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Differential Calculus on Quantum Spaces and Quantum Groups

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Differential Calculus on Quantum Spaces and Quantum Groups *

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Abstract

A review of recent developments in the quantum differential calculus. The quantum group $GL_q(n)$ is treated by considering it as a particular quantum space. Functions on $SL_q(n)$ are defined as a subclass of functions on $GL_q(n)$. The case of $SO_q(n)$ is also briefly considered. These notes cover part of a lecture given at the XIX International Conference on Group Theoretic Methods in Physics, Salamanca, Spain 1992.

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1. Introduction

In this lecture I shall describe some recent developments in the theory of differential calculi on quantum spaces and quantum groups. The general theory is due to Woronowicz [1] and a number of interesting papers (see [2, 3]) have elucidated various aspects of it. I shall emphasize techniques [4] which give explicit commutation relations and which are hopefully suitable for future physical applications. Many of the conventions and notations used here can be found in [5]. This basic paper also contains numerous references.

2. Differential Calculus on Quantum Planes

We consider basic variables x^k, for $k = 1, 2, ...n$, which satisfy commutation relations

$$B^{k\ell}_{mn} x^m x^n = 0 \tag{2.1}$$

where the $B^{k\ell}_{mn}$ are numerical coefficients. We assume that these commutation relations allow one to order in some standard way an arbitrary monomial in the variables. Functions f, g etc. of the basic variables can be defined as formal power series and form an associative algebra. We wish to define an exterior differential d satisfying the usual underformed properties such as linearity, plus

$$d^2 = 0, \tag{2.2}$$

the Leibniz rule on functions (zero-forms)

$$d(fg) = (df)g + f\ dg, \tag{2.3}$$

and

$$d(dx^k f) = -dx^k df. \tag{2.4}$$

In (2.4) f is a function or a differential form. In general the differentials dx^k of the basic variables will not commute with the variables. Here we consider the case when the commutation relations between the differentials and the variables are bilinear

$$x^k(dx^\ell) = C^{k\ell}_{mn}(dx^m)x^n, \tag{2.5}$$

where $C^{k\ell}_{mn}$ are numerical coefficients.

One can introduce derivatives on functions

$$\partial_k \equiv \frac{\partial}{\partial x^k}, \quad (\partial_k x^\ell) = \delta_k^\ell \tag{2.6}$$

in the standard way through

$$df = dx^k \partial_k f. \tag{2.7}$$

In general the derivatives do not satisfy the simple Leibniz rule of commutative algebra. We have, for an arbitrary function f,

$$\begin{aligned} d(x^k f) &= (dx^\ell \partial_\ell x^k) f + x^k dx^\ell \partial_\ell f \\ &= (dx^k) f + C^{k\ell}_{mn}(dx^m) x^n \partial_\ell f. \end{aligned} \tag{2.8}$$

which can be written as a commutation relation between derivatives and variables

$$\partial_\ell x^k = \delta_\ell^k + C^{km}_{\ell n} x^n \partial_m. \tag{2.9}$$

Applying d to (2.5) one obtains, from (2.4),

$$dx^k dx^\ell = -C^{k\ell}_{mn} dx^m dx^n. \tag{2.10}$$

In [6,7] commutation relations between derivatives and differentials where also given, in the form

$$\partial_k(dx^\ell) - D^{\ell m}_{kn}(dx^n)\partial_m = 0 \tag{2.11}$$

and among the derivatives in the form

$$\partial_n \partial_m F^{mn}_{k\ell} = 0. \tag{2.12}$$

The coefficients B, C, D, and F must satisfy certain consistency relations, discussed in [6,7]. There it was shown that it must be

$$B^{k\ell}_{rs} + B^{k\ell}_{mn} C^{mn}_{rs} = 0. \tag{2.13}$$

This equation can be written in standard tensor product notation as

$$B_{12}(I_{12} + C_{12}) = 0 \tag{2.14}$$

where I is the unit matrix. One finds also that $D = C^{-1}$, i.e.

$$D^{k\ell}_{mn} C^{mn}_{rs} = C^{k\ell}_{mn} D^{mn}_{rs} = \delta^k_r \delta^\ell_s, \tag{2.15}$$

a Yang-Baxter equations for C

$$C_{12} C_{23} C_{12} = C_{23} C_{12} C_{23} \tag{2.16}$$

and an orthogonality relation analogous to (2.14)

$$(I_{12} + C_{12}) F_{12} = 0. \tag{2.17}$$

Finally, we have two mixed Yang-Baxter equations:

$$B_{12} C_{23} C_{12} = C_{23} C_{12} B_{23} \tag{2.18}$$

and

$$C_{12} C_{23} F_{12} = F_{23} C_{12} C_{23}, \tag{2.19}$$

which are sufficient conditions for the consistency of the calculus.

The above consistency conditions (2.13-19) are obtained by combining the various commutation relations. For instance, multiply (2.1) from the left with ∂_r and commute this derivative through to the right by using (2.9) twice. One finds two terms which must vanish separately, the first proportional to a single x, the second proportional to a product of the type $xx\partial$. The vanishing of the first term gives (2.13), the vanishing of the second term is ensured by (2.18). The other conditions are obtained in a similar manner.

