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Combining Multipole Data*

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1 Introduction.

From June through December of 1986 physicists at the Magnet Test Facility (MTF) at Fermilab measured the fields of forty-eight Main Ring dipoles. Twenty-six of these were "B1" style magnets, whose physical apertures are rectangles with rounded corners, roughly 1.4 inches tall by 4.5 inches wide. A harmonic probe sampled the magnetic field of these magnets at three locations separated by one inch; the probe's radius was 0.6 inch. (See Figure 1.) As the probe rotates, "bucking coils" subtract the contribution from the magnet's dipole field from the signal. Fourier transforming the "residual" or "error" field filters out its harmonic content, the multipoles. Twenty-nine normal and skew multipoles were quoted at each of the three locations. Statistical errors associated with these data were estimated by taking one hundred measurements at one location of one magnet. Perhaps ~ 40 of the 174 multipoles recorded for each magnet, were sufficiently above the noise to be meaningful.

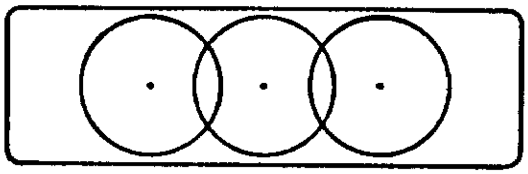

Figure 1: Geometry of the magnetic field measurements.

We address here the problem of combining the information from these three sets of data.

Before beginning, it is worthwhile to review the fundamental assumption which supports the entire discussion: that the magnetic field is well represented by a complex analytic function. Within a source-free region, horizontal and vertical components of a static magnetic field, $\vec{B}(\vec{x})$, must satisfy homogeneous Maxwell equations:

$$\begin{aligned} \frac{\partial B_2}{\partial x_1} &= \frac{\partial B_1}{\partial x_2} \ , \\ \frac{\partial B_2}{\partial x_2} &= -\frac{\partial B_1}{\partial x_1} - \frac{\partial B_3}{\partial x_3} \ . \end{aligned} \tag{1}$$

Were it not for the last term in Eq.(1) these would look identical to Cauchy-Riemann equations for a one-parameter family of analytic functions

$$G(z) \equiv B_2(z) + iB_1(z)$$

of the complex argument $z \equiv x_1 + ix_2$; the (real) variable x_3 (suppressed) would only label the individual members of this family. One way to justify ignoring the unwanted term is to assume longitudinal symmetry, so that $\partial B_3/\partial x_3 = 0$ identically. This sounds almost acceptable near the center of the magnet, but becomes less so when one end of the probe extends beyond the edge of the magnet. However, we can weaken this *local* condition to the *global* $\Delta B_3 = 0$, where ΔB_3 is the difference in B_3 between the endpoints of the probe, by integrating the field over the length of the probe. Since $B_3 \approx 0$ at both ends, this justifies representing at least the integrated transverse components with an analytic function. Arguing that the probe—and, more importantly, the particle beam—is actually sensitive only to integrated fields completes this line of reasoning.

2 Methods.

We shall describe three methods for combining multipole data which may be useful under possibly different assumptions: (1) multipole feeddown, (2) expansion in orthogonal functions, and (3) fictitious sources. All three are phenomenological—that is, they employ only the observed data—and are exceedingly simple, yet to do something more exact would require a full computer model of the magnet.

2.1 Method of Multipole Feeddown.

The "obvious" approach to this problem employs the feeddown effect for multipoles, by which translating a quadrupole induces a dipole field, translating a sextupole induces quadrupole and dipole fields, and so forth. Begin by defining complex multipoles $c_n(z_o)$, evaluated at z_o, as the coefficients of a Taylor expansion of G about z_o.

$$\begin{aligned} G(z) &= B_r(z_o) \sum_{n=0}^{\infty} c_n(z_o)(z - z_o)^n \\ c_n &\equiv b_n + ia_n \end{aligned} \tag{2}$$

We have allowed for the possibility that the reference dipole field, B_r, may depend on z_o; b_n and a_n are the usual normalized "normal" and "skew" components of the multipole. There is a linear relationship between the multipoles at the origin and those at any other point of reference.

