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USING SUBJECTIVE PERCENTILES AND TEST DATA FOR ESTIMATING FRAGILITY FUNCTIONS

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Using Subjective Percentiles and Test Data for Estimating Fragility Functions*

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ABSTRACT

Fragility functions are cumulative distribution functions (cdfs) of strengths at failure. They are needed for reliability analyses of systems such as power generation and transmission systems. Subjective opinions supplement sparse test data for estimating fragility functions. Often the opinions are opinions on the percentiles of the fragility function. Subjective percentiles are likely to be less biased than opinions on parameters of cdfs.

This paper provides solutions to several problems in the estimation of fragility functions given subjective percentiles and test data. How should subjective percentiles be used to estimate subjective fragility functions? How should subjective percentiles be combined with test data? How should fragility functions for several failure modes be combined into a

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composite fragility function? How should inherent randomness and uncertainty due to lack of knowledge be represented?

We treat subjective percentiles as independent estimates of percentiles. We derive:

1. least squares parameter estimators for normal and lognormal cdfs, based on subjective percentiles; the method is applicable to any invertible cdf,
2. a composite fragility function for combining several failure modes,
3. estimators of variation within and between groups of experts for nonidentically distributed subjective percentiles,
4. weighted least squares estimators when subjective percentiles have higher variation at higher percents, and
5. weighted least squares and Bayes parameter estimators based on combining subjective percentiles and test data.

I. Introduction

Risk analysis of nuclear power plants requires system reliability computation. The computation requires a failure model. The failure model in the Seismic Safety Margins Research Program (SSMRP) is a mechanical reliability model. Component failure occurs when a response random variable exceeds a strength random variable [1]. Component response and strength data are combined to compute system reliability [2]. The component strength data is summarized in estimates of the cumulative distribution functions (cdfs) of the strength random variables for all components.

Unfortunately, there is little relevant component strength test data. Components may have been acceptance tested but have not been tested to failure. Or they may have been tested to failure but not under earthquake loads. In order to compute system reliability given an earthquake, earthquake failure data is needed. There is little. Thus other sources of information about strength are needed. One source is expert opinion. Opinions come in many forms. The form used in this paper is expert opinion of the fragility function percentiles.

Questionnaires were sent to 253 experts to obtain strength percentile estimates of nuclear power plant components in an earthquake. Forty experts returned 120 questionnaires on 31 categories of components. The questionnaires gave the 10-th, 50-th and 90-th percentiles of the fragility functions for the three failure modes judged most likely by each expert. The questionnaires also gave expert self credibility weights.

This paper describes estimation of fragility functions from subjective percentiles and test data. The objective is to estimate a single fragility function for each of the 31 categories of components. This requires estimation of fragility functions for each failure mode from the subjective percentiles and test data. It also requires combining modal fragility function estimates. Because the estimated cdf comes from subjective percentiles and test data, the uncertainty due to lack of knowldege about the true strength must be quantified.

2. Using Subjective Percentiles for Estimating Fragility Functions

The questionnaire used for eliciting fragility information from experts asked for subjective percentiles of the fragility functions. The answers are treated as independent estimates of specified percentiles. They are used as inputs for estimating the parameters of a cdf based on the least squares criteria. The method can be used for any invertible cdf. It is applied to normal, lognormal and exponential cdfs in this paper. The estimators are easily modified to accept weighted data. The weights may be credibility ratings given by the respondents or by the person analyzing the data.

The model for subjective percentiles is the following. The subjective percentiles from each expert are uncorrelated and the percentiles from different experts are independent. Let X_{iq} denote the q-th subjective percentile given by the i-th expert. The assumed model for X_{iq} is

$$X_{iq} = x_q + E_{iq} \qquad q = .1,\ .5,\ .9$$
$$i = 1, 2, \ldots, n$$

where x_q is the q-th population percentile of the reference population, assumed to be the aggregation of subjective fragility functions for all experts; and E_{iq} is a random variable with $E(E_{iq}) = 0$ and $Var(E_{iq}) = \sigma_E^2$. Thus, for each percentile, the model assumes the i-th expert's opinion is randomly selected from the population of opinions of all experts.

This is a simplification of reality. It is doubtful that an expert's opinion about the three percentiles are uncorrelated, nor can it necessarily be expected that experts are independent sources of information. Some alternative models are discussed in Sections 5 and 6.

