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J. SCOTT PROVAN AND ANDY S. KYDES

September 1980

NATIONAL CENTER FOR ANALYSIS OF ENERGY SYSTEMS
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ECONOMIC AND SYSTEM ANALYSIS DIVISION

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ABSTRACT

The concept of determinacy measures the consistency of interaction between the operating variables or parameters of a large scale model. Determinacy is first defined in a general "input-output" model, and is related to similar concepts in economics and linear programming. It is then applied to a particular network flow model. Special techniques are then developed to detect determinacy in this model and these techniques are applied to two special classes of networks to uncover the high degree of determinacy in models of these classes.

I. INTRODUCTION

The motivation for this research came out of a series of technical memoranda by Harvey Greenberg, Measuring Complementarity and Qualitative Determinacy in Matricial Form, [1], on the subject of model organization, written for the Department of Energy. His efforts have focused on quantifying the interrelationships between variables of a well-defined model. Analysts today, utilizing the speed and storage capabilities of modern computers, are able to develop models with a high degree of realism and detail. By the time such a complex model has been completed, however--along with the necessary calibration, special case computations, and corrective factors--the basic relations between variables have often been obscured. One of the present methods of recovering the relations is through sensitivity analysis, which for these models amounts to a "black box" statistical approach. An alternative justification is desirable, particularly for those with a skeptical eye for statistical or external techniques, and it is to this end that the techniques developed in this paper are addressed.

The classically stated criterion apropos the question of the relationships of variables in a model is: "All other things being fixed, what is the effect of a change in factor X (a variable, a parameter, or perhaps even a computation) on factor Y?" Many authors make no further clarification on this "definition," often with disquieting results; the fact is that in most models it is impossible to change factor X without changing a sizable number of other factors (in addition to Y). At the other extreme, the criterion, "Under all circumstances, what is the effect of a change in X on Y?" is too broad; rarely can any consistent answer be made to this question. So the problem remains to formulate a definition of relationships between variables which is not so broad as to be meaningless and not so narrow as to be vacuous.

A reasonable and workable balance is drawn in this paper in the concept of "output realizable configurations." This is a slight modification of Greenberg's concept of configuration, made to allow meaningful relations among a sufficiently large class of variables in a model. It is derived directly from the idea of a "feasible basis" in linear programming, and in Section III this is the precise form in which it is applied. The criterion here is: "If a certain maximal number of factors are fixed, what is the effect of a change in factor X on factor Y?" Implicit in this formulation is the requirement that the fixed factors allow the model to work "realistically," that is, X, Y, and the variable factors interact within specified operational limits. Now we do not have a two-factor trade-off but a many-factor (although minimal) alteration in the model performance. We can therefore no longer speak of the isolated effect of factor X on factor Y but rather the effect of "substituting X into a configuration" on the other variables--including Y--of the configuration. In linear programming terminology, this operation is known as a "pivot."

We are now able to define the concept of determinacy in models, extending the ideas of Greenberg, and the earlier ideas of Lady [2]. The question asked here is: "To what extent can two factors X_1 and X_2 be substituted into the various configurations, in that they have similar effects on each other or on the other factors of the model?" It is here that we resort to the simplification used in economic theory of "qualitative" effect. This has been the subject of many papers--[3] - [6], to cite a few. To assess the "qualitative effect" of, say, a substitution on a set of variables, we ignore the magnitude of the change (which is often subject to calibration and judgment errors anyway), and consider only the sign of the change, i.e., we are interested only in whether affected variables increase, decrease, or remain the same in response to a specified change in model operation. With this final simplification, it becomes possible to state a workable definition of determinancy--that is, two factors are considered determinant if their qualitative effects on any configuration are either identical (substitutes) or opposite (complements). This means that one can speak of certain factors as being "cooperative" or "competitive" according to their effect on the operation of the model.

The second portion of the paper applies the definitions developed to an important type of model, called the supply-demand model. Generally, this model can be described as moving goods (information, events, etc.) from supply points (sources, initiation points, etc.) to demand points (sinks, termination points, etc.). In these senses, it underlies nearly all organization and management processes, and thus is a natural and critical place to begin determinacy studies. Further, since we are dealing with the model in a qualitative fashion, we can work with the structure of the network itself, rather than the precise amount or nature of the entities traversing the network. Tools and techniques are developed which enable analysts to uncover substantial numbers of determinant pairs of variables in a network, and these tools are applied in the special cases of "transportation" and "series-parallel" networks to give a complete description of determinant pairs. The strong series-parallel networks are, in addition, a wide class of networks for which all pairs of variables are determinant, indicating the high degree of organization these networks possess.

Determinacy and its application to networks, then, may uncover new clues to organizational behavior and management of large-scale systems or models. It can be used in constructing well-formulated models as well as evaluating determinancy in existing ones. Beyond this, though, network determinancy can provide a measure of organization in systems with interrelated factors, the degree of organization being a factor of the amount of determinancy in the system. Thus, it transcends specific input-output models and becomes important to more general descriptions and scenarios, and we hope may be useful in policy decisions and general organization outlook.

II. INPUT-OUTPUT MODELS AND QUALITATIVE DETERMINACY

In order to make a precise and general study of qualitative determinacy it is necessary first to define the type of model with which we are dealing.

Definition 2.1: An input-output model $M = M(R_X, R_Y, f)$ consists of a real vector $X = (X_1, \ldots, X_n)$ of inputs chosen from an input domain R_X, and a real vector $Y = (Y_1, \ldots, Y_m)$ of outputs chosen from an output domain R_Y which are related by the functional equation

$$Y = f(X). \tag{2.1}$$

M is called a differentiable model if f has continuous first-order partial derivatives. For a working model, X may include parameters, intermediate variables, and other factors as well as input, and Y may represent intermediate processes or terminal states in addition to the outputs. Note that it may not be true--and usually it is not--that $f(R_X)$ is contained in R_Y. This is one of the interesting properties of the model. In a particular energy system, for example, R_X might represent the set of available resources, the transportation and conversion infrastructure, and the allocation decision rules; f then translates these into final energy services. These services, however, may not satisfy specific energy service demands $[f(R_X) \subseteq R_Y]$. We do assume, for the sake of subsequent definitions, that $f(R_X)$ contains R_Y, i.e., that any set of demands can be met by the allocation of available resources. This may necessitate a restriction of the demand space, and the reader should assume for the sake of the definition that any stated demand set R_Y is actually restricted to $R_Y \cap f(R_X)$. We make the further technical assumption that the function f is defined (and differentiable when appropriate) on an open neighborhood containing R_X.

We now give three examples of input-output models for illustration and for later use.

<u>Example 2.1: A linear input-output model</u> - This is the basic model structure of interest in this paper. It models any activity where each output is a linear function of the inputs. Thus if $X \in R_X = \mathbb{R}^n$ consists of n real input factors and $Y \in R_Y = \mathbb{R}^m$ consists of m real output factors, then X and Y are related by the functional equation

$$Y = f(X) = AX, \tag{2.2}$$

where A is an $m \times n$ matrix. This is clearly also a differentiable model with the partial derivatives $\partial Y_i/\partial X_j$ equal to a_{ij}. It will also follow that examples 2.2 and 2.3 below are also differentiable.

<u>Example 2.2: A real activity linear input-output model</u> - Here in addition to the linear property indicated in Example 2.1 we require that all input and output consist of real activity--that is, the input and output are non-negative. Thus the functional Eq. (2.2) is also the equation for this model, but now

$$R_X = \mathbb{R}^n_+ \quad \{X \in \mathbb{R}^n | X_i \geq 0, \quad i = 1,\ldots,n\}$$

and

$$R_Y = \mathbb{R}^m_+ .$$

A special case of this model, called the <u>supply-demand model</u>, is studied in Section III. The reader is encouraged to refer to this model for illustration of the concepts in this section.

<u>Example 2.3: A fixed-demand model</u> - We can make a further restriction on Example 2.2 by choosing R_Y to be a single point, $R_Y = \{b = (b_1,\ldots,b_m)\}$ (with $R_X = \mathbb{R}^n_+$ as above). This corresponds to the standard linear programming model, for now the set of values of X which realize the input-output model are precisely those which satisfy the linear programming constraints

$$Ax = b$$
$$x \geq 0 .$$

Notice that no further criteria are imposed other than boundedness of the solution.

<u>Example 2.4: A linear programming model</u> - We present this model because it illustrates the flexibility of usage for the input and output vectors. Here we

assume a real fixed-demand submodel ($Ax = b$, $R_X = R_+^n$, $R_Y = \{b\}$, $b \in r_+^m$), but now we alter the "inputs," or decision parameters (X), so as to optimize some standard ($c \cdot X$) of performance for the system. (It might be, for example, that $c \cdot X$ minimizes system costs or maximizes the value to society of the energy system.) This is the classical linear programming sensitivity analysis model, where now inputs correspond to demands b as well as the optimization measure parameter c; outputs correspond to the optimal operating state as well as the optimal performance measure. Specifically, the inputs $X = (b,c) \in \mathbb{R}_+^m \times \mathbb{R}^n$ consist of the vector b of demands and the m-vector c representing the objective function. The output $Y = (x^*,d) \in \mathbb{R}_+^n \times \mathbb{R}$ consists of the optimal state x^* and its optimal objective function value d. The functional equation

$$f(b,c) = (x^*,d)$$

is defined by

$$d = cx^* = \max\{cx \mid Ax = b, \quad x \geq 0\}.$$

The function f in this example is thus, loosely speaking, an inverse of the function used in the previous three examples. Note that an input $X = (b,c)$, in order to be a factor in the functional equation, must have the region

$$Ax = b, \quad x \geq 0 \tag{2.3}$$

feasible and the functional cx bounded over (2.3). The formulation given above describes the most general sensitivity model. If one wishes to test sensitivity of specific costs or demands, one can simply restrict R_X to include just those parameters, letting the other elements be constants. It is important to note that this model is <u>not</u> differentiable, because jumps may occur for small changes in operational parameters.

Examples 2.1 to 2.4 are illustrated in Fig. 1. We now define the concept of configuration outlined in the Introduction.

<u>Definition 2.2</u>: An <u>output-realizable configuration</u>, or just <u>configuration</u> for a model $M = M(X,Y,f)$ is a partition (X_B,X_N) of the variables of X so that for <u>any</u> choice $y \in R_Y$ for Y there exists a <u>unique</u> choice x_B for X_B so that $(x_B,0) \in R_X$ and $y = f(x_B,0)$, that is, the function of $f_B(X_B) = f(X_B,0)$ is invertible on the domain subset $R_X \cap f_B^{-1}(R_Y) = \{x_B \mid (x_B,0) \in R_X,\ f_B(x_B) \in R_Y\}$. We will identify the configuration by $Z = (Z_B,Z_N)$, where $Z_B = X_B$ and $Z_N = X_N \cup Y$.

Thus Z_N corresponds to a maximal set of allowable "fixed" variables. Specifically, any desired operation of the model--as measured by output performance--can be obtained by the minimal set Z_B of operating input variables. It will be assumed

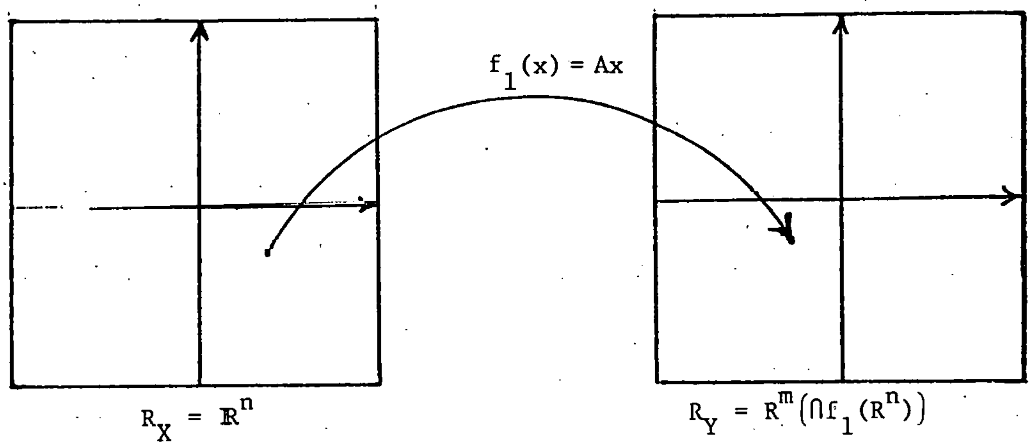


Figure 1a. Example 2.1 illustrated.