$$B_r(0) \sum_{k=0}^{\infty} c_k(0) z^k = B_r(0) \sum_{k=0}^{\infty} c_k(0)(z - z_o + z_o)^k$$

$$= B_r(0) \sum_{n=0}^{\infty} \left[\sum_{k=n}^{\infty} \binom{k}{n} z_o^{k-n} c_k(0) \right] (z - z_o)^n \tag{3}$$

Equating this to the expansion in Eq.(2) provides the connection, which is written compactly using matrix notation:

$$c(z_o) = M(z_o) c(0) \ ,$$

$$M_{nk}(z_o) = \begin{cases} 0 & k < n \\ (B_r(0)/B_r(z_o)) \binom{k}{n} z_o^{k-n} & k \geq n \end{cases} \ .$$

The full data set is expressed by adjoining the matrices $M(+1)$ and $M(-1)$.

$$\left(\frac{c(+1)}{c(-1)} \right) = \left(\frac{M(+1)}{M(-1)} \right) c(0)$$

Data reduction would then consist of truncating this system and applying linear regression, weighted by the estimated statistical errors, to fix the coefficients, $c(0)$.

2.2 Method of Orthogonal Expansion.

The power series of Eq.(2) is the natural way to expand functions analytic on a *circular* aperture: in particular, the basis functions $(z - z_o)^n$ are orthogonal over circles centered at z_o. To make this more precise, let D represent the unit disk in the complex plane and let f and g be two complex-valued functions defined on D. We define the scalar product between f and g in an obvious way:

$$(f, g)_D \equiv \int\!\!\int_D dA(z) f^*(z) g(z) \ ,$$

where $dA(z) = (i/2) dz^* \wedge dz$ is the natural area measure over D. It is easy to verify that $(,)_D$ induces a metric and that

$$(z^n, z^m)_D = \frac{\pi}{n+1} \delta_{mn} \ .$$

Thus, the analytic functions $\varphi_n(z) = \sqrt{(n+1)/\pi}\, z^n$ form an orthonormal family over D.

Far more important than orthogonality — which, after all, can be forced by a Gram-Schmidt procedure — is the property of completeness: the functions φ_n form a complete basis for expanding functions analytic over the unit disk, ***but not over a rectangle.*** (In order to simplify the geometry, we shall ignore the rounded corners and treat the aperture as a simple rectangle.)

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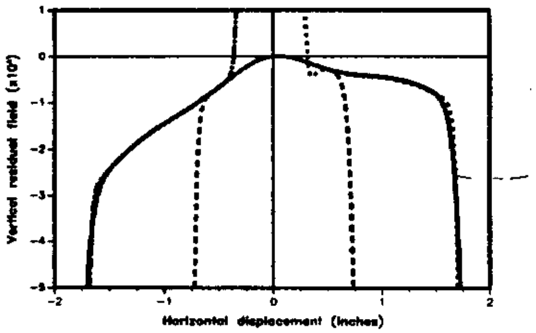

Figure 2: Vertical residual field calculated using the Method of Sources.

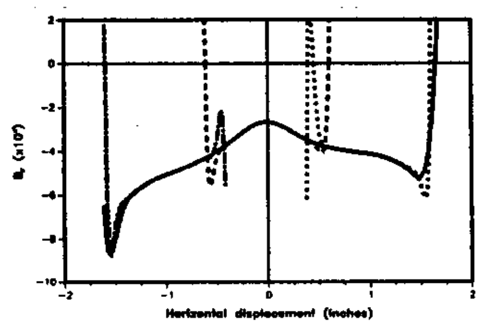

Figure 3: Vertical residual field at scan height 0.4 inches.

The problem of finding a corresponding set of basis functions, say $\psi_n(z)$, which are both orthogonal and complete over a rectangular domain, R, can be solved by constructing a conformal transformation, $u(z)$, which maps the interior of R onto the unit disk, D. If we have such a mapping, we can take

$$\psi_n(z) = \varphi_n(u(z))\frac{du(z)}{dz} \quad , \tag{4}$$

for then

$$\begin{aligned}(\psi_n, \psi_m)_R &= \iint_R dA(z) \mid du(z)/dz \mid^2 \varphi_n^*(u(z))\varphi_m(u(z)) \\ &= \iint_{u(R)=D} dA(u)\,\varphi_n^*(u)\varphi_m(u) \;=\; \delta_{nm} \quad .\end{aligned}$$

That the set of functions $\{\psi_n\}$ is complete over R follows immediately from the observation that they are conformally related, via Eq.(4), to the set $\{\varphi_n\}$, which is complete over D.