The parameter estimators for a hypothesized cdf minimize the sum of squared differences between the sample percentiles and the percentiles of the hypothesized cdf. Let $\underline{\theta}$ denote the vector of parameters of the cdf and let $F^{-1}(q,\underline{\theta})$ denote the inverse of the hypothesized cdf. The objective is to find the value of $\underline{\theta}$ that minimizes

$$\sum_{i=1}^{n} \sum_{q} (X_{iq} - F^{-1}(q, \underline{\theta}))^2.$$

It is not necessary to have the same set of q for all i. The cardinality of the set of q must be at least as large as that of $\underline{\theta}$.

First, assume the population fragility function can be approximated by the normal distribution. The parameters to be estimated are the mean and standard deviation, μ and σ. Notice the percentile $x_q = \mu + \sigma z_q$, where z_q is the percentile of the standard normal cdf. The objective function is

$$\min_{\mu,\sigma} \sum_{i=1}^{n} \sum_{q} (X_{iq} - \mu - z_q\sigma)^2.$$

The normal equations yield

$$\hat{\mu} = (\overline{X}_{.1} + \overline{X}_{.5} + \overline{X}_{.9})/3,$$

where $\overline{X}_q$ is the average of the subjective percentiles, offered by the n experts, and

$$\hat{\sigma} = \sum_{i=1}^{n} (X_{i.9}-X_{i.1})/(2nz_{.9}).$$

If the hypothesized cdf is lognormal with $E(\ell nX) = \mu$ and $Var(\ell nX) = \sigma^2$, the objective function is

$$\min_{\mu,\sigma} \sum_{i=1}^{n} \sum_{q} (\ell n X_{iq} - \mu - z_q\sigma)^2.$$

The estimators are

$$\hat{\mu} = (\overline{\ell n\ X}_{.1} + \overline{\ell n\ X}_{.5} + \overline{\ell n\ X}_{.9})/3$$

and

$$\hat{\sigma} = \sum_{i=1}^{n} (\ell n\ X_{i.9} - \ell n\ X_{i.1})/(2n\ z_{.9}) .$$

In either case, under the assumed model, the estimators $\hat{\mu}$ and $\hat{\sigma}$ are unbiased. Their variances are $\sigma_E^2/3n$ and $\sigma_E^2/(2n\ z^2_{.9})$ respectively, assuming all experts give 10-th, 50-th and 90-th percentiles. An estimate of the variation, σ_E^2, between experts is

$$\hat{\sigma}_E^2 = \sum_{i=1}^{n} \sum_{q} (X_{iq} - \hat{x}_q)^2/(3(n-1))$$

where

$$\hat{x}_q = \hat{\mu} + z_q\hat{\sigma} .$$

If the hypothesized cdf is exponential with parameter λ, the objective function is

$$\min_{\lambda} \sum_{i=1}^{n} \sum_{q} (X_{iq} + \ell n\ (1-q)/\lambda)^2 .$$

The solution to the normal equation is

$$\hat{\lambda} = -\frac{\sum_q [\ell n(1-q)]^2}{\sum_q \bar{X}_q \ \ell n(1-q)}$$

Thus, the least squares estimator of the expected value, $\theta = 1/\lambda$, of the exponential distribution is $\hat{\theta} = 1/\hat{\lambda}$.

Under the assumed model $\hat{\theta}$ is an unbiased estimator of θ and the variance of $\hat{\theta}$ is

$$\sigma_E^2 / (n \sum [\ell n(1-q)]^2).$$

3. Combining Fragility Functions for Several Failure Modes

If a component can fail in several modes, it is of interest to find a single variate fragility function which describes the component failure in the weakest mode. A graphical description of component failure, when a component can fail in two modes, is given in Figure 1.