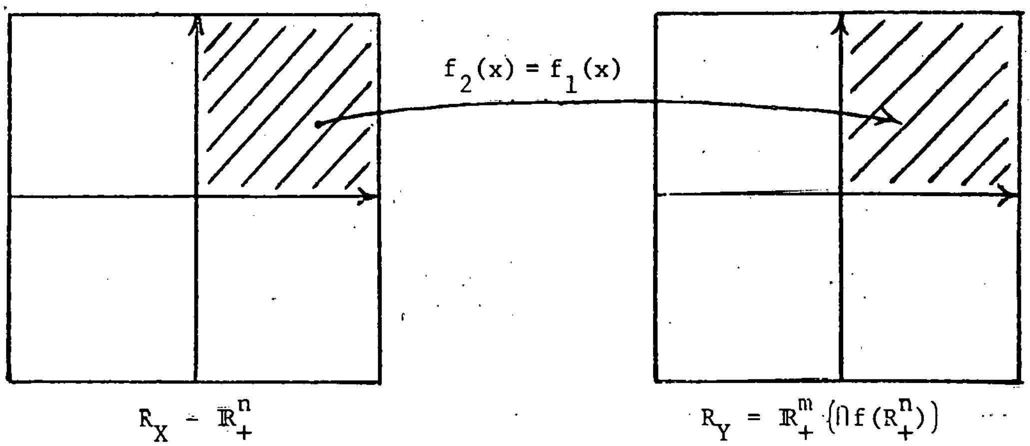


Figure 1b. Example 2.2 illustrated.

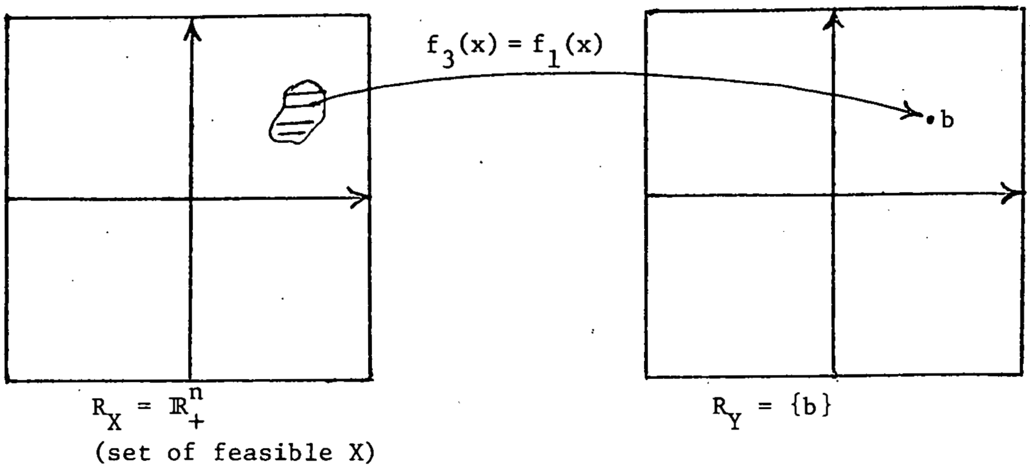


Figure 1c. Example 2.3 illustrated.

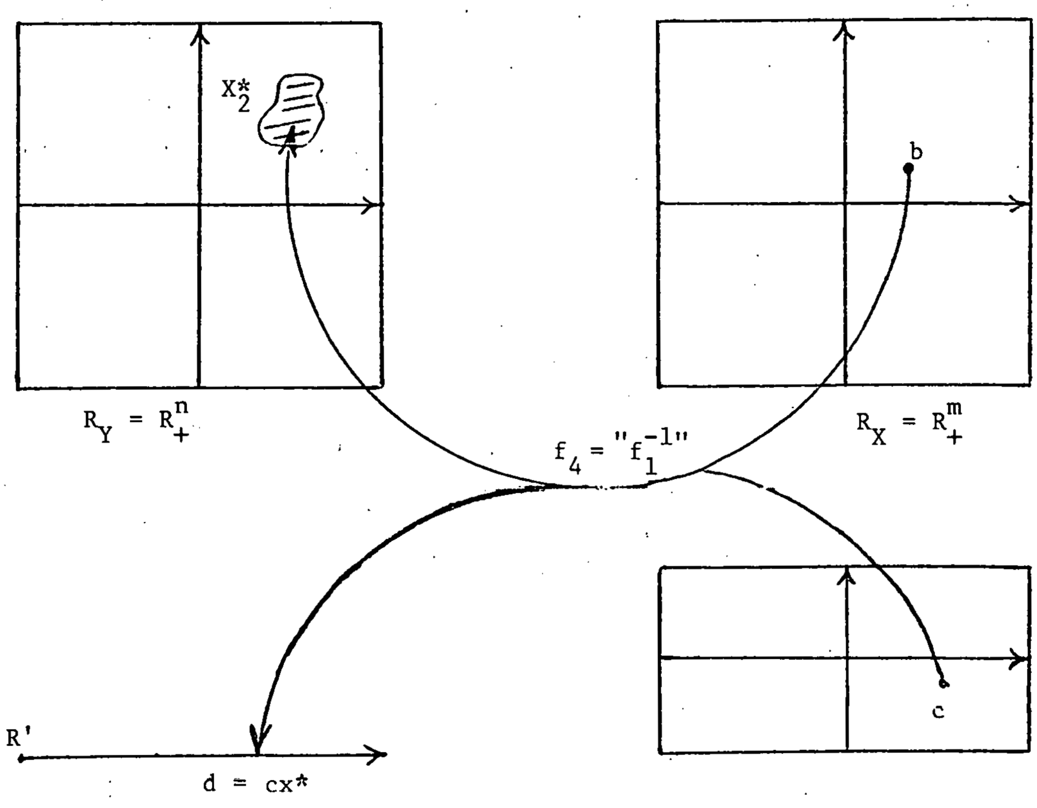


Figure 1d. Example 2.4 illustrated.

throughout the paper that every model has at least one configuration.

It is a simple matter to characterize the configurations in linear, real activity linear, and fixed-demand models (Examples 2.1 to 2.3). For the linear model we find that (Z_B, Z_N) is a configuration if and only if for every choice of values for Y there exists a unique value of X_B for which

$$A_B X_B = Y, \tag{2.4}$$

where A_B is the submatrix of A consisting of the columns of A corresponding to X_B. Thus the configurations correspond to the sets of m X variables whose columns in A are linearly independent. For the real activity linear model we insist that for every choice of <u>non-negative</u> values for Y there exists a unique non-negative value of X_B satisfying (2.4). Hence, in addition to solving (2.4) uniquely, the value $X_B = A_B^{-1} Y$ must be non-negative for each non-negative Y. The configurations, then, correspond to the sets of m X variables for which the corresponding matrix of columns has a non-negative inverse (known as a <u>strongly monotone</u> submatrix). For fixed-demand models we need only that $A_B X_B = b$ can be satisfied by a unique choice of $X_B \in \mathbb{R}_+^n$. Thus the configurations correspond to those sets of variables X_B for which A_B is invertible and $A_B^{-1} b$ is non-negative. These are precisely the feasible bases of a linear program, and the corresponding values of $X = (X_B, X_N)$ comprise the basic feasible solutions for the system

$$Ax = b$$
$$x \geq 0.$$

Configurations are the central construct used to observe variable relationships. In particular, we will be studying the effect of bringing an inoperative (non-basic) variable into a configuration. For this effect to be well defined, we need to make a technical assumption on the model.

<u>Definition 2.3</u>: A model $M = M(R_X, R_Y, f)$ is called <u>non-degenerate</u> if, for every configuration $Z = (Z_B, Z_N)$, there is an $\varepsilon > 0$ so that for each set of values $y \in R_Y$ for Y and each set of values x_N for X_N with $\|x_N\| < \varepsilon$ (the standard norm), there is a unique set of values $g(x_N, y)$ for X_B so that

$$y = f[g(x_N, y), x_N].$$

This is a strengthened form of Definition 2.2 in that we insist that the function $f(X_B, X_N)$, as a function of X_B, is invertible for X_N in a <u>neighborhood</u> of zero, rather than simply $X_N = 0$. Note that the definition of non-degeneracy does not depend on the input domain R_X. It merely avoids "kinks" or jumps in f about the values $X_N = 0$, thus allowing us to measure in a well-defined way the effect of bringing a non-basic variable into operation in a configuration. Any linear model is non-degenerate, and in a differentiable model one can show that a sufficient condition for non-degeneracy is that the function f_B defined in Definition 2.2 has its matrix of partial derivatives (with respect to X_B) both square and non-singular. Most other models can be made non-degenerate by perturbing f, R_X, or R_Y slightly--a reasonable task in view of the estimates usually built into the model.

It is easy to see that for a particular configuration of a non-degenerate model, the effect of a (small) increase in any of the non-basic variables (including the y variables) on the basic variables can be determined exactly through the function g of Definition 2.3. We are concerned specifically with the rate of change of one variable with respect to another. For this type of measure it is simplest to assume the model to be differentiable, although a more elaborate definition could be stated which covers the general non-degenerate case.

Definition 2.4: Let M be a non-degenerate differentiable model, $Z = (Z_B, Z_N)$ a configuration of M, Z_i in Z_N, and $y \in R_Y$ a set of values for Y. Then the span of Z_i with respect to Z and y is denoted

$$\alpha(Z, Z_i, y) = \frac{\partial}{\partial Z_i} g(0, y)$$

[where the right hand side is the vector of partial derivatives of g with respect to Z_i evaluated at $(Z_N, Y) = (0, y)$], and the effect of Z_i on any variable Z_j in Z_B is denoted by $\alpha_j(Z, Z_i, y)$ = the component of $\alpha(Z, Z_i, y)$ corresponding to Z_j. For Z_i in Z_B we set $\alpha_j(Z, Z_i, y)$ equal to 0 for $i \neq j$ and $\alpha_i(Z, Z_i, y) = -1$.

Note: The extension of α to Z_i in Z_B reflects the fact that

$$Z_B = X_B = g(X_N, Y)$$

has the symmetric form

$$0 = -X_B + g(X_N, Y) = \bar{g}_B(X, Y)$$

and further

$$\alpha(Z, Z_i, y) = \frac{\partial}{\partial Z_i} \bar{g}_B(0, y).$$

The significance of this extension will be further clarified as we proceed.

For the linear models the α vector is particularly easy to calculate. If the configuration Z is given, X_B and X_N the corresponding partition of X, and A is partitioned into A_B and A_N corresponding to the columns of X_B and X_N, then we can write Eq. (2.2) as

$$Y = A_B X_B + A_N X_N ,$$

or, solving for X_B,

$$X_B = A_B^{-1} Y - A_B^{-1} A_N X_N .$$

Thus for each Z_i in Z_B, $\alpha(Z,Z_i,y)$ is independent of y and corresponds to the column $A_B^{-1}(-I,A)$ corresponding to the variable Z_i (in linear programming, the pivot column of Z_i). For linear models, then, we will drop the y arguments, and denote the span simply by $\alpha(Z,Z_i)$.

We are finally able to make the main definition of the paper, namely, that of qualitative determinacy between variables in a model. This will be a measure of the consistency of interaction between the variables and is defined in terms of the span vector α. By using the term "qualitative," we emphasize that we are concerned solely with the sign of α rather than its magnitude; that is, we wish only to know whether the variables increase activity, decrease activity, or are unaffected by a given change in model operation. There is considerable historical basis for this; see, for example ref. [5], pp. 23-28. "Determinacy" will measure the effect of one variable on another or the mutual effect of a pair of variables on the other variables of the model.