We construct the transformation $u(z)$ in two steps. First, the Weierstrauss elliptic function

$$w(z) = \wp(z \mid \omega_1, \omega_2)$$

maps a rectangle of dimension $2\omega_1 \times 2\omega_2$ in the z-plane into the half-plane, $\mathrm{Im}[w] < 0$. Second, the Möbius mapping

$$u = \frac{w + i\epsilon}{w - i\epsilon}, \quad \epsilon \text{ real, positive}$$

takes the half-plane into the unit disk. Combining the two gives us the desired conformal transformation:

$$u(z) = \frac{\wp(z \mid \omega_1, \omega_2) + i\epsilon}{\wp(z \mid \omega_1, \omega_2) - i\epsilon} \quad . \tag{5}$$

The parameters ω_1 and ω_2 are fixed by the dimensions of the rectangle; the value of ϵ determines the point that maps into the origin. Riemann's famous Mapping Theorem assures us that no simpler conformal transformation exists which takes rectangles into circles.

The procedure now would be as follows. Expand $G(z)$ over the rectangular aperture according to

$$G(z) = \sum_{n=0}^{\infty} g_n \psi_n(z) \ ,$$

with ψ_n given by Equations (4) and (5). By Taylor expanding G, equivalently ψ_n, about $z_o = -1, 0, +1$ we develop linear equations relating the coefficients, g_n, to the data, $c_n(z_o)$. These are truncated, and the g_n obtained, as before, by linear regression.

2.3 Method of Sources.

Because G is an analytic function, it can be represented by a Cauchy integral,

$$G(z) = \frac{1}{2\pi i}\oint \frac{G(u)du}{u - z} \quad ,$$

around the aperture's boundary. This we approximate with a Riemann sum.

$$\begin{aligned} G(z) &\cong \frac{1}{2\pi i}\sum_k \frac{G(u_k)\Delta u_k}{u_k - z} \\ &\equiv \frac{1}{2\pi}\sum_k \frac{1}{z - u_k} I_k \end{aligned} \tag{6}$$

$$I_k \equiv iG(u_k)\Delta u_k$$

The complex numbers I_k can be thought of as fictitious sources placed on the edges of the physical aperture. If we write an individual term as

$$\begin{aligned} B_2 + iB_1 &= \frac{1}{2\pi \mid \varsigma \mid^2}\{\, \mathrm{Re}[I_k](\varsigma_1 - i\varsigma_2) + \mathrm{Im}[I_k](\varsigma_2 + i\varsigma_1)\,\} \\ \varsigma &\equiv z - u_k \end{aligned}$$

then it is obvious that $\mathrm{Re}[I_k]$ is interpreted as an electric current, and $\mathrm{Im}[I_k]$ as a line density of magnetic monopoles located at u_k. (To see this correspondence, simply apply Stokes's theorem to Maxwell's equations in the usual way.)

Measuring field multipoles at a point, z_o, amounts to Taylor expanding G about that point.

$$\begin{aligned} \frac{1}{z - u_k} &= \frac{1}{(z - z_o) - (u_k - z_o)} \\ &= -\sum_{n=0}^{\infty}\left(\frac{1}{u_k - z_o}\right)^{n+1}(z - z_o)^n \end{aligned}$$

$$G(z) = \sum_{n=0}^{\infty}\left[-\frac{1}{2\pi}\sum_k\left(\frac{1}{u_k - z_o}\right)^{n+1} I_k\right](z - z_o)^n$$

We identify the coefficient of $(z - z_o)^n$ with the n^{th} complex magnetic multipole. (See Eq.(2).)

$$B_r(z_o)c_n(z_o) = -\frac{1}{2\pi}\sum_k\left(\frac{1}{u_k - z_o}\right)^{n+1} I_k$$

The fictitious sources I_k are obtained by weighted linear regression on the data, after which the field can be evaluated using Eq.(6). As an additional constraint, we set the dipole component exactly to zero at the origin.

$$G(0) = -\frac{1}{2\pi}\sum_k I_k/u_k \equiv 0$$

This reflects the dipole subtraction from the data and focusses the numerical procedure on the residual field.