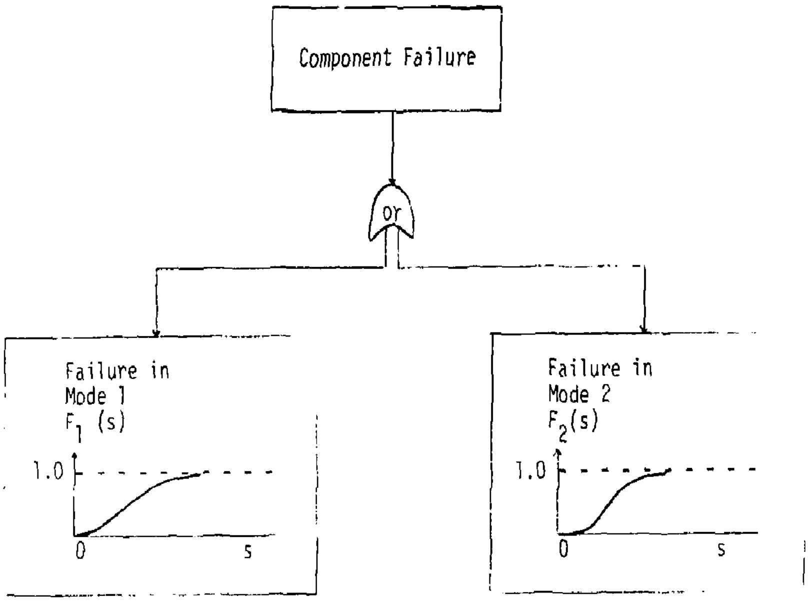

Figure 1. Component Failure Definition

In this section we discuss some methods for determining the fragility function for component failure when the marginal fragility function for each mode is given. The description is in terms of two modes of failure; it is easily extended to several modes.

If failure in either mode is caused by the same response variable, combining modes is easy. This is shown first.

Define

S_1, S_2 = strength or capability of component to resist failure in mode 1 and 2 respectively. The marginal cdfs of S_1 and S_2 are $F_1(s)$ and $F_2(s)$.

R = response seen by the component. It must have the same units as S_1 and S_2. The cdf of R is $F_R(r)$.

The probability of component failure is

$$P[\text{Component Failure}] = P[S_1 \leq R \cup S_2 \leq R]$$

$$= \int_0^{\infty} P[S_1 \leq s \cup S_2 \leq s | R = s] dF_R(s)$$

If S_1 and S_2 are independent,

$$P[\text{Component Failure}] = \int_0^{\infty} \{1 - P[S_1 > s | R = s] P[S_2 > S | R = S]\} dF_R(s)$$

or,

$$P[\text{Comp. Failure}] = \int_0^{\infty} [1 - \prod_{i=1}^{2} (1 - F_i(s))] dF_R(s)$$

Thus, the combined fragility function is

$$F_C(s) = 1 - \prod_{i=1}^{2} [1 - F_i(s)]$$

For example, suppose S_1 and S_2 are independent normal random variables with means μ_1 and μ_2 and standard deviations σ_1 and σ_2. The combined fragility function is

$$F_C(s) = 1 - \prod_{i=1}^{2} \left[1 - \Phi\left(\frac{s - \mu_i}{\sigma_i} \right) \right]$$

where $\Phi(\cdot)$ is the N(0,1) cdf. Suppose $\mu_1 = 2$, $\mu_2 = 1.5$, $\sigma_1 = .5$ and $\sigma_2 = .2$. The combined fragility function is

$$F_C(s) = 1 - [1-\Phi(2s-4)][1-\Phi(5s-7.5)].$$

The values for $F_C(s)$, for selected values of s, are in Table 1. The combined fragility function is not a normal cdf.

Table 1. A Combined Fragility Function

s	0	.5	1	1.5	2	2.5	3
$F_C(s)$	3.17×10^{-5}	.00135	.02887	.5793	.99696	1.0	1.0

If S_1 and S_2 are independent lognormal random variables the combined fragility function is

$$F_C(s) = 1-\left[1-\Phi\left(\frac{\ell n\ s-\mu_1}{\sigma_1}\right)\right]\left[1-\Phi\left(\frac{\ell n\ s-\mu_2}{\sigma_2}\right)\right]$$

where $\mu_i = E(\ell n\ S_i)$ and $\sigma_i^2 = \text{Var}\ (\ell n\ S_i)$, i=1,2.

The generalization for more than two failure modes is clear.

In many cases responses causing the two failure modes are not the same, but come from a common response. For example, R_1 and R_2 may be maximum displacement and peak velocity, both of which are related to peak acceleration, R. Let $g_i(R) = R_i$ i=1,2 be known functions relating the R_i's to the common response R. The probability of component failure is

$$P[\text{Comp. Failure}] = P[S_1 \leq R_1 \cup S_2 \leq R_2]\ .$$

Again, if S_1 and S_2 are independent, the probability of component failure is

$$P[\text{Comp. Failure}] = \int_0^\infty \{1 - \prod_{i=1}^{2} [1-F_i(g_i(s))]\} dF_R(s)$$

or, the combined fragility function is

$$F_C(s) = 1 - \prod_{i=1}^{2} [1-F_i(g_i(s))].$$

For example, suppose S_1 and S_2 have the same cdfs as the previous example. Suppose

$$g_1(R) = 0.5 + R^{1/2}$$

and

$$g_2(R) = -0.25 + 1.1R.$$

Table 2 shows values of the combined fragility function.

