CONF-780952--6

ANALYTIC SOLUTION FOR THE PROPAGATION VELOCITY IN SUPERCONDUCTING COMPOSITES*

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ABSTRACT

The propagation velocity of normal zones in composite superconductors has been calculated analytically for the case of constant thermophysical properties, including the effects of current sharing. The solution is compared with that of a more elementary theory in which current sharing is neglected, i.e., in which there is a sharp transition from the superconducting to the normal state. The solution is also compared with experiment. This comparison demonstrates the important influence of transient heat transfer on the propagation velocity.

I. INTRODUCTION

In order to calculate the time behavior of a quenching magnet, we need to know the velocity of propagation of normal zones in the conductor. From the rate of growth of the normal zones, we can calculate the variation with time of the resistance of the magnet, a quantity that enters the circuit equation describing the decay of the magnet current.

In 1967, Keilin, Klimenko, Kremlev, and Samoilov[1] described a theory of normal zone propagation in composite superconductors. The formulation of their theory included the effects of current sharing, but the problem they actually solved was one in which current sharing was neglected (i.e., in which the superconducting-normal transition occurred at a single temperature). In 1973, Altov, Kremlev, Sytchev, and Zenkevitch[2] remedied this by solving the differential equation of Keilin et al. on a computer. Both works were carried out under the assumption of constant thermophysical properties (i.e., constant specific heats, thermal conductivities, and heat transfer coefficient).

In 1976, I developed a more complete theory of propagation that included current sharing as well as the temperature variation of all the thermophysical properties.[3] In order to secure agreement with experiment, a further ad hoc correction for transient heat transfer had to be made. Surprisingly, this correction turned out to be in excellent agreement with corrections directly measured by Iwasa and Apgar.[4,10]

My theory, like that of Altov et al., also requires a computer for evaluation of the propagation velocity. While this is easy enough when only a few velocities are required, the computer program is too cumbersome to be included as a subroutine in a program like Wilson's QUENCH.[5]

In this paper, I shall show how to solve analytically the problem of Altov et al. (current sharing and constant thermophysical properties) and thus how to avoid the necessity of numerically integrating the differential equation. The propagation velocity is found as the solution of a transcendental equation. While this step involves the intervention of a computer, it is fast and well adapted to inclusion as a subroutine in another program. If errors of only a few percent are admissible, the propagation velocity can be obtained from a simple explicit formula that can be included without difficulty in just a few lines in any other program.

Manuscript received September 28, 1978.

*Research sponsored by the Office of Fusion Energy (ETM), U.S. Department of Energy under Contract No. W-7405-eng-26 with Union Carbide Corporation.

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II. THEORY

We begin with a heat balance for a unit length of composite superconductor

$$S \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + Q_J - \frac{hP}{A}(T - T_b) \tag{1}$$

(All symbols are defined in a list at the end of the paper.) We look for a traveling wave solution of Eq. (1), namely $T = T(x + vt)$, which represents a wave traveling from right to left when v is positive. Then since $\partial T/\partial t = v(\partial T/\partial x)$ for the traveling wave, Eq. (1) reduces to the ordinary differential equation

$$\frac{d}{dx}\left(k \frac{dT}{dx}\right) - vS \frac{dT}{dx} + Q_J - \frac{hP}{A}(T - T_b) = 0 \tag{2}$$

We reduce the order of this differential equation by using the substitution of Maddock, James, and Norris,[6] $s = k(dT/dx)$:

$$s \frac{ds}{dT} - vSs + k\left[Q_J - \frac{hP}{A}(T - T_b)\right] = 0 \tag{3}$$

If we introduce dimensionless variables, Eq. (3) becomes

$$y \frac{dy}{d\tau} - my + \alpha i^2 g(\tau) - \tau = 0 \tag{4}$$

where $g(\tau)$ is the function of Keilin et al.[1] and Maddock et al.[6]

$$g(\tau) = \begin{cases} 0 & 1 - i > \tau \\ (\tau + i - 1)/i & 1 - i < \tau < 1 \\ 1 & \tau > 1 \end{cases} \tag{5}$$

Use of this function is equivalent to assuming a linear relation between temperature and critical current. In the range of temperatures $\tau < 1 - i$, the conductor is superconducting, the current sharing range is $1 - i < \tau < 1$ and for $\tau > 1$, the conductor is normal.

