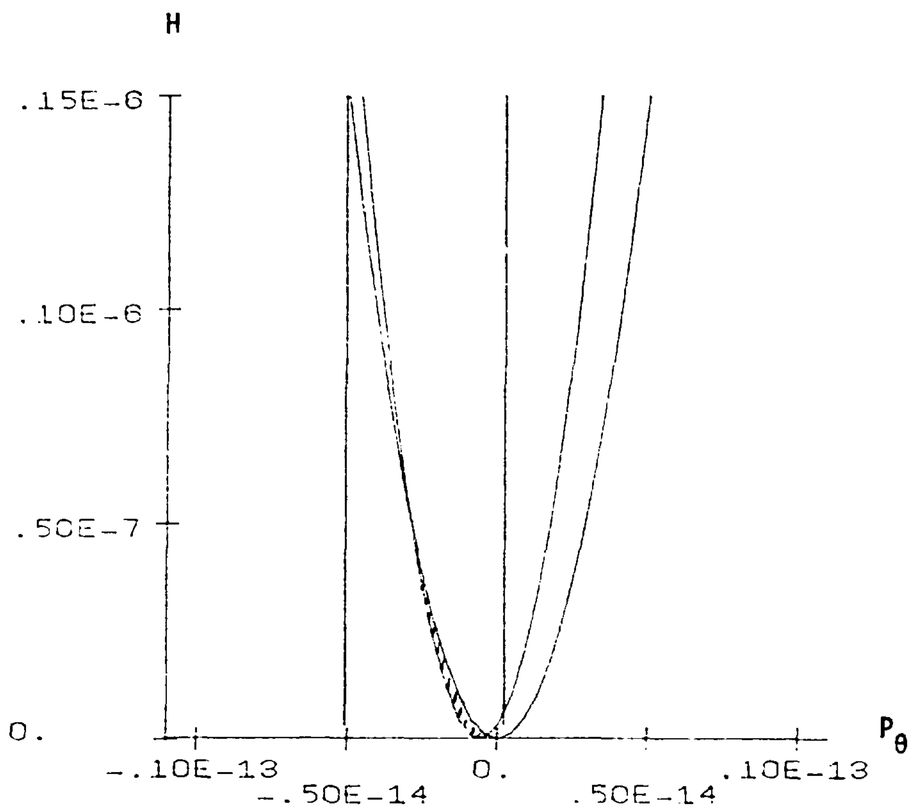

Figure 3. A typical confined region in (H, P_θ) space for the closed-field confinement. System parameters, in Gaussian units, are $B_o = 8500$, $q = 4.8 \times 10^{-10}$, $m = 3.34 \times 10^{-24}$, $R_W = 10$, $R_S = 5$, $L = 50$, $z_S = 40$, $r = 4$, $z = 0$.