3 Calculations.

The history of applying these ideas to MTF data was this: Emphasis was placed first on implementing the most obvious approach, the method of Multipole Feeddown. The one ingredient most essential to its applicability is rapid convergence of the summation that appears inside the square bracket in Eq.(3). In particular, z_o must be small enough so that the factor $z_o^{k-n}c_k(0)$ offsets the divergence of the binomial coefficient. Unfortunately, this was *not* the case: offsets of ± 1 inch were far too large to make feeddown a viable approach. Experimentation with a fading memory Kalman filter, done in the hope of developing an asymptotic procedure, proved unable to surmount this problem.

The Method of Sources was tried next, and it worked well almost immediately. After this success, the ongoing development of the Method of Orthogonal Expansion was stopped; no calculations were carried out using this third approach.

Figure 2 illustrates one of the calculations carried out on data from a Main Ring dipole (ADM285) using the Method of Sources. The solid line shows a midplane scan ($x_2 = 0$) of the vertical component of the interpolating residual field, as calculated from Eq.(6), normalized to 10^{-4} of the dipole field; the three dashed lines show the results of summing the three multipole series at $x_o = -1, 0,$ and $+1$ inches. (Dipole field offsets at ± 1 were set by the interpolating field; the $\sim 10^{-4}$ variation in $B_r(x_o)$ was ignored.) The interpolating field matches each series out to about half an inch from its center, where the series expansion abruptly fails. It also does an excellent job of smoothly splicing the three data sets together. The interpolating field itself is good only to about ± 1.5 inches; it cannot be used for extrapolation.

We can see from this picture why, apart from the convergence problem, Feeddown was doomed to failure. The two regions of overlap between the series-expanded fields are extremely small, and in one of them the expansions do not agree. It would have been impossible for the method to work under such conditions.

In Figure 3 the horizontal scan is done at a vertical height of 0.4 inches from the magnet's midplane. The solution continues to interpolate smoothly through the data even though there is now no overlap between the three series. A number of other scans of both horizontal and vertical fields produced similarly encouraging results.

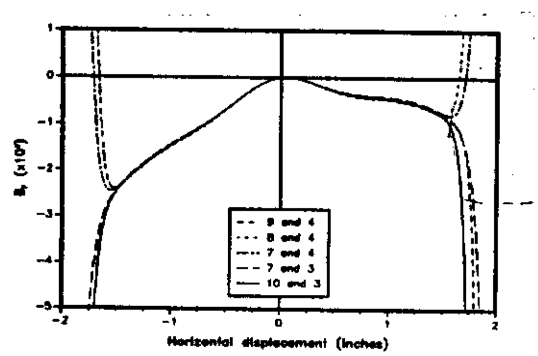

Figure 4: Comparison between five sets of fictitious sources for interpolating the residual field.

The interpolating field of Figure 2 was calculated using 26 sources, 10 associated with each horizontal edge and 3 with each vertical edge, while that of Figure 3 used a configuration of 9 (horizontal) and 4 (vertical). The results are almost independent of these numbers. To demonstrate this insensitivity further, five different arrangements of sources. are compared in Figure 4. The actual source values for the "9 and 4" case, normalized by $B_r(0)$, are

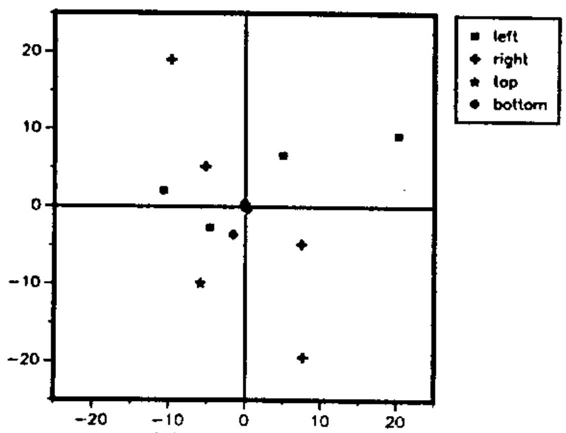

Figure 5: Values of the sources obtained using the "9 and 4" configuration.

illustrated in Figure 5. Almost all the significant sources driving the error field fell on the left and right edges of the aperture; the information contained in the original data has effectively been encoded into twenty (real) numbers. No attempt was made to optimize the placement or number of sources. My objective here was only to demonstrate that at least one of these methods could be made to work on actual data taken at MTF.

ACKNOWLEDGMENTS

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