In many concrete examples the matrices B, C, and F can be expressed [8] as functions of a single matrix $\widehat{R}$ which satisfies the Yang-Baxter equation

$$\widehat{R}_{12} \widehat{R}_{23} \widehat{R}_{12} = \widehat{R}_{23} \widehat{R}_{12} \widehat{R}_{23} \tag{2.20}$$

and a characteristic equation

$$(\widehat{R} - \mu_1)(\widehat{R} - \mu_2)...(\widehat{R} - \mu_m) = 0. \tag{2.21}$$

We assume that the eigenvalues $\mu_1, \mu_2, ...\mu_m$ are distinct; they may have different multiplicities. For any particular non vanishing eigenvalue μ_α, one can choose

$$C = -\frac{\widehat{R}}{\mu_\alpha} \tag{2.22}$$

and

$$B = F = \prod_{\beta \neq \alpha} (\hat{R} - \mu_\beta). \tag{2.23}$$

Clearly (2.16) is true, the orthogonality relations (2.14) and (2.17) are obviously satisfied and the Yang-Baxter equations (2.18) and (2.19) are also valid, because (2.20) implies

$$p(\hat{R}_{12})\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\, p(\hat{R}_{23}) \tag{2.24}$$

and

$$\hat{R}_{12}\hat{R}_{23}\, p(\hat{R}_{12}) = p(\hat{R}_{23})\hat{R}_{12}\hat{R}_{23} \tag{2.25}$$

for any polynomial $p(.)$.

Notice that now (2.10) becomes.

$$\hat{R}^{ij}_{k\ell} dx^k dx^\ell = \mu_\alpha dx^i dx^j. \tag{2.26}$$

Therefore

$$F^{ij}_{k\ell} dx^k dx^\ell = \prod_{\beta \neq \alpha} (\mu_\alpha - \mu_\beta) dx^i dx^j, \tag{2.27}$$

where α is fixed. Multiply this equation by $\partial_j \partial_i$ from the right. The left hand side vanishes by (2.12), so we obtain

$$0 = \prod_{\beta \neq \alpha} (\mu_\alpha - \mu_\beta) dx^i dx^j \partial_j \partial_i. \tag{2.28}$$

Since $\mu_\beta \neq \mu_\alpha$ for all $\beta \neq \alpha$ (α fixed),

$$d^2 = d\; dx^i \partial_i = -dx^i d\partial_i = -dx^i dx^j \partial_j \partial_i \tag{2.29}$$

vanishes in agreement with (2.2).

If all eigenvalues are different from zero one can use the inverse matrix $\hat{R}^{-1}$ which also satisfies the Yang-Baxter equation. This gives alternative consistent forms of the calculus based on

$$C = -\mu_\alpha \hat{R}^{-1} \tag{2.30}$$

and

$$B = F = \prod_{\beta \neq \alpha} (\hat{R}^{-1} - \mu_\beta^{-1}) \tag{2.31}$$

for any given eigenvalue μ_α of $\hat{R}$.

The simplest example that fits into the present scheme is that of the quantum hyperplane where $\hat{R}$ is the $\hat{R}$-matrix of $GL_q(n)$ which satisfies a characteristic equation with the two eigenvalues $\mu_1 = q, \mu_2 = -q^{-1}$. Here one can choose the eigenvalue μ_2, which gives $C = q\hat{R},\ B = F = \hat{R} - q$. The resulting calculus has been discussed in detail in [6, 7] (the alternative based on (2.30) and (2.31) was also treated there). An equivalent formulation is given in [9]. If one chooses instead the eigenvalue μ_1 one obtains $C = -q^{-1}\hat{R},\ B = F = \hat{R} + q^{-1}$, so that the commutation relations are now

$$x_1x_2 = -q\hat{R}_{12}x_1x_2, \tag{2.32}$$

$$x_1dx_2 = -\frac{1}{q}\hat{R}_{12}dx_1x_2 \tag{2.33}$$

and

$$dx_1dx_2 = \frac{1}{q}\hat{R}_{12}dx_1dx_2. \tag{2.34}$$

As $q \longrightarrow 1, \hat{R}_{12}$ tends to P_{12}, the permutation matrix

$$(P_{12})^{k\ell}_{mn} = \delta^k_n\delta^\ell_m. \tag{2.35}$$

Therefore the commutation relations become

$$x^kx^\ell = -x^\ell x^k \tag{2.36}$$

$$x^kdx^\ell = -dx^\ell x^k \tag{2.37}$$

$$dx^kdx^\ell = dx^\ell dx^k. \tag{2.38}$$

In this limit the variables x^k are fermionic but the commutation relations (2.37) involving variables and differentials differ by a sign from the standard ones for a simply graded fermionic calculus. This is a perfectly consistent alternative with double grading which goes together with the validity of (2.3) and (2.4) also for fermionic x's, and which is equivalent [10], in a well defined sense, to the more standard simply graded fermionic calculus . The q-deformation of the standard fermionic calculus is given in [7]. An equivalent formulation was presented in [11].