Definition 2.5: Let M be a differentiable model and $Z = (Z_B, Z_N)$ a configuration of M. For variables Z_i and Z_j in M, we call Z_i and Z_j Z-qualitative substitutes (complements, independents) in Z if $\alpha_K(Z,Z_i,y) \cdot \alpha_K(Z,Z_j,y)$ is non-negative (non-positive, zero) for all Z_k in Z_B and all $Y \in R_Y$. Z_i and Z_j are called qualitative substitutes (complements, independents) if they are Z-strong qualitative substitutes (complements, independents) for every configuration Z. Z_i and Z_j will be called (Z-)determinant if they are either (Z-)substitutes, (Z-)complements, or (Z-)independents.

Note that if both variables are in Z_B, then $\alpha(Z,Z_i,y) \cdot \alpha(Z,Z_y,y)$ is 0 if $i \neq j$ and -1 if $i = j$; if exactly one variable, say Z_j, is in Z_B, then

$$\alpha_k(Z,Z_i,y) \cdot \alpha_k(Z,Z_j,y) = \begin{cases} 0 & k \neq j \\ -\alpha_j(Z,Z_i,y) & k = j. \end{cases}$$

Thus two variables Z_i and Z_j are substitutes if, for every configuration, Z_i tends to "be a substitute" for Z_j, in that it will either tend to replace Z_j (if one of the variables is basic) or cause the same behavior as Z_i on each variable Z_k (if both are in Z_N). Similarly, Z_i and Z_j are complements if Z_i tends to "complement" Z_j in that the variables tend to vary similarly with respect to each other or have opposite effects on the other variables. Independence indicates no interaction or mutual action on other variables. One could isolate the two types of interactions, with weak determinacy concerning only those configurations with one variable in Z_B, and strong determinacy concerning those with both variables in Z_N (variables in Z_B are always mutually independent). This lends unnecessary complication, however, and we will mention it only when it can be done without difficulties.

It follows from the definition that two variables are (Z-)independents if and only if they are both (Z-)substitutes and (Z-)complements. Further, a variable is always a substitute for itself, and both substitution and complementarity are symmetric properties. It is not true, as we will show later, that there is a general form of transitivity of either substitution or complementarity across pairs.

For the linear model, as developed thus far, recall that the α vector is independent of Y and Z_N and simply corresponds to the appropriate column $A_B^{-1}(-I,A)$, where (A_B,A_N) is the partition of A corresponding to (Z_B,Z_N). To test the determinacy of Z_i and Z_j it is necessary to check the $(Z_i)^{th}$ and $(Z_j)^{th}$ columns of $A_B^{-1}(-I,A)$ for each configuration. If the columns always match term for term in sign whenever they are both non-zero, then Z_i and Z_j are substitutes. If the terms are opposite in sign, the two variables are complements, and if one term is always zero, the variables are independent.

Thus we could give a complete list of determinant pairs by checking every configuration of the model in the above fashion. This is a substantial amount of work, however, since the number of configurations is generally an exponential function of the number of variables. We investigate in Section III a special class of linear models called the supply-demand models, and attempt to find more tractable techniques for discovering determinant pairs.

III. THE SUPPLY-DEMAND MODEL

We present in this section an important class of linear models which represent the natural flow of goods through a directed network. We can then restate the definitions given in Section II in the context of properties of the network itself. This not only gives us a more realistic sense of determinacy, but provides techniques for discovering determinism in this type of model.

The Model

Let G = (N,A) be a direct network, defined by node set N and arc set A. Denote elements in A by (u,v), the edge directed from u to v, where u and v are elements of N (we allow multiple arcs). Specifying one element r of N as the supply node, we can define the supply-demand model M(G,r) to be the real-activity linear model with input variables

$$X = \{t(u,v) \mid (u,v) \in A\} \in R_X = \mathbb{R}_+^A, \tag{3.1}$$

output variables

$$Y = \{y(u) \mid u \in N - \{r\}\} \in R_Y = \mathbb{R}_+^{N-\{r\}}, \tag{3.2}$$

and functional equation $Y = f_G(X)$ defined by components

$$y(u) = \sum_{(x,u)\in A} t(x,u) - \sum_{(u,x)\in A} t(u,x), \qquad u \in N-\{r\}. \tag{3.3}$$

Physically, this model describes the process whereby goods are produced at a single supply point r and are shipped through the network G with t(u,v) denoting the flow from u to v. A residual amount y(u) of the material is left at the non-supply node u. The residual at each node u is defined by (3.3), and (3.1) and (3.2) indicate the fact that the shipped and residual amounts of the material must always be non-negative. The quantity y(u) could represent, for example, efficiency losses at a node u if node u represents a conversion process or demands for energy services at the end of the network.

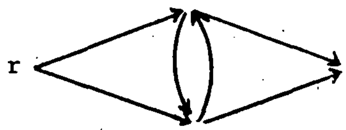
Configurations

We now define the fundamental concepts of determinacy in terms of properties of the network G. For ease of notation, we will often identify an arc interchangeably with the variable associated with that arc, and identify a node interchangeably with the variable associated with that node, where there is no confusion. Also for a given set of values t for X and set $C \subseteq A$ we denote t_C to be the values on the set C. We begin by defining the construct in a graph which corresponds to configurations in the corresponding supply-demand model. For nodes a and b in G, a <u>directed path</u> from a to b will be a <u>set of edges</u> of the form

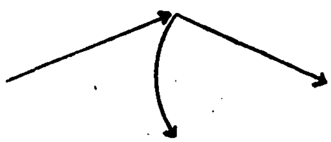
$$\{(a,u_1),(u_1,u_2),\ldots,(u_{n-1},u_n),(u_n,b)\}.$$

<u>Definition 3.1</u>: Let $G=(N,A)$ be a directed graph, and r a node in G. Then an <u>r-rooted spanning tree</u>, or simply <u>r-tree</u>, of G is any set of edges T of G with the property that for each node v of G there is a unique directed path in T from r to v.

<u>Examples</u>: For a graph



we have



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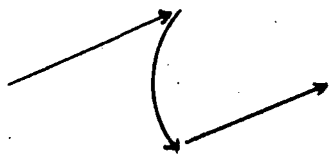
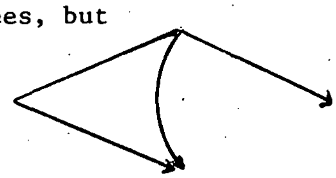


are r-trees, but



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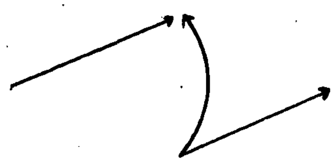


are not.

It is easy to see that an equivalent definition of an r-tree is any set of edges which contains no directed cycle (edge set of the form $\{(u_1,u_2)(u_2,u_3)\ldots,(u_n,u_1)\}$) and for which every node except r has exactly one edge pointing into that node.

<u>Proposition 3.2</u>: Let $M=M(G,r)$ be a supply-demand model and T a subset of the arc set. (Recall that f is defined by (3.1) to (3.3) above and that $X=(t_T,0)$ represents a partition of flows for which variables not in T are set to zero. See definition 2.2.) Then the following are equivalent:

(1) For some positive set of values y for Y there is a unique set of positive values t_T for T such that

$$f_G(t_T,0) = y.$$

That is, for some set of positive residuals there is a unique set of flows on the arc set T which yields these positive residuals.

(2) For each non-negative set of values y for Y there is a unique set of non-negative values t_T for T such that

$$f_G(t_T,0) = y.$$

(3) T forms an r-tree of G.

Proof: We prove (2) ⇒ (1) ⇒ (3) ⇒ (2).

(2) ⇒ (1): It is sufficient to prove that, for each set of positive values y for Y, the set of values t_T defined in (2) is positive, since the uniqueness follows. Suppose then that the t_T defined above has $t(w,z) = 0$ for some (w,z) in T. Now consider the set of values y' for Y defined

$$y'(u) = \begin{cases} y(u) & u \neq z \\ 0 & u = z \,. \end{cases}$$

Then y' is non-negative, and so by (2) there is a non-negative set of values t'_T for T with

$$f_G(t'_T,0) = y'.$$

But if we define the set of values t''_T for T by

$$t''_T(u,v) = \begin{cases} t'(u,v) & (u,v) \neq (w,z) \\ t'(w,z) + y(z) & (u,v) = (w,z) \,, \end{cases}$$

then it follows from (3.3) that

$$f_G(t''_T,0) = y \,,$$

where $t''(w,z) > 0$. Therefore t_T is not unique, contradicting the fact that such a value exists.

(1) ⇒ (3): Suppose T satisfies 1), and let $s \neq r$ be a node in G. First, suppose there is no path from r to s in T. Define S to be that set of nodes for which there is a directed path to s. Then $s \in S$, $r \notin S$, and no edge of T goes from N-S to S. Therefore, by adding together Eqs. (3.1) for every $u \in S$, we get

$$\begin{aligned} 0 < y(s) &\leq \sum_{u \in S} y(u) \\ &= \sum\{-t(u,v) \mid (u,v) \in A,\ u \in S,\ v \in N-s\} \\ &\leq 0 \,, \end{aligned}$$

a contradiction. Second, suppose that there are two paths

$$\Gamma : \ (u_0,u_1),\ldots,(u_{k-1},u_k)$$

$$\Gamma' : \ (u'_0,u'_1),\ldots,(u'_{\ell-1},u'_\ell)$$

from $u_0 = u'_0 = s$ to $u_k = u'_\ell = r$ in T. Set $\varepsilon = \min\{t(u_i,u_{i+1}) \mid i=0,\ldots,k-1\}$ and now define t'_T by

$$t'(u,v) = \begin{cases} t(u,v) & (u,v) \in \Gamma \cup \Gamma' \\ t(u,v) - \varepsilon/2 & (u,v) \in \Gamma \\ t(u,v) + \varepsilon/2 & (u,v) \in \Gamma'. \end{cases}$$

Then $t'_T > 0$, $f_G(t') = y$, and since $\Gamma \neq \Gamma'$, $t' \neq t$, contradicting the fact that t is unique, and that two such paths exist.

(3) ⇒ (2) Suppose T is an r-tree. Let $y = [y(u)]$ be a set of non-negative values for Y, and (w,v) an edge of T. Let U be a set of nodes of G which can be reached from v by a directed path of edges in T. Since T is an r-tree, the only paths from r to nodes in U must go through the edge (w,v); in particular, (w,v) is the only edge in T whose head is in U and whose tail is in V-U Thus, by summing Eq. (3.1) over $u \in U$, we get

$$\sum_{u\in U} y(u) = \sum_{u\in U}\left(\sum_{(x,u)\in T} t(x,u) - \sum_{(u,x)\in T} t(x,u)\right)$$

$$= t(u,v)$$

since $t(x,u) = 0$ when $(x,u) \notin T$, and every arc except (u,v) which appears in the sum occurs once with each sign. Thus $t(u,v) = \sum_{u\in U} y(u) \geq 0$ is the unique value for t(u,v).

Corollary 3.3: Let $M = M(G,r)$ be a supply-demand model, and $Z = (Z_B, Z_N)$ a partition of the variables of M. Then Z is a configuration for M if and only if Z_B comprise edges forming an r-tree in G.

Proposition 3.2 points out the fact peculiar to supply-demand models that the configurations found with respect to any particular positive residual value are in fact all of the configurations for the model. Thus, the set of configurations characterized above is the same even when the model is taken to be a fixed-demand model (Example 2.3) with demand $b = [y_1(u),\ldots,y_m(u)]$ positive.