Table 2. A Combined Fragility Function With Different Response Variables

s	0	1	2	2.5
$F_C(s)$	.00135	.15915	.99307	1.0

Of course, this is easily generalized for K "independent" modes of failure.

4. Using Grouped and Weighted Subjective Percentiles for Estimating Fragility Functions and Quantifying Uncertainty Due to Lack of Knowledge

The least squares procedure outlined in Section 2 assumed identically distributed errors for each of the subjective percentiles. Such estimates are not always identically distributed. Experts could be estimating percentiles for different types of components for the same generic component (e.g. different sized valves). Similarly, experts could be grouped by their background and experience. Consequently, subjective percentiles from experts in different groups may not be identically distributed. Also, opinions from the different experts could be weighted differently by the experts themselves (self or peer weighting) or by the person conducting the survey.

This section contains an analysis of a model with group effects and weighted experts. The analysis estimates the fragility function parameters and the variation between experts within a group and between groups. The latter estimates can help to quantify "uncertainty" in the parameter estimators due to differences between experts.

Assuming the normal distribution approximates the population fragility function, the model for the q-th percentile given by the j-th expert in the i-th group is

$$X_{ijq} = \mu + z_q\sigma + T_i + E_{ijq} \qquad \begin{array}{l} i = 1, 2, \ldots, I \\ j = 1, 2, \ldots, N_i \\ q = .1, .5\ \ .9 \end{array}$$

where T_i is the deviation of the q-th percentile of the fragility function for the i-th group from the q-th percentile of the fragility function over all groups; and E_{ijq} is the random deviation in the estimate of the q-th percentile by the j-th expert in the i-th group. Further, assume T_i and E_{ijq} are uncorrelated random variables with $E(T_i) = E(E_{ijq}) = 0$, and $Var(T_i) = \sigma_T^2$ and $Var(E_{ijq}) = \sigma_E^2$.

The parameter σ^2 represents the perceived (by the experts) variation in component strengths within a group of components. The parameter σ_T^2 represents the variation in the average strength between groups of components. Thus, the sum $\sigma^2 + \sigma_T^2$ represents the "subjective" variance of strengths of components between the different groups. The parameter σ_E^2 represents the variation between experts due to lack of knowledge about component strengths.

Again, this model is only an approximation to reality. It only assumes a shift in the mean, μ, between groups and assumes σ to be the same for all groups. Models in which σ also varies will be considered in the future.

To allow for perceived (by the analyst) differences in the ability and/or knowledge of the experts, let w_{ij} be a weighting factor applied to the percentiles given by the j-th expert in the i-th group. The weighted least squares estimator is based on the objective function

$$\min_{\mu,\sigma} \sum_{i=1}^{I} \sum_{j=1}^{N_i} \sum_{q} w_{ij}(X_{ijq} - \mu - z_q\sigma)^2$$

where $\sum_{i=1}^{I} \sum_{j=1}^{N_i} w_{ij} \equiv 1$. The estimators are

$$\hat{\mu} = \frac{1}{3} \sum_{i=1}^{I} \sum_{j=1}^{N_i} \sum_{q} w_{ij} \, X_{ijq}$$

and

$$\hat{\sigma} = \sum_{q} z_q \sum_{i=1}^{I} \sum_{j=1}^{N_i} w_{ij} \, X_{ijq} / \sum_{q} z_q^{\,2}.$$

For any symmetric distribution,

$$\hat{\sigma} = (\bar{X}_{...9} - \bar{X}_{...1})/(2\, z_{.9})$$

where $\bar{X}_{..q} = \sum_{i=1}^{I} \sum_{j=1}^{N_i} w_{ij} \, X_{ijq}$.