Another interesting example is that of quantum euclidian space, in which case one takes the $\hat{R}$-matrix of $SO_q(N)$, which satisfies a characteristic equation withe the three eigenvalues $\mu_1 = q, \mu_2 = -q^{-1}$ and $\mu_3 = q^{1-N}$. If one chooses the eigenvalue μ_2 and applies the general formulas (2.22) and (2.23) for $\alpha = 2$, one obtains the conventional quantum calculus on euclidian space.

In the next section we shall see that the quantum group $GL_q(n)$ itself can be treated as a quantum plane and that the calculus on $GL_q(n)$ fits into the present formulation with an $\hat{R}$-matrix having three distinct eigenvalues.

3. Calculus on $GL_q(n)$

The defining representation of the quantum group $GL_q(n)$ is given in terms of $n \times n$ matrices A whose matrix elements satisfy the commutation relations

$$\hat{R}_{12} A_1 A_2 = A_1 A_2 \hat{R}_{12} \tag{3.1}$$

where the $\hat{R}$ matrix of $GL_q(n)$ is given in [5] as

$$\hat{R}^{ij}_{k\ell} = \delta^i_\ell \delta^j_k (1 + (q-1)\delta^{ij}) + \lambda \delta^i_k \delta^j_\ell \theta_{ji}. \tag{3.2}$$

Here

$$\lambda = q - \frac{1}{q} \tag{3.3}$$

and

$$\theta_{ji} = \begin{cases} 1 & j > i \\ 0 & j \leq i \end{cases}. \tag{3.4}$$

We shall take q to be a generic complex number, not too far from 1. The quantum determinant $det_q A$ of the matrix A is defined [5] as

$$det_q A = \sum_\sigma (-q)^{\ell(\sigma)} A_{1\sigma_1} A_{2\sigma_2} \ldots A_{n\sigma_n}, \tag{3.5}$$

where the sum is over all permutations $(\sigma_1, \sigma_2, \ldots \sigma_n)$ of the integers $(1, 2, \ldots n)$ and $\ell(\sigma)$ is the length (number of inversions) of the permutation σ. The quantum determinant of A commutes with all elements of A as a consequence of (3.1). We assume that it does not vanish, so that we can define, at least in a formal sense, the matrix A^{-1}.

Let us consider the matrix elements of A as basic coordinates on group space. With the notation $\binom{i}{j} = \alpha$, $\binom{k}{\ell} = \beta$ etc., we can write $A^i_j = x_\alpha$ etc.

The commutation relations (3.1) can be written in a form similar to (2.1) if we introduce a "large" $\widehat{\mathbf{R}}$-matrix defined by

$$\widehat{\mathbf{R}}_{(12)(34)} = \frac{1}{q}\hat{R}_{13}\hat{R}_{24}. \tag{3.6}$$

$\hat{R}$ satisfies the characteristic equation

$$(\hat{R} - q)(\hat{R} + \frac{1}{q}) = 0. \tag{3.7}$$

Since its eigenvalues are q and $-q^{-1}$, those of $\widehat{\mathbf{R}}$ are $\mu_1 = q, \mu_2 = -q^{-1}$ and $\mu_3 = q^{-3}$

$$(\widehat{\mathbf{R}} - q)(\widehat{\mathbf{R}} + \frac{1}{q})(\widehat{\mathbf{R}} - \frac{1}{q^3}) = 0. \tag{3.8}$$

If we choose the eigenvalue μ_2 and apply the formulas of the previous section we obtain

$$\mathbf{C} = q\widehat{\mathbf{R}}, \tag{3.9}$$

$$\mathbf{B} = \mathbf{F} = (\widehat{\mathbf{R}} - q)(\widehat{\mathbf{R}} - \frac{1}{q^3}) \tag{3.10}$$

and we are led to the commutation relations

$$\left[(\widehat{\mathbf{R}} - q)(\widehat{\mathbf{R}} - \frac{1}{q^3})\right]_{\alpha\beta,\gamma\delta} x_\gamma x_\delta = 0, \tag{3.11}$$

$$x_\alpha dx_\beta = q\widehat{\mathbf{R}}_{\alpha\beta,\gamma\delta} dx_\gamma x_\delta \tag{3.12}$$

and

$$dx_\alpha dx_\beta = -q\widehat{\mathbf{R}}_{\alpha\beta,\gamma\delta} dx_\gamma dx_\delta. \tag{3.13}$$

It is not hard to check that (3.11) is equivalent to (3.1) and that (3.12) and (3.13) can be written respectively as

$$A_1 dA_2 = \hat{R}_{12} dA_1 A_2 \hat{R}_{12} \tag{3.14}$$

and

$$dA_1 dA_2 = -\hat{R}_{12} dA_1 dA_2 \hat{R}_{12}. \tag{3.15}$$

This is the form given by Schirrmacher [12] and Sudbery [13, 14] to the commutation relations of Maltsiniotis [15, 16] and Manin [17, 18] for the calculus on $GL_q(n)$. We see that they agree with the general formulation of Sec. 2 for the calculus on a quantum plane (notice that the characteristic equation (3.8) is the same as for $SO_q(4) \sim SL_q(2) \times SL_q(2)$).