Determinacy

Using Proposition 3.1 we are in a position to calculate, for any configuration, the effect on the basic variables of a change in one of the non-basic variables. For configuration $Z = (Z_B, Z_N)$ we know that the edges of Z_B form an r-tree of G. Thus for any node u in G there is a directed path $\gamma(Z_B, u)$ from r to u of edges in Z_B. For variable Z_i of M we define $\phi(Z_B, Z_i)$--the forward edges of Z_i with respect to Z_B--and $\beta(Z_B, Z_i)$--the backward edges of Z_i with respect to Z_B--as follows: if $Z_i = t(u,v)$, then $\phi(Z_N, Z_i) = \gamma(Z_B, u)$ and $\beta(Z_B, Z_i) = \gamma(Z_B, v)$; if $Z_i = y(v)$, then $\phi(Z_B, Z_i) = \gamma(Z_B, v)$ and $\beta(Z_B, Z_i) = \emptyset$. We can state the following lemma:

Lemma 3.4: Let $M = M(G,r)$ be a supply demand model, $Z = (Z_B, Z_N)$ a configuration for M, and Z_i a variable in M. Then

$$\alpha_j(Z_B, Z_i) = \begin{cases} +1 & \text{if } Z_j \in S^+(Z_B, Z_i) \\ -1 & \text{if } Z_j \in S^-(Z_B, Z_i) \\ 0 & \text{otherwise,} \end{cases}$$

where

$S^+ \equiv \phi(Z_B, Z_i) - \beta(Z_B, Z_i)$ (i.e., the residual forward flow)

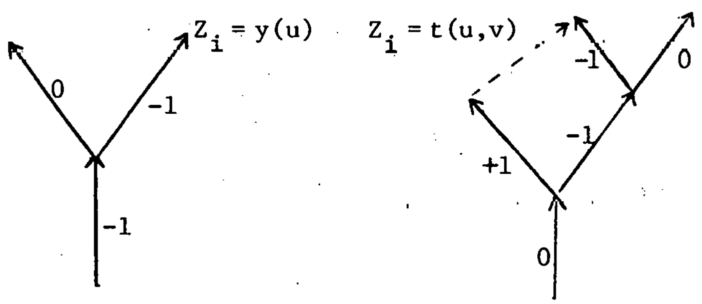
$S^- \equiv \beta(Z_B, Z_i) - \phi(Z_B, Z_i)$ (i.e., the residual backward flow).

Recall that the third parameter, y, of α has been dropped for linear models.

Proof: Consider the effect of a unit increase of Z_i on the variables of Z_B. This will increase the values of $\phi(Z_B, Z_i)$ one unit and decrease the values in $\beta(Z_B, Z_i)$ one unit, and the net effect on elements of $\phi(Z_B, Z_i) \cap \beta(Z_B, Z_i)$ is zero. The lemma follows.

Graphically, we can describe $\alpha(Z_B, Z_i)$ as follows. If $Z_i = y(u)$ then $\alpha(Z_B, Z_i)$ is -1 on the edges of the unique path from r to u, and 0 on the other edges of Z_B--that is, $S^+(Z_B, Z_i)$ comprises the entire path from r to u, and $S^-(Z_B, Z_i)$ is empty. If $Z_i = t(u,v)$, we know that (u,v) forms a unique undirected circuit C with Z_B; and $\alpha(Z_B, Z_i)$ is then +1 on the edges of C facing the same direction as (u,v) on the circuit, i.e., $S^+(Z_B, Z_i)$; -1 on the edges facing opposite (u,v) on the circuit, i.e., $S^-(Z_B, Z_i)$; and 0 on all other edges of Z_B.

Examples:



We remark that if Z_i is in Z_B (and therefore an edge) this definition is still consistent with that of Definition 2.4, since then $S^-(Z_B,Z_i) = \{Z_i\}$ and $S^+(Z_B,Z_i) = \emptyset$.

Lemma 3.5: Let $Z = (Z_B,Z_N)$ be a configuration for the supply-demand model $M(G,r)$, and Z_i and Z_j two variables in M. Then Z_i and Z_j are Z-substitutes if and only if

(i) $S^+(Z_B,Z_i) \cap S^-(Z_B,Z_j) = \emptyset$

and

(ii) $S^-(Z_B,Z_i) \cap S^+(Z_B,Z_j) = \emptyset$.

That is, the residual forward flow in Z_i does not intersect the residual flow in Z_j and vice versa. They are complements if and only if

(iii) $S^+(Z_B,Z_i) \cap S^+(Z_B,Z_j) = \emptyset$

and

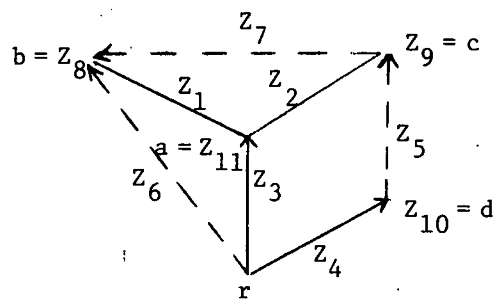
(iv) $S^-(Z_B,Z_i) \cap S^-(Z_B,Z_j) = \emptyset$.

That is, the residual forward flows in Z_i and Z_j do not intersect. Similarly for the backward flows, they are independents if and only if

(v) $[S^+(Z_B,Z_i) \cap S^-(Z_B,Z_i)] \cap [S^+(Z_B,Z_j) \cap S^-(Z_B,Z_j)] = \emptyset$.

Proof: Follows immediately from Lemma 2 and Definition 2.5.

We illustrate with an example:



$Z_B = \{Z_1,Z_2,Z_3,Z_4\}$ and $Z_N = \{Z_5,Z_6,Z_7,b,c,d\}$. Then

$$\beta(Z_B,Z_2) = \beta(Z_B,Z_5) = \phi(Z_B,Z_7) = \phi(Z_B,c) = \{Z_2,Z_3\}.$$

For example, increase in flow along any of the edges into node c (Z_2 or Z_5) results in a potential decrease in the flow from the source r to c through the spanning tree--in this case through the edges Z_2 and Z_3. On the other hand, an increase of flow along any of the edges out of c (Z_7) or of the demand at c(Z_9) results in a potential increase in flow along spanning tree path.

$\beta(Z_B,Z_1) = \beta(Z_B,Z_6) = \beta(Z_B,Z_7) = \phi(Z_B,b) = \{Z_1,Z_3\}$

$\beta(Z_B,Z_4) = \phi(Z_B,Z_5) = \phi(Z_B,d) = \{Z_4\}$

$\beta(Z_B,Z_3) = \phi(Z_B,a) = \{Z_3\}$

$\phi(Z_B,Z_6) = \beta(Z_B,b) = \beta(Z_B,c) = \beta(Z_B,d) = \beta(Z_B,a) = \emptyset$.

The S^+, S^- and α's* are calculated as follows:

i	$S^+(Z_B,Z_i)$	$S^-(Z_B,Z_i)$	$\alpha_1(Z_B,Z_i)$	$\alpha_2(Z_B,Z_i)$	$\alpha_2(Z_B,Z_i)$	$\alpha_4(Z_B,Z_i)$
1	$\{Z_3\}$	$\{Z_1,Z_3\}$	-1	0	0	0
2	$\{Z_3\}$	$\{Z_1,Z_3\}$	0	-1	0	0
3	$\emptyset$	$\{Z_3\}$	0	0	-1	0
4	$\emptyset$	$\{Z_4\}$	0	0	0	-1
5	$\{Z_4\}$	$\{Z_2,Z_3\}$	0	-1	-1	+1
6	$\emptyset$	$\{Z_1,Z_3\}$	-1	0	-1	0
7	$\{Z_2,Z_3\}$	$\{Z_1,Z_3\}$	-1	+1	0	0
8	$\{Z_1,Z_3\}$	$\emptyset$	+1	0	+1	0
9	$\{Z_2,Z_3\}$	$\emptyset$	0	+1	+1	0
10	$\{Z_4\}$	$\emptyset$	0	0	0	+1
11	$\{Z_3\}$	$\emptyset$	0	0	+1	0

*Note that the α's are only defined for the basic variables Z_1, Z_2, Z_3, Z_4.
The determinacies are shown in the following table.

	Strict Substitutes	Strict Complements	Independents
Z_1	Z_1,Z_6,Z_7	Z_8	$Z_2,Z_3,Z_4,Z_5,Z_9,Z_{10},Z_{11}$
Z_2	Z_2,Z_5	Z_7,Z_9	$Z_1,Z_3,Z_4,Z_6,Z_8,Z_{10},Z_{11}$
Z_3	Z_3,Z_5,Z_6	Z_8,Z_9,Z_{11}	Z_1,Z_2,Z_4,Z_7,Z_{10}
Z_4	Z_4	Z_5,Z_{10}	$Z_1,Z_2,Z_3,Z_6,Z_7,Z_8,Z_9,Z_{11}$
Z_5	Z_2,Z_3,Z_5,Z_6,Z_{10}	Z_4,Z_7,Z_8,Z_9,Z_{11}	Z_1
Z_6	Z_1,Z_3,Z_5,Z_6,Z_7	Z_8,Z_9,Z_{11}	Z_2,Z_4,Z_{10}
Z_7	Z_1,Z_6,Z_7,Z_9	Z_2,Z_5,Z_8	Z_3,Z_4,Z_{10},Z_{11}
Z_8	Z_8,Z_9,Z_{11}	Z_1,Z_3,Z_5,Z_6,Z_7	Z_2,Z_4,Z_{10}
Z_9	Z_7,Z_8,Z_9,Z_{11}	Z_2,Z_3,Z_5,Z_6	Z_1,Z_4,Z_{10}
Z_{10}	Z_5,Z_{10}	Z_4	$Z_1,Z_2,Z_3,Z_6,Z_7,Z_8,Z_9,Z_{11}$
Z_{11}	Z_8,Z_9,Z_{11}	Z_3,Z_5,Z_5	Z_1,Z_2,Z_4,Z_7,Z_{10}

where strict substitutes (complements) are substitutes (complements) which are not independent. This table also shows the nontransitivity of determinacy. Although both of the pairs (Z_5,Z_6) and (Z_6,Z_7) are Z-substitutes (and, in fact, substitutes in every configuration), Z_5 and Z_7 are not Z-substitutes. The table also suggests that in any given configuration, every pair of variables is determinant. This is shown by Proposition 3.6 which implies that every pair of variables in a given configuration of a supply-demand model have qualitatively predictable effects--either complements or substitutes.

Proposition 3.6: Let $Z = (Z_B,Z_N)$ be a configuration for the supply-demand model M and Z_i a variable in M. Then every variable of M is Z-determinant with Z_i.

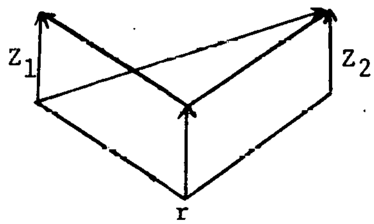
Proof: Choose variables Z_j in M. We must prove that at least one of the pairs of states (i) and (ii) or (iii) and (iv) of Lemma 3.5 hold for Z_i, Z_j, and Z. Suppose on the contrary that (i) and (iii) are both violated. (The other cases are symmetric.) Then there must be an edge (u_1,v_1) in $S^+(Z_B,Z_i) \cap S^-(Z_N,Z_j)$ and an edge (u_2,v_2) in $S^+(Z_B,Z_i) \cap S^+(Z_B,Z_j)$. If (u_1,v_1) is further away from r than (u_2,v_2) on $\phi(Z_B,Z_i)$, then there are two paths from r to u_1--one going up $S^-(Z_B,Z_j)$ and one going up $S^+(Z_B,Z_j)$ to (u_2,v_2) and then up $\phi(Z_B,Z_i)$. Similarly, if (u_2,v_2) is further away from r than (u_1,v_1), then there are two paths from r to u_2. In either case we have a contradiction, and so such a situation cannot exist. This proves the lemma.

Although we will not show it here, the results of Proposition 3.6 hold for a much broader class of configurations, namely, the configurations of any linear model where the underlying matrix is totally unimodular, (see ref. [7]). A matrix is totally unimodular if every square submatrix of size $m \geq 1$ has determinant ± 1 or 0. This includes more general network models, as well as certain models involving matching and assignment problems.

