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Plasma Transport in Stochastic Magnetic Fields II: Principles and Problems of Test Electron Transport

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Plasma confinement in toroidal devices may be significantly degraded because of flux surface destruction and consequent stochastic wandering of magnetic lines. In this study a model stochastic differential equation is considered which describes guiding center electron motion in a statistically specified spectrum of turbulent magnetic fluctuations. The fluctuation intensity is assumed to satisfy the Chirikov criterion (resonance overlap) for onset of stochasticity. In this limit typical lines diffuse and are adequately described by a quasilinear diffusion coefficient D_m. However, quasilinear theory does not describe an important mechanism for loss of particle correlations: particles collisionally diffuse from one line to an adjacent one which diverges rapidly from the first, carrying the particles away. The scale length L_K for line divergence is related to the inverse of the Kolmogorov-Sinai entropy. An attempt is made to determine L_K from a simplified Eulerian vertex renormalization. The exponentiation length which emerges is $L_K \sim L_s(\bar{k}_\theta^2 D_m'' L_s)^{-1/3}$, where L_s is the shear length, $\bar{k}_\theta$ is a typical azimuthal wavenumber, and D_m'' is of order D_m. In a particular limit of weak shear, the particle diffusion coefficient can then be estimated as $D \sim \Delta r^2/\tau_c$, where $\Delta r^2 \sim D_m z(\tau_c)$, $z(\tau)$ is the distance traveled along the lines in time τ, and for static fluctuations $\tau_c \sim \tau(L_\delta)$, where L_δ is L_K multiplied by a logarithmic factor involving the perpendicular collisional diffusion coefficient. The problems of more refined quantitative computations from the renormalized kinetic equation are severe, and further study is necessary.

1. Introduction

The physics of transport of hot plasma across a strong magnetic field is an area very rich in nonlinear phenomena, most of which are as yet understood only poorly. The plasma is described theoretically

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in terms of either the (quadratically) nonlinear Liouville (Klimontovich) equation or related nonlinear fluid equations, variants of the Navier-Stokes equation. As is well known, such equations admit a variety of stochastic or turbulent solutions, and it is widely held that some form of turbulence is responsible for the anomalous losses observed in many confinement experiments. We will be concerned here with one particular mechanism for anomalous cross-field transport: the resonant destruction of magnetic flux surfaces, stochastic wandering of magnetic lines, and consequent plasma losses by rapid particle motion along the lines.

Study of this stochastic transport mechanism is motivated in part by the results of experiments on a particularly promising magnetic confinement device, the tokamak.[1] In these experiments, the energy confinement time τ is observed to be distinctly anomalous: although for stable magnetohydrodynamic equilibria the ion physics appear to be nearly (neo-)classical,* the electron energy losses far exceed the neoclassical predictions. Now the neoclassical theory assumes the existence of well formed, nested surfaces of constant magnetic flux. This existence can be proven rigorously in situations of special symmetry such as when the plasma retains the toroidal symmetry of the confining vessel. However, plasma fluctuations with magnetic components perpendicular to the equilibrium flux surfaces can spontaneously break this symmetry and destroy those surfaces. (Examples of such fluctuations are drift and tearing modes, which are most likely linearly unstable and are apparently ubiquitous in tokamaks.) Of course, for sufficiently small perturbations the Kolmogorov-Arnol'd-Moser theorem[2] applies: most surfaces are slightly

*Neoclassical theory describes the physics of random walk by Coulomb collisions in toroidal systems with magnetic field gradients and the attendant magnetically trapped particles. For a review, see F.L. Hinton and R.D. Hazeltine, Rev. Mod. Phys. 48 (1976), 239.

distorted but not destroyed. Because the equations for the magnetic lines $d\underline{x} \times \underline{B} = 0$ follow from a Hamiltonian with two degrees of freedom, the phase space remains in this limit partitioned by invariant KAM tori and transport is inhibited. However, for larger perturbations such that the Chirikov criterion[3] $S \equiv$ (typical island width)/(typical island separation) > 1 is satisfied, the well-known stochastic instability sets in: tori are destroyed over a large portion of the phase space and magnetic lines wander randomly over a sizable fraction of the confinement volume. The stochastic lines form an effective channel for <u>radial</u> loss via <u>parallel</u> motion: the effective radial velocity of the guiding center of a particle is $V_r \quad V_{\parallel}(B_r/B)$, and parallel mobility is high.[4,5] In this mechanism energy losses are dominated by electrons because of their higher thermal velocity; this is in qualitative agreement with the observations. The macroscropic plasma diffusion is limited by the ambipolarity constraint.