It is convenient to introduce the numerical diagonal matrix

$$D = diag(1, q^2, ..., q^{2(n-1)}). \tag{3.16}$$

This matrix satisfies a number of useful relations which are listed in [4, 5, 19]. In particular, for any $n \times n$ matrix M, it is

$$\mathrm{tr}_1(D_1^{-1} \hat{R}_{12}^{-1} M_2 \hat{R}_{12}) = \mathrm{tr}(D^{-1}M) I_2 \tag{3.17}$$

where tr_1 is the trace with respect to the indices relative to the first space in the tensor product and I_2 is the unit matrix in the second space. Also

$$\mathrm{tr}_1(D_1^{-1} \hat{R}_{12}^{-1}) = q^{1-2n} I_2. \tag{3.18}$$

If A satisfies (3.1) then

$$D^{-1} A^t D (A^{-1})^t = (A^{-1})^t D^{-1} A^t D = I, \tag{3.19}$$

where ^{t} denotes transposition. It follows that, if the matrix elements of M commute with those of A, then

$$\mathrm{tr}(D^{-1} A^{-1} M A) = \mathrm{tr}(D^{-1} M). \tag{3.20}$$

For this reason, $\mathrm{tr}(D^{-1}M)$ is called the quantum invariant trace of M.

As we know from the previous section, one can introduce derivatives which, according to (2.9) and (2.12) satisfy now

$$\partial_\alpha x_\beta = \delta_{\alpha\beta} + q \widehat{\mathbf{R}}_{\beta\delta,\alpha\gamma} x_\gamma \partial_\delta \tag{3.21}$$

and

$$\partial_\beta \partial_\alpha \left[(\widehat{\mathbf{R}} - q)(\widehat{\mathbf{R}} - \frac{1}{q^3}) \right]_{\alpha\beta,\gamma\delta} = 0. \tag{3.22}$$

These equations can be written in a form analogous to (3.1), (3.14) and (3.15). Let us introduce a matrix of derivatives by $\tilde{\partial}^i_j = \partial_\alpha$ and $\partial^i_j = (D^{-1})^j_k \tilde{\partial}^k_i$. Using (3.17) and (3.18) one finds

$$\partial_1 \hat{R}_{12}^{-1} A_1 = q^{1-2n} I_{12} + A_2 \hat{R}_{12} \partial_2 \tag{3.23}$$

and

$$\hat{R}_{12}\partial_2\partial_1 = \partial_2\partial_1\hat{R}_{12}. \tag{3.24}$$

Equations (3.1), (3.14) and (3.15) go into themselves under the left coaction $A \to A'A$ and the right coaction $A \to AA'$ where A' is a constant (i.e. $dA' = 0$), $GL_q(n)$ matrix which satisfies (3.1). Equations (3.23) and (3.24) also go into themselves if one transforms the derivative matrix respectively as $\partial \to \partial(A')^{-1}$ and $\partial \to (A')^{-1}\partial$ (the constancy of the matrix A' implies that its matrix elements commute with those of dA and of ∂).

The Cartan-Maurer form

$$\Omega = A^{-1}\, dA \tag{3.25}$$

is left-invariant and right-covariant i.e. $\Omega \to \Omega$ and $\Omega \to (A')^{-1}\Omega A'$ under the respective coactions above. The 1-form

$$\xi = -q^{2n-1}\,\mathrm{tr}(D^{-1}\Omega) \tag{3.26}$$

is both left- and right-invariant, see (3.20). Ω satisfies the following equations due to (3.1), (3.14) and (3.15)

$$\Omega_1 A_2 \;= A_2 R_{12}^{-1}\Omega_1 R_{21}^{-1} \tag{3.27}$$

$$\Omega_1 dA_2 \;+ dA_2 R_{12}^{-1}\Omega_1 R_{12} = 0 \tag{3.28}$$

$$\Omega_1 R_{21}^{-1}\Omega_2 R_{21} \;+ R_{21}^{-1}\Omega_2 R_{12}^{-1}\Omega_1 = 0. \tag{3.29}$$

Here and in the followings we use the R-matrix of $GL_q(n)$, which is related to the $\hat{R}$-matrix used above by

$$(R_{12})^{ij}_{k\ell} = (P_{12}\hat{R}_{12})^{ij}_{k\ell} = (\hat{R}_{12})^{ji}_{k\ell}. \tag{3.30}$$

Thus (3.1) becomes

$$R_{12}A_1A_2 = A_2A_1R_{12}. \tag{3.31}$$

From the properties of D and the characteristic equation (3.7) one can show [4] that the above equations imply

$$dA = \lambda^{-1}(\xi A - A\xi) \tag{3.32}$$

and

$$d\Omega = -\Omega^2 = \lambda^{-1}(\xi\Omega + \Omega\xi). \tag{3.33}$$

Thus, if f is any form

$$df = \lambda^{-1}[\xi, f]_\pm, \tag{3.34}$$

where $[\ ,\ ]_\pm$ is a commutator for even degree forms, an anticommutator for odd degree forms (λ is given in (3.3)).