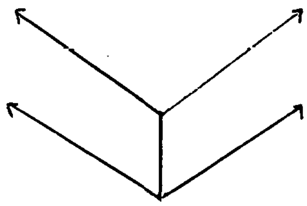
Proposition 3.6 has an important corollary. In the economic literature, the economic correlation between two (domain) variables in a linear function can be defined as the dot product of their column vectors in the corresponding matrix [8]. Since the relationship between basic and non-basic variables in any configuration of a linear model is a linear function, we can talk about the correlation between non-basic variables in a configuration. It is clear from the definition that substitutes (complements) in a configuration are positively (negatively) correlated with respect to that configuration. In the particular case of the supply-demand model, the i-th column of the matrix associated with the configuration $Z = (Z_B, Z_N)$ consists of the elements $\alpha_j(Z_B, Z_i)$ for Z_j in Z_B. Proposition 3.6 asserts that no two terms in the correlation dot product can have opposite signs. Corollary 3.7 follows immediately.

Corollary 3.7: Let $Z = (Z_B, Z_N)$ be a configuration of the supply-demand model, and Z_i and Z_j be two variables of M. Then Z_i and Z_j are Z-substitutes (Z-complements) if and only if their correlation (with respect to Z) is non-negative (non-positive).

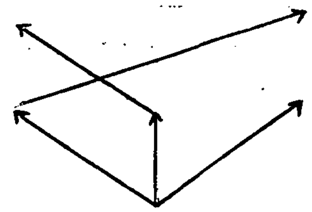
Now if there were only one configuration, then all variables would be determinant. Unfortunately, variables are not necessarily determinant between configurations, as the following example shows.



Z_1 is a substitute for Z_2 in the configuration



but is a complement in the configuration



We have, however, the following general result, which concerns "local" determinancy in networks.

Proposition 3.8: Let M = M(G,r) be a supply-demand model. Then the following variables are determinant.

(1) Variable pairs of the form

(i) [t(u,v),t(w,v)]: flows to the same node,

(ii) [t(u,v),t(u,w)]: flows from the same node,

(iii) [y(u),t(u,v)] : residuals and flows from same node,

(iv) [y(u),y(v)] : any two residuals

are substitutes;

(2) variable pairs of the form

(v) [t(u,v),t(v,w)]: flow in and out from same node,

(vi) [y(v),t(u,v)] : flow in and residual flow out same node,

(vii) [y(v),t(r,u)] : "demands" and "supplies"

are complements.

Proof: The key observation here is that for each variable pair (Z_i,Z_j) listed in the theorem one of the following cases holds for every configuration (Z_B,Z_N):

(a) $\phi(Z_B,Z_i) = \phi(Z_B,Z_j)$

(b) $\beta(Z_B,Z_i) = \beta(Z_B,Z_j)$

(c) $\phi(Z_B,Z_i) = \beta(Z_B,Z_j)$

(d) $\beta(Z_B,Z_i) = \phi(Z_B,Z_j)$.

Applying Lemma 3.4 we get in cases (a) and (b) corresponding to (1) above, that Z_i and Z_j are substitutes, and in cases (c) and (d) corresponding to (2) above, that Z_i and Z_j are complements.

It is interesting to note here that in the "weak" sense of determinacy, as in Section II, y variables are determinant to all other variables. Since they are never basic variables, they are weakly independent of each other. Further, for y variable Z_i, t variable Z_j, and any configuration $Z = (Z_B,Z_N)$ with $Z_j \in Z_B$, we have $S^-(Z_B,Z_i) = S^+(Z_B,Z_j) = \emptyset$, so that y and t variables are always weak complements.

Series-Parallel Decompositions

We now consider two types of decompositions of a network G which allow us to transfer determinacy in components to determinacy in all of G.

Definition 3.9: Let $G = G(N,A)$ be a network and r and s two nodes in G. Then an s-series decomposition of G is any pair of edge non-empty subnetworks $G' = G(N',A')$ and $G'' = G(N'',A'')$ of G for which

(S1) $A' \cup A'' = A$, $A' \cap A'' = \phi$, i.e. edge disjoint, and

(S2) $N' \cup N'' = N$, $N' \cap N'' = \{s\}$.

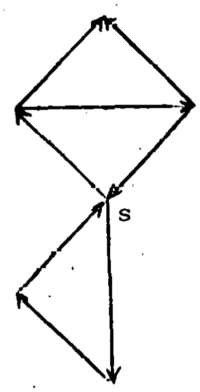
An (r,s)-parallel decomposition of G is any pair of edge non-empty subnetworks $G' = G(N',A')$ and $G'' = G(N'',A'')$ of G for which

(P1) $A' \cup A'' = A$, $A' \cap A'' = \phi$

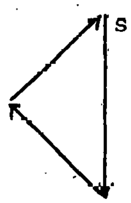
(P2) $N' \cup N'' = N$, $N' \cap N'' = \{r,s\}$

(P3) No edge of A points out of s, and at least one edge in each of A' and A'' points into s. (Equivalently, G' and G'' do not form as r-series decomposition of G.)

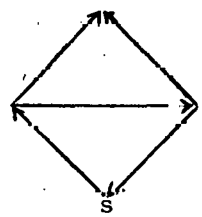
Example:

G = 

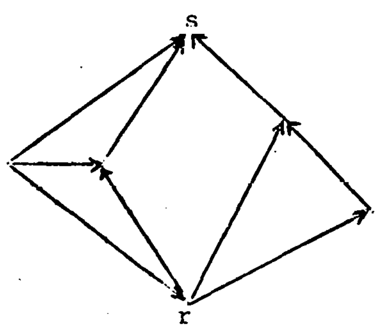
has an s-series decomposition into

G' = 

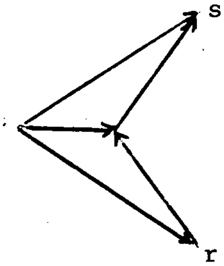
and

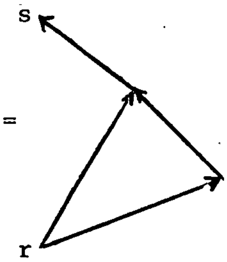
G'' = 

and

G = 

has an (r,s)-parallel decomposition into

$G' =$ 

and $G'' =$ 

Lemma 3.10: Let $G' = G(N',A')$ and $G'' = (N'',A'')$ form an s-series decomposition of the network $G = G(N,A)$, r a node of G' and T a subset of A. Then T is an r-tree of G if and only if $T \cap A'$ is an r-tree of G' and $T \cap A''$ is an s-tree of G''.

Proof: If T is an r-tree of G, and u is a node in G, then there is a unique path from r to u. If u is in N' then the path lies entirely in A' and so comprises a unique path from r to u in $T \cap A'$. If u is in N'', then the path must pass through s, and so the final part of the path comprises a unique path from s to u in $T \cap A''$. Conversely, if $T \cap A'$ and $T \cap A''$ are r- and s-trees of G' and G'', respectively, then for any node u in G there exists a unique path from r to u, which for $u \in N'$ is the unique path from r to u in $T \cap A'$, and for $u \in N''$ and the unique path from r to s in $T \cap A'$ followed by the unique path from s to u in $T \cap A''$. This proves the lemma.

Proposition 3.11: Let $M = M(G,r)$ be a supply-demand model, u a node in G, and G' and G'' an s-series decomposition of G for which r is a node in G'. Set $M' = M(G,r)$ and $M'' = M(G'',u)$. Then for variables Z_i and Z_j in M:

(1) If Z_i and Z_j are in M', then Z_i and Z_j are substitutes (complements) in M if and only if they are substitutes (complements) in M'.

(2) If Z_i and Z_j are in M'' and both are not y-variables, then Z_i and Z_j are substitutes (complements) in M if and only if they are substitutes (complements) in M''.

(3) If Z_i and Z_j are y-variables in M'', then they are independent if and only if they are independent in M'' and $r = u$ (and are strict substitutes otherwise).

(4) If Z_i is any variable in M' and Z_j is a t-variable in M'', then Z_i and Z_j are independent.

(5) If $r = u$, Z_i is a y-variable in M', and Z_j is a y-variable in M'', then Z_i and Z_j are independent.

(6) If $r \neq u$, Z_i is any variable in M', and Z_j is a y-variable in M'', then Z_i and Z_j are substitutes (complements) if and only if Z_i and y(u) are substitutes (complements) in M'.

Proof: Let $Z = (Z_B, Z_N)$ be a configuration of M. Then from Lemma 3.10 we have

(a) If Z_i is in M', then
$$S^+(Z_B, Z_i) = S^+(Z_B \cap A', Z_i)$$
$$S^-(Z_B, Z_i) = S^-(Z_B \cap A', Z_i)$$
(b) If Z_i is a t-variable in M'', then
$$S^+(Z_B, Z_i) = S^+(Z_B \cap A'', Z_i)$$
$$S^-(Z_B, Z_i) = S^-(Z_B \cap A'', Z_i)$$
(c) If Z_i is a y-variable in M'', then
$$S^+(Z_B, Z_i) = \emptyset$$
$$S^-(Z_B, Z_i) = S^-(Z_B \cap A'', Z_i) \cup S^-[Z_B \cap A', y(s)].$$

The lemma follows.

If $r \neq s$, the component M'' of Proposition 3.11 will be called the lower component of the decomposition, and M'' the upper component.

For parallel decompositions we have similar results:

Lemma 3.12: Let G' = G(N',A') and G'' = G(N'',A'') form an (r,s)-parallel decomposition of the network G = G(N,A), and T a subset of A. Then T is an r-tree of G if and only if there exists an r-tree T' of G' and an r-tree T'' of G'' such that T = T' ∪ T'' - {(v,s)}, where (v,s) is the unique edge adjacent to r in either T' or T''.

Proof: First let T be a tree in G, and by symmetry suppose the unique edge (v',s) adjacent to s in T is in A'. Then for any u' in N', there exists a unique path from r to u' in T which lies entirely in A', and for any u'' in N''-{s} there exists a unique path from r to u'' in T which lies entirely in A''. Thus T ∩ A' is an r-tree in G', and for any edge (v'',s) in A'' adjacent to r (by definition there must be at least one), (T ∩ A') ∪ {(v',s)} is an r-tree in G''.

Conversely, let T' and T'' be r-trees in G' and G'', respectively, and (v'',s) the edge in T'' adjacent to s. Set T = T' ∪ T'' - {(v'',s)}, and choose any node u in G. If u is in N' then there is a unique path from r to u in T' and no path from r to u in T containing any edges of T", and if u is in N"-{s}, then there is a unique path from r to u in T"-{(v",s)}, and no path from r to u in T containing any edges of T'. Thus T is an r-tree of G, and this completes the lemma.

We call a node or arc <u>interior</u> to a component of an (r,s)-parallel decomposition if it is neither equal to nor adjacent to s. The parallel analog to Proposition 3.11 is:

<u>Proposition 3.13</u>: Let M = M(G,r) be a supply-demand model, and let G' = G(N',A') and G" = G(N",A") comprise an (r,s)-parallel decomposition of G. Set M' = M(G',r) and M" = M(G",r). Then for variables Z_i and Z_j in M:

(1) If Z_i and Z_j are interior to M', then they are substitutes (complements) in M if and only if they are substitutes (complements) in M'.

(2) If Z_i is interior to M' and Z_j is interior to M", then Z_i and Z_j are independent.

(3) If Z_i = y(s) and Z_j is interior to M', then Z_i and Z_j are substitutes (complements) in M if and only if they are substitutes (complements) in M'.

(4) If Z_i = t(u,s) is in M' and Z_j is interior to M', then Z_i and Z_j are substitutes (complements) in M if and only if the pairs (Z_i,Z_j) and $[y(u),Z_j]$ are both substitutes (complements) in M'.

(5) If Z_i = t(u,s) is in M' and Z_j is interior to M", then Z_i and Z_j are substitutes (complements) in M if and only if y(s) and Z_j are complements (substitutes) in M".

(6) If neither Z_i nor Z_j is interior to either component, then they are strict substitutes if they are the same type of variable and strict complements if they are different types.

Symmetrical statements hold for Z_i and Z_j in M", and for Z_i in M" and Z_j in M'.

<u>Proof</u>: The idea of the proof will be to break each configuration into its parallel component edges and thus extend determinacy with respect to these components into determinacy with respect to the entire configuration. By Lemma 3.11 we know that $Z = (Z_B,Z_N)$ is a configuration for M if and only if $Z_B = Z'_B \cap Z''_B - \{(v,s)\}$, where $Z' = (Z'_B, N' \cap A' - Z'_B)$ and $Z'' = (Z''_B, N'' \cap A'' - Z''_B)$ are configurations in M' and M", respectively, and (v,s) is in $Z'_B \cap Z''_B$. Now for Z_i in M', it is easy to verify the following.