To estimate the variance components, $\sigma_E^{\,2}$ and $\sigma_T^{\,2}$, define

$$SSE = \sum_{q} \sum_{i=1}^{I} \sum_{j=1}^{N_i} w_{ij}(X_{ijq} - \hat{\mu} - z_q\hat{\sigma})^2,$$

$$SST = \sum_{q} \sum_{i} \sum_{j} w_{ij}(X_{ijq} - \bar{X}_{i.q})^2,$$

and

$$SSM = \sum_{q} \sum_{i} \sum_{j} w_{ij}(\bar{X}_{..q} - \hat{\mu} - z_q\hat{\sigma})^2$$

where $\bar{X}_{i.q} = \sum_{j} w_{ij} \, X_{ijq}/w_{i.}$ and $w_{i.} = \sum_{j=1}^{N_i} w_{ij}$.

Assume a symmetric distribution. Then the sums of squares have expectations

$$E(SSE) = 3\sigma_T^2 \left(1-\sum_i w_{i.}^2\right) + 3\,\sigma_E^2 \left(1-\frac{2}{3}\sum_i \sum_j w_{ij}^2\right),$$

$$E(SST) = 3\,\sigma_E^2 \left[1-\sum_i \frac{\sum_j w_{ij}^2}{w_{i.}}\right],$$

and

$$E(SSM) = \sigma_E^2 \sum_i \sum_j w_{ij}^2.$$

Thus, σ_E^2 can be estimated using either SST or SSM, and SSE gives an estimate of σ_T^2. These estimators can be used to construct subjective "confidence" limits for individual percentiles or bands for the fragility functions. Alternatively, if it is necessary to have a single fragility function which includes perceived variation within and between groups as well as variation between experts, an adjusted standard deviation

$$\hat{\sigma}_A = (\hat{\sigma}^2 + \hat{\sigma}_E^2 + \hat{\sigma}_T^2)^{1/2}$$

can be used for [illegible] estimator of the variance of the fragility function. *Using $\hat{\sigma}_A$ instead of $\hat{\sigma}$ biases the system failure probabilities upward.*

Since the model for the q-th percentile, $x_q = \mu + z_q\sigma$, is a linear *function of z_q, standard least squares methodology (assuming the distribution* of the variations, T_i and E_{ijq} can be approximated by a normal distribution) can be used to construct confidence bands for the percentiles as shown in *Figure 2*

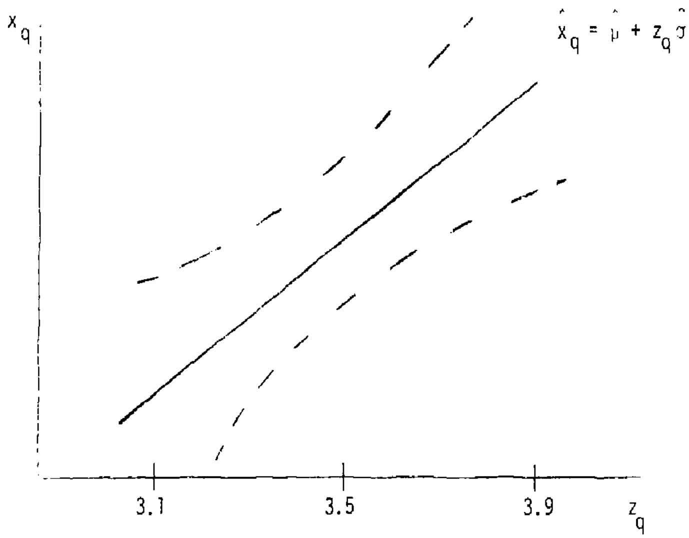

Figure 2. Confidence Bands for Percentiles, x_q.

Alternatively, simultaneous confidence bands, called the Working-Hotelling [3] *confidence bands are given by*

$$\hat{\mu} + z_q\hat{\sigma} \pm (2F_{(1-\alpha),2,\nu})^{1/2}\,\hat{\sigma}(\hat{\mu} + z_q\hat{\sigma})$$

where ν is the "degrees of freedom" for the estimated variance components, and the estimated standard deviation is

$$\hat{\sigma}(\hat{\mu} + z_q\hat{\sigma}) = \left[\hat{\sigma}_T^2 \sum_i w_i^2 + \frac{1}{3}\hat{\sigma}_E^2 \sum_i \sum_j w_{ij}^2 + z_q^2\left(\frac{\hat{\sigma}_E^2}{\sum_q z_q^2} \sum_i \sum_j w_{ij}^2\right)\right]^{1/2}$$

In terms of the fragility function, the estimated fragility function and the associated confidence bands are shown in Figure 3