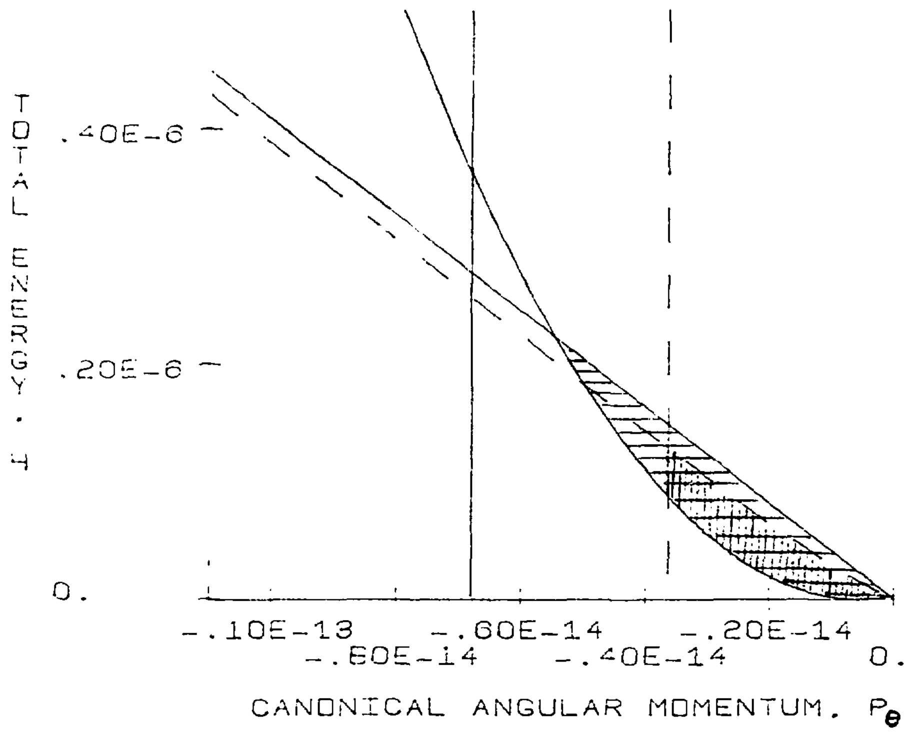

Figure 4. Typical confined regions in (H, P_θ) space, at a position (r, z) in the closed-field region, for the exact axial open-field confinement (dashed lines) and the approximate limiting case (solid lines). All parameters are the same as in Fig. 3.

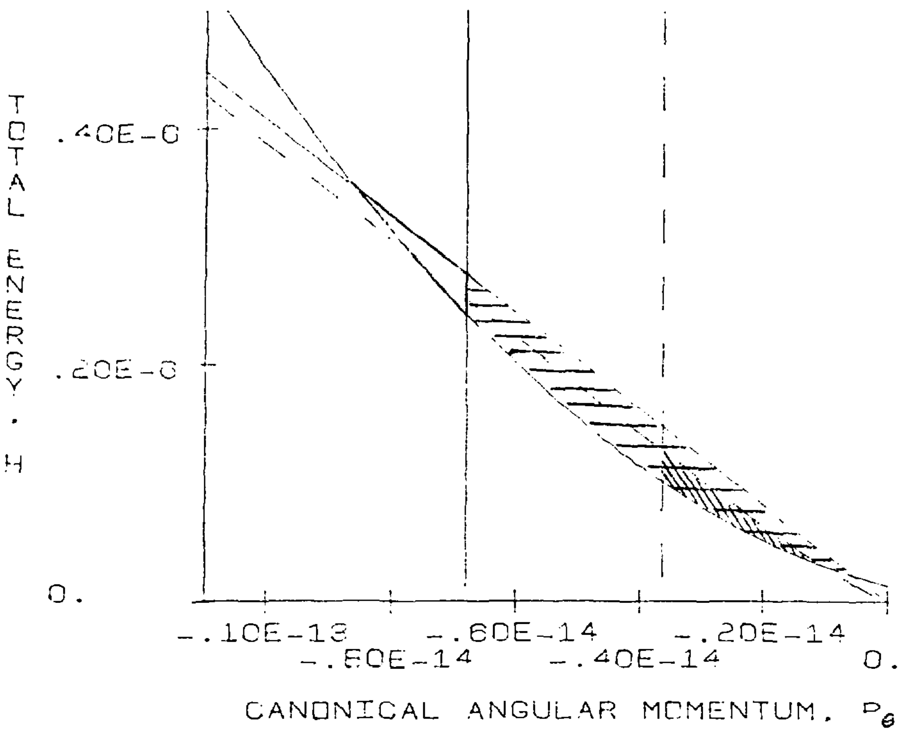

Figure 5. Typical confined regions in (H,P_θ) space, at a position (r, z) in the open-field region, for the exact axial open-field confinement (dashed lines) and the approximate limiting case (solid lines). All parameters are the same as in Fig. 4 except $r = 7$.