(i) If (v,s) is in Z''_B, or (v,s) is in Z'_B and Z_i is interior to M', then

$$S^+(Z_B,Z_i) = S^+(Z'_B,Z_i)$$

$$S^-(Z_B,Z_i) = S^-(Z'_B,Z_i);$$

(ii) if (v,s) is in Z'_B and $Z_i = y(s)$, then

$$S^+(Z_N,Z_i) = S^+(Z''_B,Z_i)$$

$$S^-(Z_B,Z_i) = \emptyset;$$

(iii) if (v,s) is in Z'_B and $Z_i = t(u,s)$, then

$$S^+(Z_B,Z_i) = S^+[Z_B,y(u)]$$

$$S^-(Z_B,Z_i) = S^+[Z_B,y(s)];$$

and symmetrically for Z_j in M". Conclusions (1) and (2) follow immediately, and (6) follows from Proposition 3.8 $\{$since $\beta[Z_B,t(u,s)] = \beta(Z_B,y(s)] = \gamma(Z_B,s) \neq \emptyset$ for all configurations $Z = (Z_B,Z_N)\}$. For (3) to (5), we prove the "if" parts by contradiction. Suppose Z_i and Z_j are of the type indicated in (3), (4), or (5), and are not substitutes (complements). Then there exists a configuration $Z = (Z_B,Z_N)$ of M for which Z_i and Z_j are not Z substitutes (complements). Let Z', Z", and (v,s) be defined as above, so that Z_i satisfies (i) (or its M" counterpart). We consider two cases:

<u>Case 1</u> [(v,s) is in Z''_B]: Now Z_i also satisfies (i). For (3) and (4), then, Z_i and Z_j not being Z-substitutes (complements) implies that they cannot be Z'-substitutes (complements). Condition (5) cannot occur, since $S^+(Z_B,Z_i)$ and $S^-(Z_B,Z_i)$ are contained in A', and $S^+(Z_B,Z_j)$ and $S^-(Z_B,Z_j)$ are contained in A".

<u>Case 2</u> [(v,s) is in Z'_B]: Now Z_i satisfies (ii) above if it is of type (3), and (iii) if it is of type (4) or (5). The situation in (3) cannot therefore occur, since $S^+(Z_B,Z_i)$ is contained in A" and $S^+(Z_B,Z_j)$ and $S^-(Z_B,Z_j)$ are contained in A'. For (4), Z_i and Z_j not being Z-substitutes (complements) implies that y(u) and Z_j are not Z-substitutes (complements), and for (5) Z_i and Z_j not being Z-substitutes (complements) implies that y(s) and Z_j are not Z"-complements (substitutes). This completes the "if" parts of (3) to (5).

For the "only if," we take (3), (4), and (5) separately.

(3): Suppose Z_i and Z_j are not substitutes (complements) in M'. Then there exists a configuration $Z' = (Z'_B,Z'_N)$ of M' for which Z_i and Z_j are not Z'-substitutes (Z'-complements). Now for any configuration $Z'' = (Z''_B,Z''_N)$ of M", set $Z = (Z_B, Z_N)$, where $Z_B = Z''_B \cup Z''_B - \{(v,s)\}$ and (v,s) is the edge in Z''_B adjacent to s. Z_i and Z_j are not Z-substitutes (Z-complements), and hence not substitutes (complements) in M.

(4): If Z_i and Z_j are not substitutes (complements) in M', then by the same argument as above they are not substitutes (complements) in M. If y(u) and Z_j are not substitutes (complements) in M', then there must be a configuration $Z' = (Z'_B, Z'_N)$ in M' for which y(u) and Z_j are not Z'-substitutes (Z'-complements). Now let $Z'' = (Z''_B, Z''_N)$ be any configuration for M'', and set $Z = (Z_B, Z_N)$, where $Z_B = Z'_B \cup Z''_B - \{(v,s)\}$ and (v,s) is the edge in Z'_B adjacent to s. Now $S^+(Z_B, Z_i) = S^+[Z'_B, y(u)]$ and so since Z_j is not adjacent to s, then Z_i and Z_j are not Z-substitutes (Z-complements) and hence not substitutes (complements).

(5): If Z_i and y(s) are not complements (substitutes) in M'', then by a symmetric argument to the one above, Z_i and Z_j are not substitutes (complements) in M.

This completes the proposition.

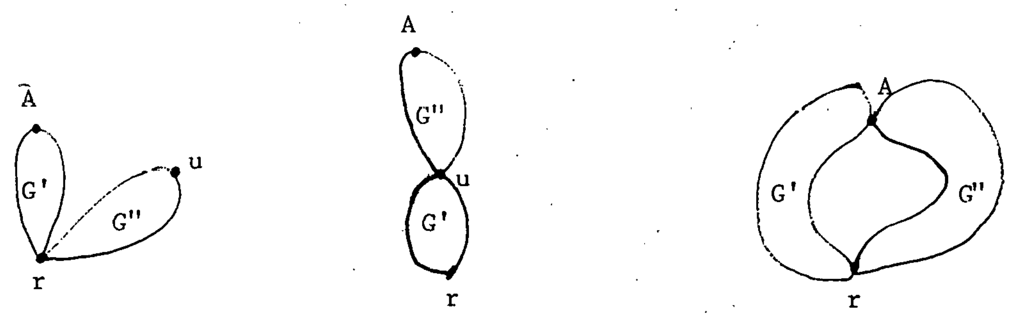
The main consequence of the preceding discussion relates to the important class of series-parallel networks. We state the directed version here.

Definition 3.14: Let G = G(N,A) be a network and r and s two nodes of G. G is called an (r,s)-series-parallel network if G is comprised of the single edge (r,s) or, one of the following holds:

(1) There exists an r-series decomposition of G into networks G' and G'' such that G' is (r,s)-series-parallel, and G'' is (r,u)-series-parallel for some node u in G''.

(2) There exists a u-series decomposition of G, $u \neq r$, into lower part G' and upper part G'' such that G' is (r,u)-series-parallel and G'' is (u,s)-series-parallel.

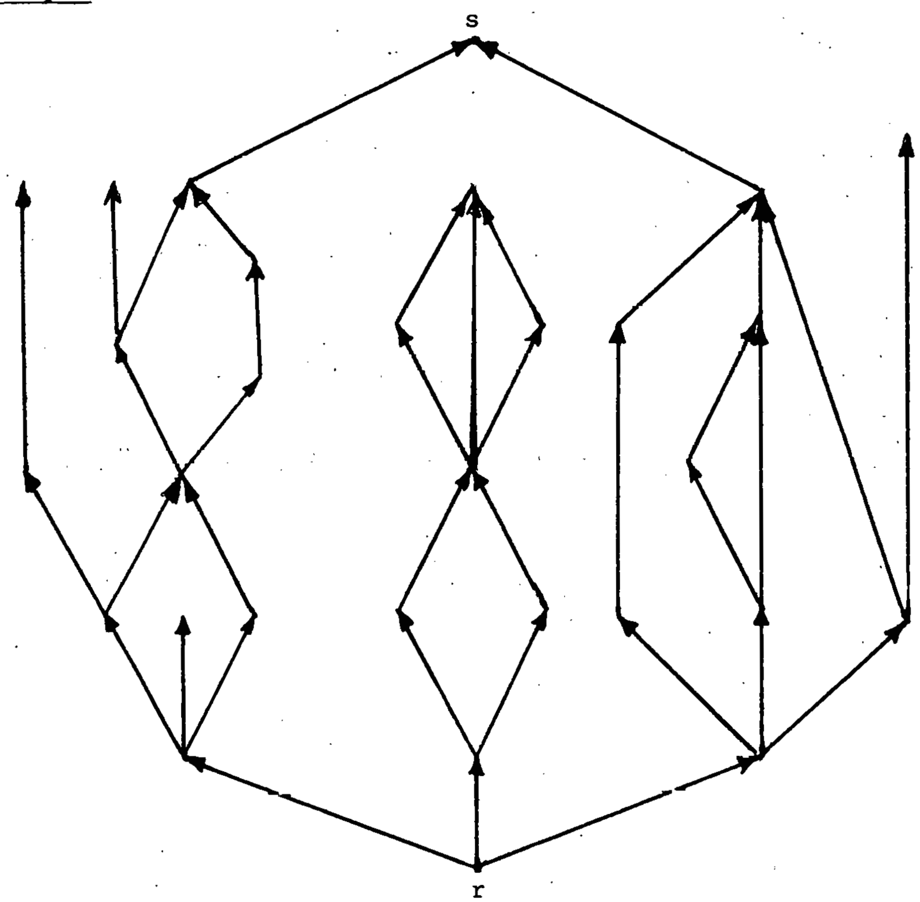
(3) There exists an (r,s)-parallel decomposition of G into networks G' and G'', both of which are (r,s)-series-parallel.



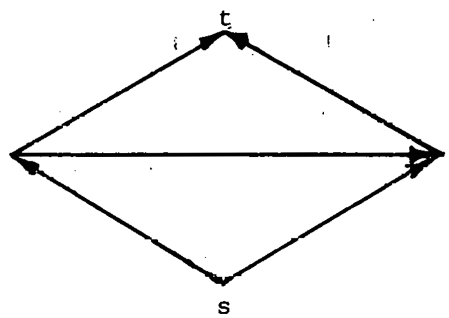
(1) (2) (3)

The decomposition in (1) will be called an improper series decomposition, and that of (2) a proper series decomposition.

Example:



is an (r,s)-series-parallel network, but



is not.

Series-parallel networks are a standard in organized flow processes (in fact the Brookhaven National Laboratory reference energy system, is described by a flow which is to a large degree series-parallel). The following theorem illustrates just how organized these systems are.

Theorem 3.15: Let M = M(G,r) be a supply-demand model for which G is an (r,s)-series-parallel network. Then the following variable pairs in M are determinant:

(1) y variables are substitutes;

(2) distinct t variables are

  (a) substitutes if they lie in different components of some parallel decomposition,

  (b) complements otherwise;

(3) a particular y and t variable are

  (a) substitutes if they lie in different components of a proper series decomposition with the t variable in the upper part,

  (b) complements if they are in different components of a proper series decomposition with the t variable in the lower part,

  (c) complements if they lie in different components of an (r,v)-parallel decomposition,

  (d) complements if they are of the form y(v), t(u,v).

Further,

(4) Any y-variable in one component of an r-series decomposition is independent of any variable not in that component.

(5) Any t-variable in a component of an improper series decomposition is independent of any variable not in that component.

(6) Any y variable interior to one component of an (r,v)-parallel decomposition is independent of any variable interior to the other component.

(7) Any t variable interior to one component of a (u,v)-parallel decomposition is independent of any variable interior to the other component.

(8) Any t variable interior to the upper component of a u-series decomposition is independent of any variable in the other component.

(9) A variable t(x,w) in the upper component of a u-series decomposition for which all edges of G into w lie in that component is independent of any variable in the lower component.

All other pairs satisfying (1) to (3) are strict substitutes or strict complements.

Proof: Suppose first that G is comprised of the single edge (r,s). Then the only distinct variables of M(G,r) are y(s) and t(r,s). By Proposition 3.8

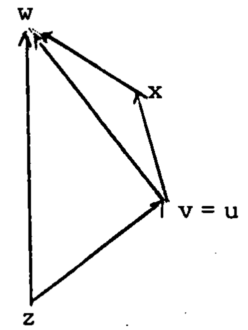
these are complements, since $\beta[Z_B, t(r,s)] = \phi[Z_B, y(s)] = \gamma(Z_B, s) = \emptyset$ for all configurations $Z = (Z_B, Z_N)$ and they are strict complements. Thus they satisfy (3d) of the theorem. Otherwise, G must have a decomposition of one of the three types described in Definition 3.14. We take each type separately.