The quantum determinant $det_q A$ of the matrix A is a zero form. We know that it commutes with all elements of A. The above equations imply that

$$\Omega(det_q A) = q^{-2}(det_q A)\Omega \tag{3.35}$$

and

$$d(det_q A) = -q^{-1}(det_q A)\xi = -q\xi(det_q A). \tag{3.36}$$

A consequence of these equations is that both $d\xi$ and ξ^2 vanish. The elements of Ω form a linearly independent basis for 1-forms, and we shall use them instead of the elements of dA from now on.

4. Inner Derivations and Lie Derivatives for $GL_q(n)$.

Following [4], we now introduce the inner derivation, which we take to be a left action mapping k-forms to $(k-1)$-forms. Its action on the n^2 elements of A and Ω is given by introducing n^2 vector fields $X^i{}_j$, and the associated n^2 inner derivations are the entries in the matrix i_X whose elements are

$$(i_X)^i{}_j = i_{X^i{}_j}. \tag{4.1}$$

i_X must act on 0- and 1-forms in a way preserving the commutation relations (3.31) and (3.27-29); the appropriate actions are

$$\begin{aligned} i_{X_1} A_2 &= A_2 R_{21} i_{X_1} R_{12}, & (4.2)\\ R_{21} i_{X_1} R_{12}\Omega_2 + \Omega_2 R_{21} i_{X_1} R_{12} &= \frac{1 - R_{21}R_{12}}{\lambda}. & (4.3) \end{aligned}$$

These two equations imply that when evaluated on 0- and 1-forms,

$$(i_X f) = 0,\ (i_{X_1}\Omega_2) = -q^{1-2n} D_2 P_{12}, \tag{4.4}$$

where f is any function of the elements of A. Equation (4.3) gives

$$i_X\xi + \xi i_X = I. \tag{4.5}$$

Notice that by using the characteristic equation $\frac{1-R_{21}R_{12}}{\lambda}$ could be replaced by $-\hat{R}_{12}$. On det_qA, the inner derivation acts as

$$i_X(det_qA) = q^2(det_qA)i_X. \tag{4.6}$$

The commutation relations between the inner derivation matrices are

$$R_{12}^{-1}i_{X_1}R_{12}i_{X_2} + i_{X_2}R_{21}i_{X_1}R_{12} = 0. \tag{4.7}$$

It is easy to see that i_X is left-invariant and right-covariant under the respective coactions on A.

We may now introduce the Lie derivative matrix L_X in the same way as in the classical theory, i.e. a left action taking k-forms to k-forms given by

$$L_X \equiv i_Xd + di_X, \tag{4.8}$$

where L_X is a matrix with elements $L_{X^i_j}$ which by definition transforms in the same way as i_X does. The equations already given for d and i_X imply the following relations involving L_X:

$$\begin{aligned}
L_Xd &= dL_X, && (4.9)\\
R_{21}L_{X_1}R_{12}i_{X_2} - i_{X_2}R_{21}L_{X_1}R_{12} &= \lambda^{-1}(R_{21}R_{12}i_{X_2} - i_{X_2}R_{21}R_{12}), && (4.10)\\
R_{21}L_{X_1}R_{12}L_{X_2} - L_{X_2}R_{21}L_{X_1}R_{12} &= \lambda^{-1}(R_{21}R_{12}L_{X_2} - L_{X_2}R_{21}R_{12}), && (4.11)\\
L_{X_1}A_2 &= A_2R_{21}L_{X_1}R_{12} + A_2(\frac{1-R_{21}R_{12}}{\lambda}), && (4.12)\\
R_{21}L_{X_1}R_{12}\Omega_2 - \Omega_2R_{21}L_{X_1}R_{12} &= \lambda^{-1}(R_{21}R_{12}\Omega_2 - \Omega_2R_{21}R_{12}), && (4.13)\\
L_X\xi &= \xi L_X, && (4.14)
\end{aligned}$$

and for the determinant,

$$L_X(det_qA) = q^2(det_qA)L_X - q(det_qA). \tag{4.15}$$

Many of these relations take a much simpler form if we introduce the Lie derivative valued operator Y given by

$$Y = 1 - \lambda L_X, \tag{4.16}$$

which, of course, has the same transformation properties as L_X. Using this, we obtain

$$\begin{aligned} Yd &= dY, & (4.17)\\ R_{21}Y_1R_{12}\,i_{X_2} &= i_{X_2}R_{21}Y_1R_{12}, & (4.18)\\ R_{21}Y_1R_{12}Y_2 &= Y_2R_{21}Y_1R_{12}, & (4.19)\\ Y_1A_2 &= A_2R_{21}Y_1R_{12}, & (4.20)\\ R_{21}Y_1R_{12}\Omega_2 &= \Omega_2R_{21}Y_1R_{12}, & (4.21)\\ Y\xi &= \xi Y, & (4.22) \end{aligned}$$

and

$$Y(det_qA) = q^2(det_qA)Y. \tag{4.23}$$

A matrix satisfying (4.19) was introduced on several occasions in the literature (see [20, 21]) and is often called L instead of Y; (4.19) is often called the "reflection equation".