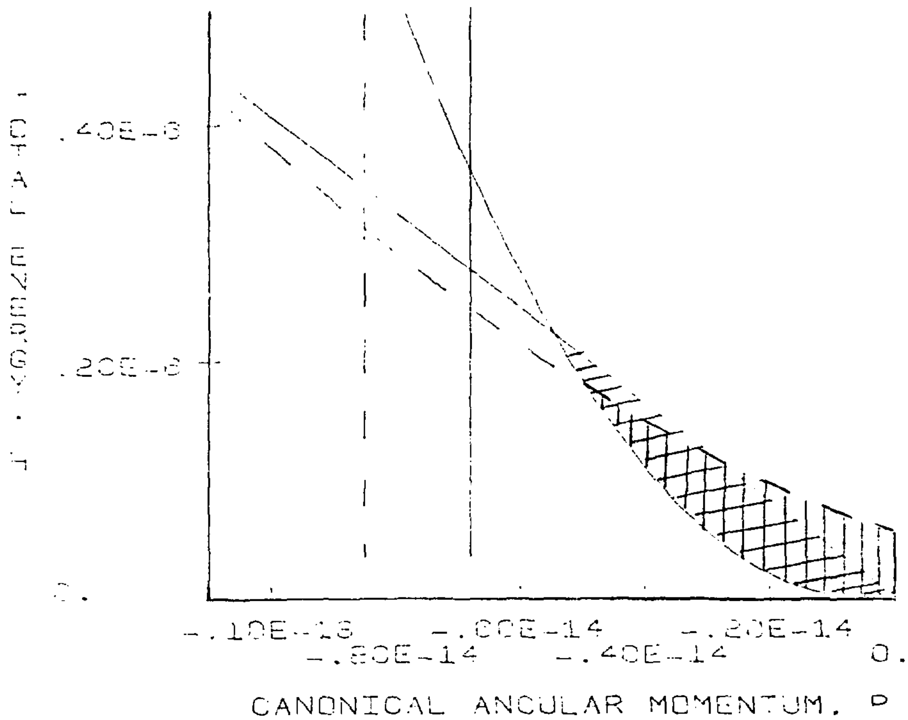

Figure 6. Typical confined regions in (H, P_θ) space, at a position (r,z) in the closed-field region, for the exact radial open-field confinement (dashed lines) and the approximate limiting case (solid lines). All parameters are the same as in Fig. 4.

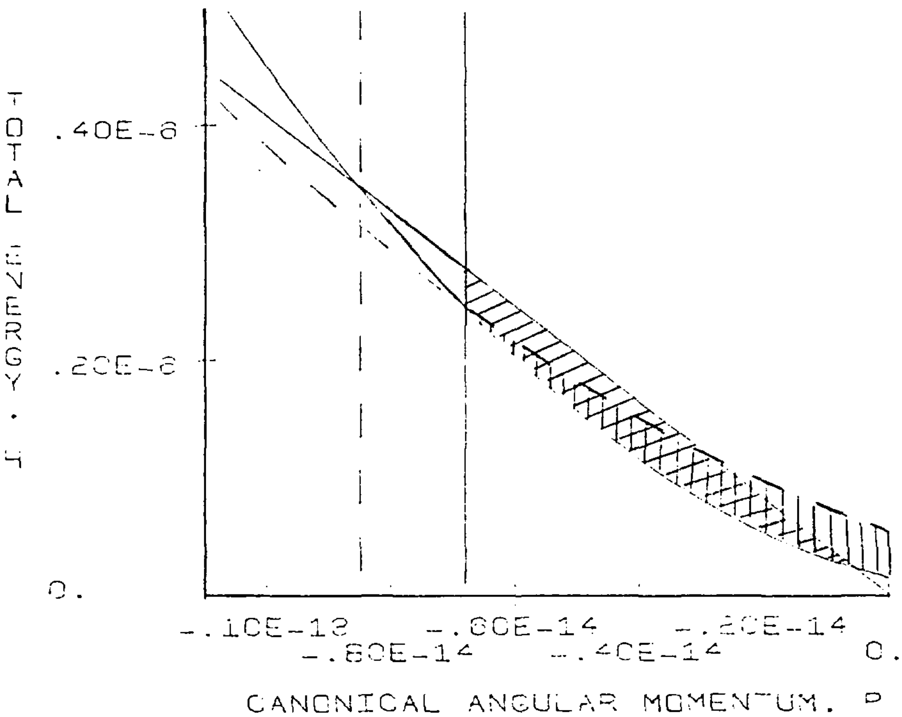

Figure 7. Typical confined regions in (H, P_θ) space, at a position in open-field region, for the exact radial open-field confinement (dashed lines) and the approximate limiting case (solid lines). All parameters are the same as in Fig. 5.

the lower bound of P_θ/q represents a new constraint which has not been imposed in previous studies.[1,4]

(v) Since only part of (H, P_θ) space is energetically allowed, the energetic constraint, Eq. (30) is another necessary condition needed to be satisfied. This constraint has not been stated explicitly in previous study.[1]

IV. REFERENCES

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