Case 1: Suppose G has an r-series decomposition into components G' and G" such that G' is (r,s)-series-parallel and G" is (r,u)-series parallel for some node u in G". Then by induction we may assume that the variables in M' = M(G',r) and M" = M(G",r) satisfy (1) to (9) of the theorem. Let Z_i and Z_j be variables in M, and now apply Proposition 3.11. Proposition 3.11 (4) and (5) imply (4) and (5) of the theorem. In any other case Z_i and Z_j are in the same component, say M', and Proposition 3.11 (1) to (3) insure that they are substitutes, complements, or independent exactly as they are in M'. But M' satisfies the theorem, and hence so must M. This completes Case 1.

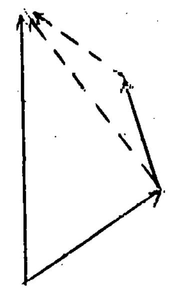
Case 2: Suppose G has a proper u-series decomposition into lower part G' and upper part G" such that G' is (r,u)-series-parallel and G" is (u,s)-series parallel. Again by induction we may assume that the variables in M' = M(G',r) and M" = M(G",r) satisfy (1) to (9) of the theorem. Let Z_i and Z_j be variables in M, and apply Proposition 3.11. Proposition 3.11 (3) implies (1) of the theorem, and 3.11 (4) implies (8) and (9) of the theorem. Any other case when Z_i and Z_j are in different components must satisfy 3.11 (6), that is, Z_i is in M', Z_j is a y variable in M", and Z_i and Z_j are substitutes (complements) if and only if Z_i and y(u) are substitutes (complements) in M'. But then Z_i and Z_j satisfy (4) or (5) of the theorem in M if and only if Z_i and y(u) satisfy (4) or (5) of the theorem, respectively, in M', and neither pair can satisfy (6) to (9) or (2a), and must satisfy (3b), of the theorem. The theorem then follows for Z_i and Z_j. Finally, if Z_i and Z_j are in the same component, then Proposition 3.11 (1) to (3) hold, and, as in Case 1, the theorem follows for M. This completes Case 2.

Case 3: Suppose G has an (r,s)-parallel decomposition into components G' and G", both of which are (r,s)-series-parallel. Again by induction we may assume that the variables in M' = M(G',r) and M" = M(G",r) satisfy the conditions of the theorem. Let Z_i and Z_j be variables in M', and now apply Proposition 3.13. Proposition 3.13 (2) implies (6) and (7) of the theorem, and 3.11 (6) implies (2) and (3) of the theorem when neither Z_i nor Z_j is interior to their components (if they are both t variables, then they must eventually fall in different parallel components). If Z_i and Z_j satisfy Proposition 3.13 (4), then $Z_i = t(u,s)$ is in M', Z_j is interior to M', and Z_i and Z_j are substitutes (complements) in M if and only if the pairs (Z_i,Z_j) and $[y(u),Z_j]$ are both substitutes (complements) in M'. For both pairs, (4) and (5) of the theorem are true in M if and only if they are then in M', and (6) to (9) cannot occur for either pair in M. So suppose (4) and (5) do not occur. If Z_j is a t variable, then (2a) holds for Z_i and Z_j in M if and only if it holds for them in M', and also occurs if and only if (7) occurs for y(u) and Z_j in M'. Thus if Z_i and Z_j are substitutes in M' they are strict substitutes in M' (since (9) cannot occur) and thus are strict substitutes in M. If (2b) holds for Z_i and Z_j in M', then (3a) must hold in M' for y(u) and Z_j. Further (6) to (8) cannot occur for y(u) and Z_j; so that they are strict substitutes in M'. Thus Z_i and Z_j are strict substitutes in M. If Z_j is a y variable, then (3a) to (3d) hold for Z_i and Z_j in M if and only if they hold for them in M'. Further, if (3a) holds then (6) cannot occur for y(u) and Z_j in M' so that Z_i and Z_j are strict substitutes in M. If (3c) holds then y(u) and Z_j must be in different components of an (r,s)-parallel decomposition in M', and so by (6) y(u) and Z_j are independent. Further (6) to (9) cannot hold for Z_i and Z_j, so that Z_i and Z_j are strict complements. (3b) cannot occur. The only other cases of Z_i and Z_j being in the same component are covered by 3.11 (1) and (3), and again, as in Cases 1 and 2, the theorem follows for M. The only other case of Z_i and Z_j being in different components is that they satisfy Proposition 3.13 (5), that is, $Z_i = t(u,s)$ is in M', Z_j is interior to M", and Z_i and Z_j are substitutes (complements) if and only if y(s) and Z_j are complements (substitutes) in M". Again, Z_i and Z_j satisfy (4) or (5) of the theorem in M if and only if y(u) and Z_j satisfy (4) and (5) the theorem, respectively, in M" and neither pair can satisfy (6) to (9). For (2) and (3) of the theorem, if Z_j is a t variable then Z_i and Z_j satisfy (2a) in M and y(u) and Z_j must satisfy (3b) in M" and if Z_j is a y variable, then Z_i and Z_j satisfy (3b) in M and y(u) and Z_j satisfy (1) in M". In either case, the theorem follows by induction. This completes Case 3, and hence the theorem.

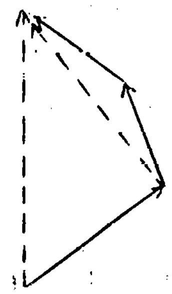
The one case when variables are not determinant in a series-parallel network is when one variable is of the form t(v,w) and the other is y(x), where x is in a different (u,w)-parallel component from (v,w) and this component in turn is in a (z,w)-parallel decomposition with $z \neq u$. For example, the (z,w)-series parallel network



has y(x) and t(v,w) Z-substitutes in this configuration



but Z-complements in this configuration



Every other case is covered in (1) to (9). The following corollary follows immediately.

Corollary 3.16: In a supply-demand model whose underlying graph is series-parallel, all pairs of t variables are determinant. A particular y and t variable are determinant if whenever they are in different parallel components, the t variable is interior to its component.

As indicated in the discussion at the end of the last subsection, Corollary 3.16 implies that all variables in a series-parallel supply-demand model are determinant in the "weak" sense.

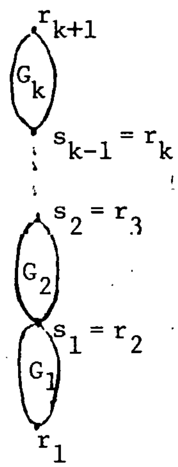
In fact, the indeterminacy problems in Theorem 3.15 disappear if we modify Definition 3.14 slightly.

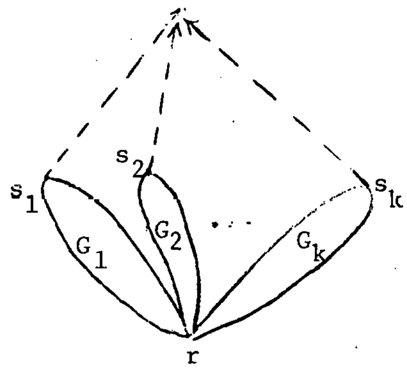
Definition 3.16: Let $G = (N,A)$ be a network and r and s two nodes of G. Then G is called a strong (r,s)-series-parallel network if G consists of a single edge, or, inductively, then exist edge disjoint subnetworks $G_1,\ldots,G_k$ of G for which G is a strong (r_i,s_i)-series-parallel network, $i = 1,\ldots,k$, and either

1) $r_i = s_{i+1}$, $i = 1,\ldots,k-1$, the G_i are otherwise disjoint, and $G = \bigcup_{i=1}^{k} G_i$; i.e. the parallel components are in series;

2) $r_i = r$, $i = 1,\ldots,k$ and $G = \bigcup_{i=1}^{k} G_i \cup \bar{G}$; i.e., all source nodes are the same;

where $\bar{G}$ is comprised of some subset of the edges (s_i,s), $i = 1,\ldots,k$. A decomposition satisfying (1) will be called a strong series decomposition with components G_i, $i = 1,\ldots,k$, and G_i is said to be above G_j if $i > j$. A decomposition satisfying (2) will be called a strong parallel decomposition with components $G_i \cup \bar{G}_i$, when $\bar{G}_i$ is that portion of $\bar{G}$ containing the node s_i. Pictorially, this says that G looks like either, where each G_i may be like (1) or (2) below.

(1) 

or (2) 

The example after Definition 3.14 is also an example of a strong series-parallel network. Certainly these are series-parallel networks, so that Theorem 3.15 holds. Further, separating s_i from s by an edge ensures that the only (z,s)-parallel components must have $z = r$, and so no variables could satisfy the conditions outlined above for indeterminant pairs in the series-parallel model.

In particular statement (3c) of Theorem 3.15 can be amended for strong series-parallel networks

(3c) ...complements if they lie in different components of a (u,v)-parallel decomposition

so that we have the following corollary to Theorem 3.15:

Theorem 3.17: Let $M = M(G,r)$ be a supply-demand model for which G is a strong (r,s)-series-parallel network. Then all variable pairs are determinant. In particular

(1) y variables are substitutes;

(2) distinct t variables are

(a) substitutes if they lie in different components of some strong parallel decomposition,

(b) complements otherwise;

(3) particular y and t variables are

(a) complements if they lie in different components of some strong series decomposition with t above y,

(b) substitutes otherwise.

Multiple Sources

Up to this point we have been considering supply-demand models where goods are shipped from a single supply point to the other nodes in the network. It is often the case that there are several supply points from which goods can be shipped to satisfy demands, so we extend our definition accordingly.

Definition 3.17: Let $G = G(N,A)$ be a network and the set $S \subseteq N$ denote the set of supply nodes. Then the multiple source supply-demand model $M(G,S)$ is defined by input variables

$$X = \{t(u,v) \mid (u,v) \in A\} \in R_X = \mathbb{R}_+^A , \tag{3.4}$$

output variables

$$Y = \{y(u) \mid u \in N - S\} \in R_Y = \mathbb{R}_+^{N-S}, \tag{3.5}$$

and functional relation

$$y(u) = \sum_{(x,u)\in A} t(x,u) - \sum_{(u,x)\in A} t(u,x) \quad u \in N-S . \qquad (3.6)$$

Here the nodes in S produce the good, which is then shipped through the arcs of the network, leaving residual amount y(u) at each node u in N-S.

Configurations are defined according to Definition 2.2. To describe configurations in multiple source models, we first need to prove a lemma about these configurations.

Lemma 3.18: Let M(G,S) be a multiple-source supply-demand model, and $Z = (Z_B,Z_N)$ a configuration for M. Then for every set of non-negative values $y \in S_Y$ and each $v \in S$ and $(t_B,0) \in S_X$ satisfying (3.6) for $u \in N-S$, we have that the net flow from source nodes is non-negative; that is,

$$\sum_{(x,v)\in A} t(x,v) - \sum_{(v,x)\in A} t(v,x) \le 0.$$

Proof: First note that Z_B contains no directed path between any nodes of S, since then, for any set of values of Y and Z_B satisfying (3.6) $u \in N-S$, we can add a positive value ε to each of the arcs of this path producing a second, and hence non-unique set of values for Z_B satisfying these equations. Now let y and t_B be as specified by the lemma, and suppose for some $v \in S$,

$$d(v) = \sum_{(x,v)\in A} t(x,v) - \sum_{(v,x)\in A} t(v,x) > 0. \qquad (3.7)$$

Construct set W of nodes inductively as follows: v is in W, and if x is in W and t(y,x) > 0, then y is in W. Suppose first that some $s \in S-\{v\}$ is in W. Then the construction of W insures that there is a directed path from s to v consisting of arcs of Z_B, a contradiction. On the other hand, suppose $W \cap S = \{v\}$. Then since every edge going into W has t(u,v) = 0, we have, by adding the Eq. (3.6) for $u \in W-\{v\}$ and (3.7),

$$d(v) + \sum_{u\in W-\{v\}} y(u) = \sum_{\substack{u\in W\\(x,u)\in A}} t(x,u) - \sum_{\substack{u\in W\\(u,x)\in A}} t(u,x)$$

$$= - \sum_{\substack{u\in W\\x\notin W\\(u,x)\in A}} t(u,x),$$

so that $\sum_{u\in W-\{v\}} y(u) < 0$, a contradiction. Thus (3.7) cannot occur, and the lemma is proved.