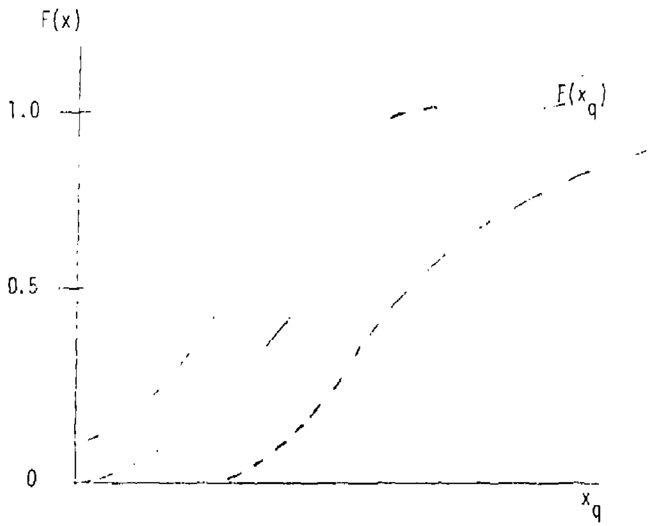

Figure 3. Confidence Bands for Fragility Function

Note that the confidence bands for the fragility function are not symmetric and not members of the family of fragility functions estimated by least squares. To alleviate this, one can consider using parallel confidence bounds for x_q based on the methods proposed by Graybill and Bowden [4].

The use of the term confidence is not consistent with its use in classical statistics. Rather, the variation reflected by the confidence bands is (1) the variation in the average strength between groups of components of the same generic type and (2) the variation between experts within a group. Thus, the confidence bands give some measure of the uncertainty associated with the estimated population fragility function, i.e. the average fragility function perceived by the population of experts. If this function can be assumed to approximate the actual fragility function for the given component, then eliciting opinions from a sample of experts provides input for a reasonable estimator of the component fragility function.

5. Least Squares Estimation With Percentile Dependent Errors

In the fragility questionnaires, subjective percentiles had greater variation at higher percentiles. This is because experts have less experience at high percentiles. Thus, the model for subjective percentiles should take this into account. Typical responses for various percentiles are shown in Figure 4. Points are numbered by respondent.

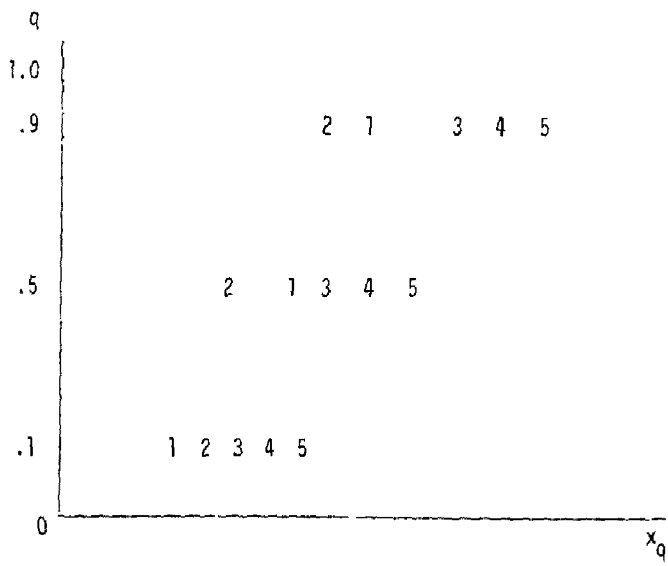

Figure 4. Typical Spread in Estimates as Functions of Percentiles.

The model for subjective percentiles with heteroscedastic variation is the following. It is assumed that the subjective percentiles from each expert are uncorrelated estimates and independent as in Section 2. The model assumes σ_E^2 increases in q. In this section, we will derive the least squares estimators of μ and σ, assuming a normal fragility function, for the model

$$X_{iq} = x_q + qE_{iq}$$

where $Var(E_{iq}) = \sigma_E^2$, but $Var(qE_{iq}) = q^2\sigma_E^2$.

Assume that all respondents give opinions on the same percentiles. The objective function is

$$\min_{\mu,\sigma} \sum_{i=1}^{n} \sum_{q} [X_{iq} - (\mu+\sigma z_q)]^2.$$

Weighted least squares is used because the variation in $X_{i.1}$ is not equal to the variation in $X_{i.5}$, etc.