In the range $\tau < 1 - i$, the third term in Eq. (4) is absent because $g(\tau) = 0$. The origin is a saddle point crossed by two separatrices L_+ and L_-. The separatrices, which are also solutions of Eq. (4), are straight lines. This can most easily be seen by noting that when $g = 0$, Eq. (4) is homogeneous, i.e., invariant to the one-parameter group of transformations $y' = \sigma y$, $\tau' = \sigma\tau$, $0 < \sigma < \infty$. The separatrices must be invariant curves of this group and the invariant curves are straight lines.[7] Substitution of $y = a\tau$ in Eq. (4) shows that the slopes of L_+ and L_- are respectively,

$$a_{\pm} = (m \pm \sqrt{m^2 + 4})/2 \tag{6}$$

When $\tau = 0$, $y = 0$ and when $\tau > 0$, $y > 0$. For, when $\tau = 0$, we are far to the left of the thermal wave (which is propagating from right to left), and thus $y \sim dT/dx = 0$. Furthermore, as we advance toward the wave front by moving in the positive x direction, T increases, so that $y \sim dT/dx > 0$. Thus the solution of Eq. (4) we seek when $\tau < 1 - i$ is the separatrix with the positive slope L_+: $y = \alpha_+\tau$.

When $\tau > 1$, $g(\tau) = 1$. If we introduce $\tau - \alpha i^2$ as a new independent variable, we get the same differential equation as when $g = 0$. When $\tau = \alpha i^2$, i.e., far to the right of the propagating wave front, $y \sim dT/dx = 0$. This time, however, we approach the origin from the negative side, i.e., $\tau - \alpha i^2 < 0$. However, since y is always > 0, we must approach $\tau - \alpha i^2 < 0$ along the separatrix with the negative slope L_-: $y = \alpha_-(\tau - \alpha i^2)$.

The two parts of the solution of Eq. (4) already obtained show that at the two edges of the current sharing region, the solution of Eq. (4) obeys the boundary conditions

$$y = \alpha_+ (1 - i) \text{ at } \tau = 1 - i \qquad (6a)$$

$$y = \alpha_- (1 - \alpha i^2) \text{ at } \tau = 1 \qquad (6b)$$

In the current sharing region, Eq. (4) is more difficult to solve. If we introduce the auxiliary variables $c = \alpha i - 1$, $b = \alpha(1 - i)i$, and $w = c\tau - b$, we get the homogeneous equation

$$cy \frac{dy}{dw} - my + w = 0 \qquad (7)$$

According to a theorem of Lie,[8] because of the homogeneity of Eq. (7), $[cy^2 + w(w - my)]^{-1}$ is an integrating factor for Eq. (7). The integration is straightforward but tedious and gives

$$\phi = \frac{1}{2} \ln [cy^2 + w(w - my)] + \frac{m}{\sqrt{4c - m^2}} \arctan \left(\frac{2cy/w - m}{\sqrt{4c - m^2}}\right) \qquad (8)$$

where ϕ is a constant of integration. Equation (8) can be verified by differentiation. The dimensionless velocity m must be chosen so that the pairs of values of y and τ given in Eqs. (6a) and (6b) both give the same value of ϕ. Shown in Fig. 1 is the dimensionless velocity m calculated using Eq. (8) as a function of i with α as parameter.

III. APPROXIMATE RESULTS — NO CURRENT SHARING

Keilin et al.[1] have already given a closed formula for m as a function of α and i. In their theory, the sudden transition from the superconducting to the normal state occurred at $\tau = 1 - i$, the current sharing threshold. Keilin's formula then gives $i = (-1 + \sqrt{1 + 2\alpha})/\alpha$ as the value of i at which the velocity $m = 0$ (minimum propagating current). On the other hand, from the equal-area theorem of Maddock, James, and Norris,[5] we know that the velocity calculated from Eq. (8) will be zero when $i = (-1 + \sqrt{1 + 8\alpha})/3\alpha$. Owing to the steepness of curves of m vs i (especially for large α), a slight horizontal displacement of a curve causes a large change in the value of m for fixed i and α. Hence, approximate results for the case of no current sharing should be calculated on the assumption that the sudden transition occurs at the equal-area point $\tau = 1 - i/2$. Then the approximate curves and those based on Eq. (8) will cross the value $m = 0$ at the same i. This means the lines L_+: $y = \alpha_+\tau$ and L_-: $y = \alpha_-(\tau - \alpha i^2)$ must intersect at $y = 1 - i/2$. Some simple algebra then shows that

$$m = (C - 1)/\sqrt{C}, \quad C = [\alpha i^2 - (i - i/2)]/(1 - i/2) \qquad (9)$$