There is a simple transformation which reduces the multiple-source model to the standard single-source model while preserving determinant and non-determinant pairs. Let $M = M(G,S)$ be a multiple-source supply-demand model. Define graph $G' = G(N',A')$ by adding to N the extra "super" supply node r and to E the arcs (r,s), where $s \in S$. The single-source supply-demand model $M' = M(G',r)$ then has input variables

$$X' = X'_A \cup X'_S = \{t'(u,v) \mid (u,v) \in A\} \cup \{t'(r,s) \mid s \in S\} \in R_{X'} = \mathbb{R}_+^{A'}, \quad (3.8)$$

output variables

$$Y' = Y'_{N-S} \cup Y'_S = \{y'(u) \mid u \in N-S\} \cup \{y'(u) \mid u \in S\} \in R_{Y'} = \mathbb{R}_+^{N}, \quad (3.9)$$

and functional relation

$$y'(u) = \begin{cases} \sum_{(x,u) \in A} t'(x,u) - \sum_{(u,x) \in A} t'(u,x) & u \in N-S \\ t'(r,u) + \sum_{(x,u) \in A} t'(x,u) - \sum_{(u,x) \in A} t'(u,x) & u \in S. \end{cases} \quad (3.10)$$

The configurations of M and M' are related as follows:

Proposition 3.19: Let $M = M(G,S)$ be a multiple-source supply-demand model and $M' = M(G',r)$ the corresponding single-source supply-demand model. Then $Z = (Z_B, Z_N)$ is a configuration for M if and only if $Z'_B = (Z'_B, Z'_N)$ is a configuration for M', when

$$Z'_B = Z_B \cup X'_S$$

$$Z'_N = Z_N \cup Y'_S$$

(the variables in Z_B and Z_N are taken to be primed here). Further, every configuration of M' is of this form.

Proof: Let $Z = (Z_B, Z_N)$ be a partition of the variables in M, and $Z' = (Z'_B, Z'_N)$ the corresponding partition in M' as defined by the theorem. First suppose that Z is a configuration. Let $y' \in R_{Y'}$ be a set of values for Y'. We know that y'_{N-S} is also a set of values for Y in R_Y, so that there is a unique set of values of Z_B satisfying (3.6), $u \in N-S$. Define

$$t'(u,v) = \begin{cases} t(u,v) & , \text{ if } t(u,v) \in Z_B \\ y'(v) - \sum_{(x,v) \in A} t(x,v) + \sum_{(v,x) \in A} t(v,x), & \text{if } u = r \\ 0 & , \text{ otherwise}. \end{cases}$$

Then $t' \in R_{X'}$, since by Lemma 3.18

$$- \sum_{(x,v)\in A} t(x,v) + \sum_{(v,x)\in A} t(v,x) \geq 0$$

for all $v \in S$. Further, t' satisfies (3.10), $u \in N$, and these are the only values Z'_B can have which satisfy these equations.

Conversely, if (Z'_B, Z'_N) is a configuration for M', and $y \in R_Y$ is a set of values for Y, then if we define

$$y'(v) = \begin{cases} y(v) & v \in N-S \\ 0 & v \in S, \end{cases}$$

there is a unique set of values $t'(u,v)$ of Z'_B which satisfies (3.10), $u \in N$. It we let t be the values of t' restricted to A, then $t \in R_X$, y and t satisfy (3.6), $u \in N-S$, and these are the only values Z_B can have which satisfy these equations.

Finally, let (Z''_B, Z''_N) be any configuration for M', and consider (Z_B, Z_N) to be defined

$$Z_B + Z''_B - X'_S$$

$$Z_N = Z''_N - (X'_S \cup Y'_S).$$

Then by the converse argument above, (Z_B, Z_N) is a configuration of M, and this completes the proposition.

Determinant pairs in multiple-source models can now be found by considering the corresponding single-source model and applying Proposition 3.19.

Corollary 3.20: Let $M = M(G,S)$ be a multiple-source supply-demand model, and $M' = M(G',r)$ the corresponding single-source model. Then two variables are complements (substitutes, independents) in M if and only if the corresponding variables are substitutes (complements, independents) in M'.

Another useful piece of information can be gained from this transformation. Suppose we wish to treat the supplies at each node $u \in S$ as variables in the model by adding equations

$$s(u) = \sum_{(u,x)\in A} t(u,x) - \sum_{(x,u)\in A} t(x,u), \qquad u \in S. \qquad (3.11)$$

Then the variable s(u) in M corresponds precisely to the variable t(r,u) in M'. With this correspondence, Corollary 3.20 extends easily to single- or multiple-source supply-demand models with supply variables.

The Transportation Network

To finish the section, we consider a special class of multiple source models, namely, those whose only edges join supply and demand nodes.

Definition 3.21: A transportation model is any multiple-source supply-demand model $M = M(G,S)$, when $G = G(N,A)$ has the property that every arc in A is of the form (u,v), $u \in S$, $v \in N-S$.

Corollary 3.20 and the succeeding discussion allow us to give a complete description of determinacy in transportation models.

Theorem: Let $M = M(G,S)$ be a transportation model. Then the determinacy of variables in M can be described as follows:

I. s-variables arc independent;

II. variables y(u) and y(v) are
  (i) substitutes if u and v are adjacent to the same node,
  (ii) independent otherwise;

III. variables s(u) and y(v) are
  (i) complements if (u,v) is an arc,
  (ii) independent otherwise;

IV. variables s(u) and t(w,z) are
  (i) complements if u = w,
  (ii) substitutes if $u \neq w$ and (u,z) is an arc,
  (iii) independent otherwise;

V. variables y(v) and t(w,z) are
  (i) complements if v = z,
  (ii) substitutes if $v \neq z$ and (w,v) is an arc,
  (iii) complements if $v \neq z$ and v and z are adjacent to a common node distinct from w,
  (iv) indeterminant if both (ii) and (iii) hold,
  (v) independent otherwise;

VI. variables t(u,v) and t(w,z) are

(i) substitutes if u = w or v = z,

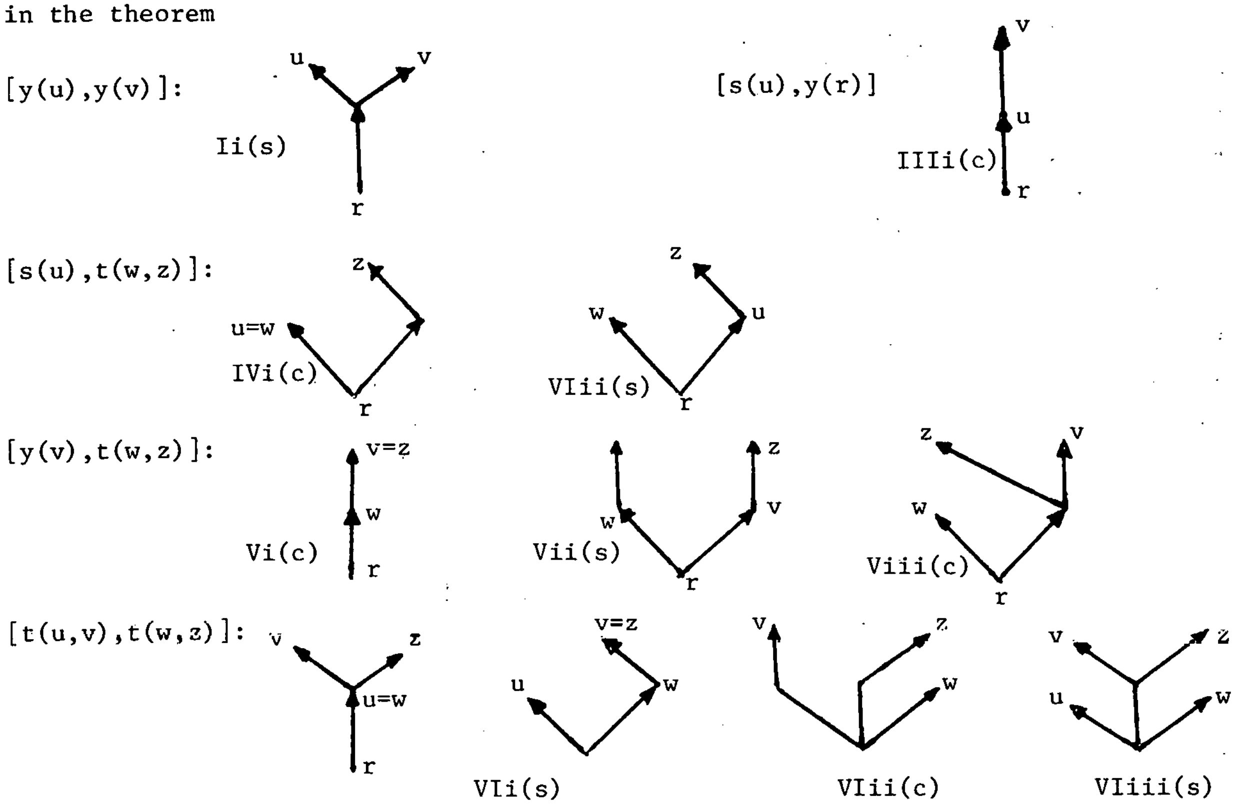
(ii) complements if n ≠ w, v ≠ z, and (u,z) or (w,v) is an arc,

(iii) substitutes if u ≠ w, v ≠ z, and v and z are adjacent to a node distinct from u and w,

(iv) indeterminant if both (ii) and (iii) hold,

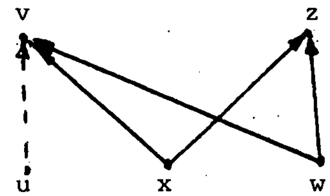
(v) independent otherwise.

Proof: Transform M into the corresponding single-source model M' = M(G',r) as in Proposition 3.19. Now we can list the eleven configurations in which any of the conditions (i) to (iv) of Lemma 3.5 fails, as they appear in the theorem



Any of these subnetworks can be extended to a configuration Z for M'. In the cases denoted by S the corresponding pairs are strict Z-substitutes, and in cases denoted by C they are strict Z-complements. In every other case, the variables must be independent, and in the case where either V(ii) and V(iii) or VI(ii) and VI(iii) both hold, the variables must be indeterminant. This proves the theorem.

Note that the only indeterminant pairs are [y(v),t(w,z)] or [t(u,v),t(w,z)] in the following configuration:



IV. CONCLUSION

The definition of determinacy which we have developed is the same in spirit as that of Greenberg, namely one which correlates variables by measuring their mutual affect on other variables under certain minimal operating conditions (configurations). For a particular configuration of a linear model, the definitions match exactly. We have imposed further restrictions on the allowable configurations with the aim of establishing a realistic, and at the same time a more easily satisfiable, measure of correlation between variables in a large-scale model. For our purposes, the definition has served to uncover determinacy in network models, specifically those relating to supply-demand or series-parallel networks.

The concept of configuration, however, will clearly be dependent on the class of models being investigated. A configuration could be taken to mean for instance: optimal solutions to a linear program, parito optimal solutions to a multi-objective program, or basic feasible solutions to a set of linear inequalities. The resulting determinacy can highlight different perspectives in relationships between the variables in a model. Furthermore, the definition of determinacy itself is subject to modification. Determinacy, as it now stands, takes into account only the mutual affect of a pair of variables. Studying determinacy in terms of the effect of another variable on this pair produces an entirely different viewpoint for variable relationships. One might, in fact, define determinacy as a hybrid of these types of relationships. It is also possible to consider a definition of determinacy which is more continuous, that is, which measures the degree of a relationship rather than the substitute-dependent-complement trichotomy.

A concept such as determinacy, however, is best developed in practice. It will be interesting to see what measure of determinacy emerges as the concept is put to work on models used for policy analysis and decision making. To this end, we have given as general a framework as possible for defining determinacy, since the more ways one has of looking at correlation and relationships in models, the more insight one can gain into their structure, imbedded biases and operating characteristics.

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