Define $Y_{iq} = X_{iq}/q = x_q/q + E_{iq}$. The variance in each Y_{iq} is the same for all q for each respondent.

Since

$$Y_{iq} = x_q/q + E_{iq} = \frac{(\mu+z_q\sigma)}{q} + E_{iq},$$

the objective function is

$$\min_{\mu,\sigma} \sum_{i=1}^{n} \sum_{q} \left[Y_{iq} - \frac{\mu+z_q\sigma}{q}\right]^2.$$

The solution for $\hat{\sigma}$ is

$$\hat{\sigma} = \frac{\sum \bar{Y}_q/q - (\sum z_q/q^2)\ (\sum \bar{Y}_q{}^2/q)/\sum 1/q^2}{\sum z_q{}^2/q^2 - (\sum z_q/q^2)/\ \sum 1/q^2}$$

where all summations are over q and $\bar{Y}_q = \sum_{i=1}^{n} Y_{iq}/n$. The estimator of μ is

$$\hat{\mu} = \frac{\sum_q (\bar{Y}_q\ q - z_q\ \hat{\sigma})/q^2}{\sum 1/q^2} .$$

The estimator of the variance, σ_E^2, of the variation E_{iq} is based on averaging the squares of the residuals

$$\hat{E}_{iq} = \frac{X_{iq} - \hat{x}_q}{q} = \frac{X_{iq} - (\hat{\mu} + z_q\hat{\sigma})}{q} \qquad i = 1, 2, \ldots n \quad q\epsilon[0,1]$$

over the respondents and the percentiles. This can then be used to estimate the variance of the estimators $\hat{\mu}$ and $\hat{\sigma}$.

6. Using Subjective Percentiles and Test Data For Estimating Fragility Functions by Least Squares and Bayesian Methods

This section describes a least squares and a Bayesian method for combining subjective percentiles and test data. The least squares method treats the test data as an empirical cdf with percentiles at each observed failure strength. These percentiles are put in the weighted sum of squared deviations just like subjective percentiles. The Bayesian method uses the subjective percentiles to estimate a prior cdf of the fragility function parameters. The test data is assumed to be a random sample from the fragility function. The Bayes method gives the posterior cdf of the fragility function parameters. Substituting the expected values (Bayes estimates) of the parameters into the fragility function yields a posterior Bayes estimate of the fragility function.

The least squares method is as follows. The inputs consist of subjective percentiles and test data. The measurement unit of the test data must be the same as that of the subjective percentiles. The objective is to estimate the parameters of the fragility function. The parameter estimators minimize the weighted sum of squared deviations between the fragility function and the percentiles of the empirical cdf (test data) or the subjective percentiles. For this method it is assumed that the population fragility function (over all experts) is an unbiased estimator of the actual fragility function.

The inputs are denoted as follows. The observed strengths at failures are $X_1 \leq X_2 \leq \ldots \leq X_k$. The original test sample size may have been larger, $m > k$, and the survivors have strength greater than X_k. The subjective percentiles are X_{iq}.

The objective function for estimating the mean and variance of strength at failure, assuming strength is normally distributed is

$$\min_{\mu,\sigma} \left[W_D \sum_{i=1}^{k} (X_i - \mu - \sigma z_{i/m})^2 + W_S \sum_{i=k+1}^{n} \sum_{q} (X_{iq} - \mu - \sigma z_q)^2 \right].$$

The weights W_S and W_D must be specified by the user.

For the following derivation, assume $W_S = W_D = 1$ and the 10th, 50th, and 90th percentiles are given by each respondent. The objective function is

$$\min_{\mu,\sigma}\left[\sum_{i=1}^{k}[X_i - \mu - \sigma z_{i/m}]^2 + \sum_{i=k+1}^{n}\sum_{q}[X_{iq} - \mu - z_q\sigma]^2\right].$$

The normal equations are

$$-\sum_{i=1}^{k} 2(X_i - \mu - \sigma z_{i/m}) - 2\sum_{i=k+1}^{n}\sum_{q}(X_{iq} - \mu - z_q\sigma) = 0$$

and

$$-\sum_{i=1}^{k} 2(X_i - \mu - \sigma z_{i/m}) z_{i/m} - 2\sum_{i=k+1}^{n}\sum_{q} z_q(X_{iq} - \mu - z_q\sigma) = 0$$