Comparison of Eq. (9) with the exact results in Fig. 1 shows that Eq. (9) underestimates the velocity m by about 22% when $\alpha = 2$, by about 7% when $\alpha = 4$, and by smaller amounts for larger α. These errors apply over most of the range of i, but near the limiting values $i = 1/\sqrt{\alpha}$ and $i = 1$ the errors are larger. We can reduce the error in the estimate of Eq. (9) by applying the empirical factor $1 + 0.561\ \alpha^{-1.45}$. After multiplication by this factor, the estimates provided by Eq. (9) agree with the values obtained from Eq. (8) to within 1% or better over most of the range $1/\sqrt{\alpha} < i < 1$, with substantial errors occurring only near the ends of the range.

IV. COMPARISON WITH EXPERIMENT

Shown in Fig. 2 are propagation and recovery data taken by Miller and Lue[3] on a $3.3 \times 1.7\ mm^2$ composite with a Cu/SC ratio of 10, a RRR of 145, and critical currents of 1070, 870, and 700 A at 2, 3, and 4 T, respectively. At 3 T, the measured minimum propagating current is 670 A, corresponding to $i = 0.77$ and $\alpha = (2 - i)/i^2 = 2.1$. At 2 T and 4 T, the minimum propagating currents correspond to $i = 0.71$ and 0.85, and $\alpha = 2.6$ and 1.6, respectively. With these values of α, we find h = 690, 820, and 970 $W\ m^{-2}\ K^{-1}$ for 2, 3, and 4 T, respectively. Shown in Fig. 2 are fits to the velocity data obtained using m as calculated from Eq. (9) and corrected with the empirical factor. The ratios $v/m = \sqrt{hkP/A}/S$ were taken to be 1.0, 0.82, and 0.80 m/s for 2, 3, and 4 T, respectively. Then S = 19.7, 23.1, and 23.2 $mJ\ cm^{-3}\ K^{-1}$ for 2, 3, and 4 T, respectively. (It is not possible to fit the recovery data below 600 A since that is the full recovery current corresponding to $\alpha = 2.1$. When a more realistic boiling curve is used instead of a constant heat transfer coefficient, the full recovery current and the minimum propagating current do not uniquely determine each other.)

We have achieved a reasonably good fit of the data by choosing $h \sim 0.08\ W\ cm^{-2}\ K^{-1}$ and $S \sim 22\ mJ\ cm^{-3}\ K^{-1}$. This value of the heat transfer coefficient h is typical of the film boiling region and is what we might expect. The specific heat S, on the other hand, is too large.

The points in Fig. 2 correspond to thermal waves in which the maximum conductor temperature behind the wave front lies in the ranges 9.7-11.7 K, 8.9-9.3 K, and 7.9-9.1 K for 2, 3, and 4 T, respectively, depending on the current. The specific heat of a 10:1 composite cannot exceed 15.3, 8.3, and 7.8 $mJ\ cm^{-3}\ K^{-1}$, respectively, in these three temperature ranges.

This paradox can be resolved in the following way. Recently, Iwasa and Apgar[4] suggested that in the film boiling region, a term $a(T)\ dT/dt$ proportional to the time rate of change of temperature should be added to the steady-state heat flux to account for transient effects. In another place,[10] I showed the equivalence of Iwasa and Apgar's transient correction term with one I used earlier in the analysis of normal zone propagation data. If this term were included in Eq. (1), it could be combined with the left-hand side, so that S would be replaced by $S + Pa/A$. The largest values that $S + Pa/A$ assumes in the three temperature ranges above are now roughly 51.6, 27.5, and 26.1 $mJ\ cm^{-3}\ K^{-1}$. These are sufficiently large so that there is no longer any contradiction with the constant value of roughly 22 $mJ\ cm^{-3}\ K^{-1}$ which is needed to fit the propagation data.