The matrix Y is invertible, at least in a formal sense. It is also possible [4, 22] to define a quantum determinant Det Y which commutes with the elements of Y. Perhaps the simplest way to introduce it is to observe that the matrix AY satisfies the same commutation relations (3.1, 31) as the matrix A itself as can be seen using (3.1, 31), (4.19) and (4.20). We can define

$$DetY = q^{n(n-1)}[det_q\ A]^{-1}[det_q(AY)], \tag{4.24}$$

where the right hand side involves only the standard quantum determinant. Alternatively, if we observe that YA^{-1} satisfies the same commutation relations as A^{-1}, we can write

$$DetY = q^{n(n-1)}[det_{q^{-1}}(YA^{-1})][det_{q^{-1}}A^{-1}], \tag{4.25}$$

which gives an equivalent result. This determinant is invariant under transformations of Y (i.e. $Y \mapsto Y$ for $A \mapsto AA'$ and $Y \mapsto (A')^{-1}YA'$ for $A \mapsto A'A$, with Y and A' having commuting elements), and satisfies the following relations:

$$d(Det\,Y) = (Det\,Y)d, \tag{4.26}$$

$$(Det\,Y)i_X = i_X(Det\,Y), \tag{4.27}$$

$$(Det\,Y)A = q^2A(Det\,Y), \tag{4.28}$$

$$(Det\,Y)\Omega = \Omega(Det\,Y), \tag{4.29}$$

$$(Det\,Y)\xi = \xi(Det\,Y), \tag{4.30}$$

and

$$(DetY)(det_qA) = q^{2n}(det_qA)(Det\,Y). \tag{4.31}$$

The above equations for $Det\,Y$ suggest the definition of an operator H_0 as

$$Det\,Y \equiv q^{2H_0}. \tag{4.32}$$

H_0 commutes with Y, d, i_X, Ω, and ξ, and satisfies

$$[H_0, A] = A, \quad [H_0, det_qA] = n(det_qA). \tag{4.33}$$

5. Calculus on the Quantum Group $SL_q(n)$

There seems to be an obvious way to specify the calculus on the quantum group $SL_q(N)$: take the matrix A and set its quantum determinant to unity. However, although det_qA commutes with the elements of A, it does *not* commute with such quantities as Ω and Y. Therefore, instead of imposing $det_qA = 1$, we *define* matrices T as

$$T = (det_qA)^{-1/n}A. \tag{5.1}$$

With det_qT defined as in (3.5), the centrality of det_qA automatically gives T determinant unity. This matrix T is what we identify as an element of the defining representation of $SL_q(N)$, since it also satisfies (3.1) with A replaced by T. As we will see in the next section, it becomes convenient to introduce the matrix

$$\mathcal{R}_{12} = q^{-1/n}R_{12}, \tag{5.2}$$

which we identify as the R-matrix for $SL_q(N)$. Thus, we shall write (3.1) as

$$\mathcal{R}_{12}T_1T_2 = T_2T_1\mathcal{R}_{12}. \tag{5.3}$$

The exterior derivative on $SL_q(n)$ can be taken to be the same as that introduced on $GL_q(n)$; this is because T is a function of the elements of A, so its differentials are given by

$$dT = \lambda^{-1}[\xi, T]. \tag{5.4}$$

Note that this implies that the Cartan-Maurer form $\tilde{\Omega}$ for $SL_q(n)$ is given by

$$\tilde{\Omega} \equiv T^{-1}dT = q^{2/n}\Omega + q\ [1/n]_q\xi,^* \tag{5.5}$$

where

$$[x]_q = \frac{1-q^{2x}}{1-q^2}. \tag{5.6}$$

In the classical limit $q \longrightarrow 1$, $\tilde{\Omega}$ is traceless, giving the appropriate reduction from n^2 to n^2-1 independent elements in the Cartan-Maurer matrix 1-form for $SL(n)$.

We have thus found a way to set the determinant of our $SL_q(n)$ matrices to unity; for the calculus on the group, we must do something similar, namely impose a constraint so that the number of independent differential operators is reduced from n^2 to n^2-1. In a way, we have already done this, because (4.33) and (5.1) together imply

$$[H_0, T] = 0, \tag{5.7}$$

so that H_0 commutes with everything of interest in $SL_q(n)$, i.e. matrices, forms, exterior derivative, etc. Thus, within the context of $SL_q(n)$, H_0 is irrelevant, reducing the number of generators from n^2 to n^2-1, as desired. Explicitly, this restriction is accomplished by defining a new Lie derivative valued operator Z by

$$Z \equiv q^{-2H_0/n}Y.^\dagger \tag{5.8}$$

*This relation implies that the matrix of differential forms introduced in [19] is equal to $-q^{2n-1}\Omega$.

†When restricted to acting on 0-forms, this operator is identical to the operator Y in [19].

Note that the determinant of Z, computed using e.g. (4.24), is unity. This is equivalent to the introduction of a set of n^2 "vector fields" $V^i{}_j$ through $Z = 1 - \lambda L_V$, so that

$$L_V = L_X + q^{-1}[H_0/n]_{q^{-1}} - q^{-1}\lambda L_X[H_0/n]_{q^{-1}}. \tag{5.9}$$

The fact that $Det\, Z = 1$ implies that only $n^2 - 1$ of the elements of L_V are actually independent, which is precisely what we require for $SL_q(n)$. In the classical limit, $H_0 = -tr(L_X)$, so L_V becomes traceless; thus, V contains only $n^2 - 1$ linearly independent vector fields, as we would expect.