The solution requires solving

$$\hat{\hat{\sigma}} = \frac{2\hat{\sigma}(1.28)^2 + (\overline{X}_D - \hat{\hat{\mu}})\sum_{i=1}^{k} z_{i/m}}{\sum_{i=1}^{k} z_{i/m}^2 - (1.28)^2 (n-k-1)/n}$$

and

$$\hat{\hat{\mu}} = \frac{\overline{X}_D}{4} + \frac{3\hat{\mu}}{4} - \hat{\hat{\sigma}}\sum_{i=1}^{k} z_{i/m}$$

simultaneously for $\hat{\hat{\sigma}}$ and $\hat{\hat{\mu}}$. The variable $\overline{X}_D$ is the average of the observed failure times, $\sum_{i=1}^{k} X_i/k$, and $\hat{\mu}$ and $\hat{\sigma}$ are the least squares estimators of μ and σ from Section 2. The solutions are

$$\hat{\hat{\mu}} = \frac{\frac{\overline{X}_D}{4} + \frac{3\hat{\mu}}{4} - \sum z_{i/m}\frac{2\hat{\sigma}(1.28)^2 + \overline{X}_D\sum z_{i/m}}{\sum z_{i/m}^2 - (1.28)^2(n-k-1)/n}}{1 + (\sum z_{i/m})^2/\{\sum z_{i/m}^2 - (1.28)\ (n-k-1)/n\}}$$

and

$$\hat{\hat{\sigma}} = \frac{2\hat{\sigma}(1.28)^2 + (\overline{X}_D - \hat{\hat{\mu}}) \sum z_{i/m}}{\sum z_{i/m}^2 - (1.28)^2 (n-k-1)/n} .$$

To describe a Bayesian procedure, assume a lognormal fragility function with parameters $\xi = E(\ell n\ x) = \ell n\ M$ and $\beta = Var(\ell n\ X)$, where X is the strength at failure and M denotes the median of X.

Assume the joint conjugate prior distribution for (ξ,β),

$$g(m,b;\ \theta_1,\theta_2,\nu) = \frac{1}{b}\sqrt{\frac{n}{2\pi}}\ e^{-1/2\left(\frac{m - \theta_1}{b/\sqrt{n}}\right)^2} \left[2\left(\frac{\nu\theta_2^{\ 2}}{2}\right)^{1/2} \frac{1}{\Gamma(\frac{\nu}{2})}\right] b^{-(\nu+1)}\ e^{-\frac{\nu\theta_2^{\ 2}}{2b^2}}; \qquad b > 0,\ -\infty < m < \infty$$

where θ_1 and θ_2 are the estimates of ξ,β derived from the expert opinions and ν is the degrees of freedom associated with θ_2, based on the n experts.

If $Y_1 \leq Y_2 \leq \ldots \leq Y_k$ is the ordered test data, where $Y_i = \ell n\ X_i$, let $\overline{Y} = \sum_{i=1}^{k} Y_i/k$ and $S^2 = \sum_{i=1}^{k} (Y_i - \overline{Y})^2/(k-1)$. Assume the sample is not censored or truncated. Then the posterior distribution of (ξ,β) has the same form as the prior distribution with parameters θ_1', θ_2', n' based on the relationships

$$n' = \nu + k + 1,$$

$$n'\ \theta_1' = k\overline{Y} + n\theta_1, \text{ and}$$

$$(n'-1)\theta_2'^{\,2} + (n'-1)\theta_1' = [(k-1)S^2 + k\ \overline{Y}^2] + [\nu\theta_2^{\ 2} + (\nu+1)\theta_1^{\ 2}] .$$

The posterior marginal means and variances are

$$E(\xi) = \theta_1' ;$$

$$Var(\xi) = \theta_2'^2 \frac{(n'-1)}{n'(n'-2)} ;$$

$$E(\beta) = \theta_2' \left(\frac{n'-1}{2}\right)^{1/2} \frac{\Gamma[(n'-3)/2]}{\Gamma[(n'-2)/2]} , \qquad n' > 3 \text{ ; and}$$

$$Var(\beta) = \theta_2'^2 \left(\frac{n'-1}{n'-4}\right) - \left[\varepsilon(\beta)\right]^2 , \qquad n' > 4 .$$

The Bayesian method involves using the subjective percentiles to determine initial values for θ_1 and θ_2, the parameters of the prior distribution, and then using the test data to evaluate the posterior marginal means for (ξ,β). These Bayes estimators are then used as estimators for the parameters of the log-normal fragility function.

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