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SYMBOLS

A = conductor cross sectional area (m^2)
a = coefficient of Iwasa and Apgar (Ref. 4)
a_+ = see Eq. (5)
b_- = $\alpha i(1 - f)$
C = $[\alpha i^2 - (1 - i/2)]/(1 - i/2)$
c = $\alpha i - 1$
f = volume fraction of matrix in the conductor
g = function defined in Eq. (5)
h = heat transfer coefficient ($Wm^{-2}\ K^{-1}$)
I = transport current (A)
I_{cr} = critical current (A)
i = I/I_{cr}
k = thermal conductivity ($Wm^{-1}\ K^{-1}$)
m = $(A/hkP)^{1/2}Sv$
P = cooled perimeter (m)
Q_J = joule power density (Wm^{-3})
s = $k\ dT/dx$
S = conductor specific heat ($Jm^{-3}\ K^{-1}$)
T = conductor temperature (K)
T_b = helium bath temperature (K)
T_{cr} = critical temperature (K)
t = time (sec)
v = propagation or recovery velocity ($msec^{-1}$)
w = $c\tau - b$
x = distance along the conductor (m)
y = $(A/hkP)^{1/2}\ s/(T_{cr} - T_b)$
α = $\rho I_{cr}^2/fAPh(T_{cr} - T_b)$ (Stekly's parameter)
τ = $(T - T_b)/T_{cr} - T_b)$
ρ = matrix resistivity (ohm m)
ϕ = constant of integration in Eq. (8)

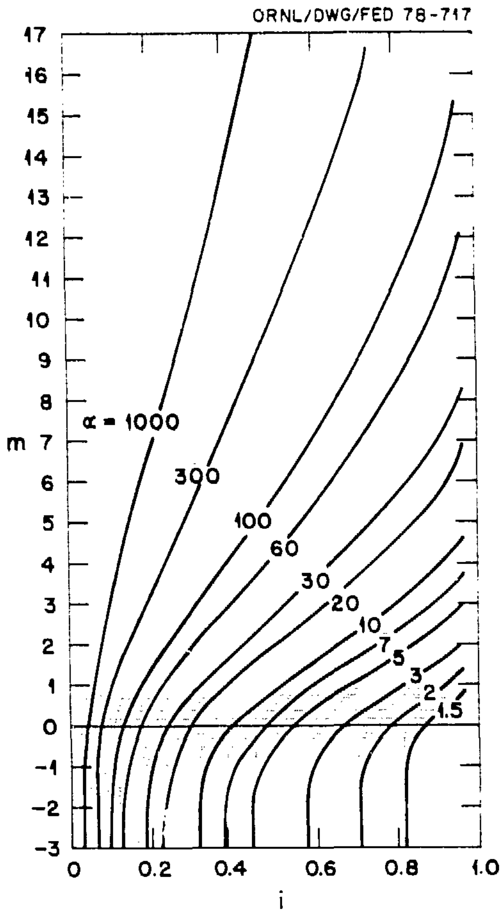

Fig. 1. Dimensionless velocity as a function of dimensionless current i with the Stekly number α as parameter.

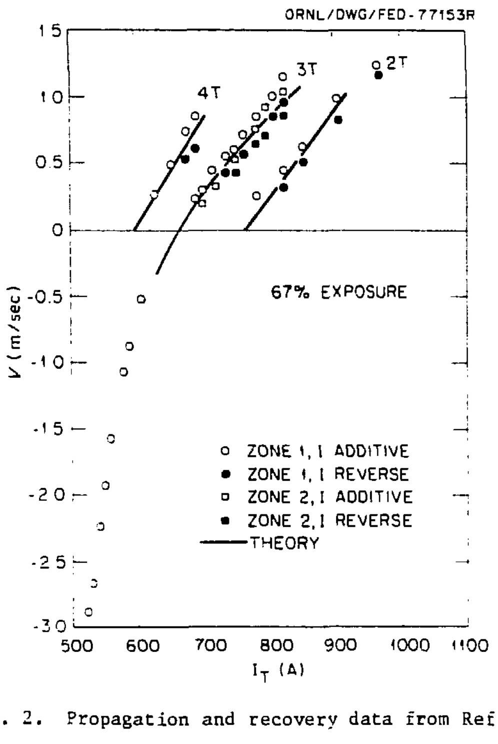

Fig. 2. Propagation and recovery data from Ref. 9.