Now that we have obtained all these quantities, we want to find the various relations they satisfy. The commutation relations between Ω and T are given by

$$\Omega_1 T_2 = q^{2/n} T_2 R_{12}^{-1} \Omega_1 R_{21}^{-1} = T_2 \mathcal{R}_{12}^{-1} \Omega_1 \mathcal{R}_{21}^{-1}. \tag{5.10}$$

Here we see the appearance of $\mathcal{R}_{12}$, as promised. Ω remains unchanged, so (3.29) is still valid: it does not have $\mathcal{R}_{12}$ in place of R_{12}. L_V satisfies

$$\mathcal{R}_{21} L_{V_1} \mathcal{R}_{12} L_{V_2} - L_{V_2} \mathcal{R}_{21} L_{V_1} \mathcal{R}_{12} = \lambda^{-1}(\mathcal{R}_{21}\mathcal{R}_{12} L_{V_2} - L_{V_2}\mathcal{R}_{21}\mathcal{R}_{12}), \tag{5.11}$$

The actions of the various operators on the 0- and 1-forms of $SL_q(n)$ are given by

$$L_{V_1} T_2 = T_2 \mathcal{R}_{21} L_{V_1} \mathcal{R}_{12} + T_2\left(\frac{1 - \mathcal{R}_{21}\mathcal{R}_{12}}{\lambda}\right), \tag{5.12}$$

and

$$\mathcal{R}_{21} L_{V_1} \mathcal{R}_{12} \Omega_2 - \Omega_2 \mathcal{R}_{21} L_{V_1} \mathcal{R}_{12} = \lambda^{-1}(\mathcal{R}_{21}\mathcal{R}_{12}\Omega_2 - \Omega_2 \mathcal{R}_{21}\mathcal{R}_{12}). \tag{5.13}$$

As a consequence, ξ satisfies

$$L_V \xi = \xi L_V. \tag{5.14}$$

The relations for Z corresponding to (4.17-22) are

$$\mathcal{R}_{21} Z_1 \mathcal{R}_{12} Z_2 = Z_2 \mathcal{R}_{21} Z_1 \mathcal{R}_{12}, \tag{5.14}$$

$$Z_1 T_2 = T_2 \mathcal{R}_{21} Z_1 \mathcal{R}_{12}, \tag{5.16}$$

$$\mathcal{R}_{21} Z_1 \mathcal{R}_{12} \Omega_2 = \Omega_2 \mathcal{R}_{21} Z_1 \mathcal{R}_{12} \tag{5.17}$$

and

$$Z\xi = \xi Z. \tag{5.18}$$

Notice that the invariant form constructed with the Cartan-Maurer form $\tilde{\Omega}$ is

$$\tilde{\xi} = tr\ D^{-1}\tilde{\Omega} = -q^{1-2n}(q^{2/n} - q^{2n}\left[\frac{1}{n}\right]_q [n]_{1/q})\xi. \tag{5.19}$$

It vanishes as $q \longrightarrow 1$ as it should.

6. Conclusion

In Sec. 3 we have seen that the differential calculus for $GL_q(n)$ is a special case of the differential calculus on quantum planes. We chose there the $\widehat{\mathbf{R}}$ version of the $GL_q(n)$ calculus, the $\widehat{\mathbf{R}}^{-1}$ version could be developed in a similar way.

In Sec. 5 we derived the differential calculus on $SL_q(n)$ by defining the functions on $SL_q(n)$ as a subclass of functions on $GL_q(n)$. While for $GL_q(n)$ there are n^2 independent Lie derivative operators Y and n^2 independent 1-forms Ω, for $SL_q(n)$ the number of Lie derivatives is reduced to $n^2 - 1$ by the relation $DetZ = 1$. However, the number of Cartan Maurer 1-forms is still n^2. One of them is the invariant form ξ which generates the differentiation through (5.4) and (3.32), and which has no classical analogue. This is related to the fact that, in spite of the restriction $DetZ = 1$, it is not possible to find n^2-1 Lie derivatives which satisfy a bicovariant deformed Lie algebra with only quadratic relations. The n^2 elements of Z are of this type and the relation $DetZ = 1$ is consistent with the commutation relations, but it is a polynomial relation. If one drops the requirement of bicovariance, for $SL_q(2)$ there exist a right invariant and also a left invariant calculus with $n^2 - 1 = 3$ Lie derivatives satisfying quadratic commutation relations. However, this seems to be a special property of $n = 2$. For higher n no such calculi with $n^2 - 1$ Lie derivatives are known, even if one drops the requirement of bicovariance.

An important lesson one can derive from the developments of the previous sections is that it is very useful to consider the larger algebra which has as generators for $GL_q(n)$ the matrix elements of A and of Y together (or of T and Z for $SL_q(n)$), their commutation relations being given by (3.31) and (4.19, 20). While the functions on the group form a Hopf algebra and the enveloping algebra of the Y is also a Hopf algebra, the A, Y larger algebra is not a Hopf algebra; still, it contains all the necessary information. This point of view, which allows multiplication of elements of A with elements of Y, leads to the simple definition of the $DetY$ given in (4.24, 25).

The consideration of the larger A, Y (or T, Y) algebra is useful for other quantum groups as well. For instance, for $SO_q(N)$, there is an orthogonality relation for the T matrices

$$T^t C T = C, \tag{6.1}$$

where the metric matrix C is defined in [5]. For this quantum group, the matrix product $q^{N-1}TZ$ satisfies all relations for a quantum orthogonal matrix, including (6.1). Indeed, one can verify that

$$q^{2(N-1)}(TZ)^t C(TZ) = C \tag{6.2}$$

gives rise to the correct relations for the Z matrix of $SO_q(N)$, i.e. [22, 24]

$$q^{N-1} C_{k\ell} Z^{\ell}_{m} R^{mk}_{in} Z^{n}_{j} = C_{ij}, \tag{6.3}$$

where R^{mk}_{in} is here the R-matrix of $SO_q(N)$ given in [5]. This can be easily seen using (6.1), the relation

$$C_{ij} R^{ji}_{k\ell} = q^{1-N} C_{k\ell}. \tag{6.4}$$

and the $Z-T$ commutation relation, which for $SO_q(N)$, is still

$$Z_1 T_2 = T_2 R_{21} Z_2 R_{12}. \tag{6.5}$$

For $SO_q(N)$ the situation described earlier for $SL_q(n)$ is even more extreme. The number of independent Lie derivatives is reduced from n^2 to $n(n-1)/2$ by the polynomial relations (6.3). However, the number of independent Cartan-Maurer 1-forms is still n^2. Of these, one is the invariant 1-form ξ which plays a special role analogous to that for $GL_q(n)$ or $SL_q(n)$, but now there are $n^2 - 1 - n(n-1)/2 = n(n+1)/2 - 1$ additional 1-forms which cannot be eliminated in the bicovariant calculus. Only as $q \to 1$ these 1-forms vanish [23] in the combination $\tilde{\Omega} = T^{-1}dT$. Except for the case of $GL_q(n)$ (and for the nonbicovariant calculi on $SL_q(2)$), the introduction of all the additional 1-forms seems unavoidable. The elegant commutation relations for Lie derivatives involving only quadratic (and linear) terms seems possible only at the price of introducing more of them than in the classical $q = 1$ case and then restricting their number by means of polynomial relations. These facts are worth emphasizing, since they are mostly ignored in the literature.

References

1. S.L. Woronowicz, Commun. Math. Phys. **122** 125 (1989).

2. D. Bernard, Prog. Theoretical Phys. Supp. **102** 49 (1990).

3. B. Jurčo, Lett. Math. Phys. **22** 177 (1991).

4. P. Schupp, P. Watts, and B. Zumino, Lett. Math. Phys. **25** 139 (1992).

5. N. Yu. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, Leningrad Math. J. **1** 193 (1990).

6. J. Wess and B. Zumino, Nucl. Phys. B (Proc. Suppl.) **18B** 302 (1990).

7. B. Zumino, Mod. Phys. Lett. **A 13** 1225 (1991).

8. L. Hlavatý, J. Phys. A: Math. Gen. **25** 485 (1992).

9. W. Pusz and S.L. Woronowicz, Reports on Math. Phys. **27** 231 (1989).

10. B. Zumino, in: Unification of the Fundamental Particle Interactions (S. Ferrara, J. Ellis and P. van Nieuwenhuizen Eds.), Proc. Europhysics Conf., Erice (1980), Plenum Press 1980, p. 101.

11. W. Pusz, Reports on Math. Phys. **27** 349 (1989).

12. A. Schirrmacher, in: Groups and Related Topics (R. Gielerak et al. Eds.), Kluwer Academic Publishers (1992) p. 55.

13. A. Sudbery, York preprint PRINT-91-0498 (YORK).

14. A. Sudbery, York preprint YORK-92-1 .

15. G. Maltsiniotis, C. R. Acad. Sci. Paris, **331** 831 (1990).

16. G. Maltsiniotis, "Calcul differentiel sur le groupe linéaire quantique", ENS exposé (1990).

17. Yu. Manin, Bonn preprint MPI/91-47.

18. Yu. Manin, Bonn preprint MPI/91-60.

19. B. Zumino, in: Math. Phys. X (K. Schmüdgen Ed.), Proc. X-th IAMP Conf., Leipzig (1991), Springer-Verlag (1992), p.20.

20. N. Yu. Reshetikin, M. A. Semenov-Tian-Shansky, Lett. Math. Phys. **19** 133 (1990).

21. P.P. Kulish, R. Sasaki, C. Schwiebert, Preprint YITP/U-92-07.

22. P. Schupp, P. Watts and B. Zumino, Berkeley preprint, LBL-32315 and UCB-PTH-92/14.

23. U. Carow-Watamura, M. Schlieker, S. Watamura, and W. Weich, Commun. Math. Phys. **142** 605 (1991).

24. B. Drabant, B. Jurčo, M. Schlieker, W. Weich and B. Zumino, Preprint MPI-PH-92-39, LBL-32354 and UCB-PTH 92/16.

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