We will review here the most recent attempts[6] to describe this process quantitatively. Though we are motivated by practical considerations of confinement and scaling, it is important to recognize that the theoretical problem is generic in its nonlinear aspects to many problems involving stochasticity, Hamiltonian and non-Hamiltonian mechanics. Although these fields are undergoing rapid development, it is clear that no complete description will likely be forthcoming for quite some time; our methods are in some ways very primitive. However, we do attempt to employ modern advances in stochastic differential equations and turbulence theory. We would welcome and benefit from a more vigorous dialogue between plasma theorists and workers in the more traditional fields of statistical mechanics and turbulence.

2. A Model Stochastic Differential Equation

We consider the motion of a small number of test electrons which move in specified stochastic magnetic fields. This is analogous to the problem of turbulent advection of a passive scalar in fluids and ignores the important nonlinear physics of the self-consistent generation of stochastic fields by random plasma currents flowing in those fields. Let the unperturbed field be a standard cylindrical model:* $(B_r, B_\theta, B_\phi) = [0, r/Rq(r), 1]B_0$, with $r/R \ll 1$, $q = O(1)$. This field has both circular flux surfaces, centered around $r = 0$, and magnetic shear, $s \equiv d(\ell n\, q)/d(\ell n\, r) = O(1)$. The advective term for guiding center motion in this field, $V(\underset{\sim}{B}/B)\cdot\underset{\sim}{\nabla} = V[(Rq)^{-1}(\partial/\partial\theta) + R^{-1}(\partial/\partial\phi)]$, transforms under variations $\sim \exp i(m\theta - n\phi)$ to $ik_\parallel(r)V$, where $k_\parallel(r) \equiv R^{-1}[m/q(r) - n]$. Resonances occur at all r for which $k_\parallel(r)$ vanishes. Motivated by microturbulence theory and observation, we consider short-wavelength fluctuations, $m\rho_i/r = O(1)$ where ρ_i is the ion gyroradius, and expand $k_\parallel(r)$ around a rational surface $q_0 \equiv q(r_0) = m_0/n_0$: $k_\parallel(r) \approx k_\parallel(0) - [(r - r_0)/L_s]k_\theta$, where the shear length L_s is defined as $L_s \equiv Rq_0/s$, and $k_\theta \equiv m/r_0$. One can then arrive[6] at the following model equation for the probability density $P(x,y,z,V,t)$ or test electrons of parallel velocity V at time t in slab coordinates $x \equiv r - r_0$, the radial distance from a rational surface r_0; $y \equiv r_0(\theta - \phi/q_0)$, the distance orthogonal to x and to the unperturbed lines; and $z \equiv R\phi$, essentially the distance along the unperturbed lines:

$$\frac{\partial}{\partial t} P + \left[V\left(\frac{\partial}{\partial z} - \frac{x}{L_s}\frac{\partial}{\partial y}\right) - D_\perp \nabla_\perp^2 - C\right]P - Vb\frac{\partial}{\partial x}P = 0 \quad . \tag{1}$$

Here $\nabla_\perp^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, $D_\perp$ represents a slow collisional diffusion across the field, C is a collision operator on the parallel velocity V, and $b(y,z;t) \ll 1$ is the ratio of the perturbing radial field to the total field. The perturbation b has a Fourier representation:

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*The cross-section of the cylinder is parametrized by polar coordinates (r,θ), the axial direction by $z \equiv R\phi$. The cylinder is periodic mod 2π in both θ and ϕ.

$b(\theta,\phi;t) = \sum_{mn} b_{mn}(t) \exp i(m\theta - n\phi)$, or in the slab coordinates

$$b(y,z;t) = \sum_{\mu} b_{\mu}(t) \exp i(k_\theta y + k_\parallel z) \quad ,$$

where μ denotes the set $\{m,n\}$ or, equivalently, $\{k_\parallel , k_\theta\}$. The parallel wavenumber $k_\parallel \approx k_\parallel(0)$ is assumed to vary over a range symmetric around $k_\parallel = 0$, of width $\Delta k_\parallel = 2\pi/L_o$, $L_o \approx 2\pi L_s (k_\theta \delta r)^{-1}$, where δr describes the localization (characteristic width) of the background radial eigenfunctions around the rational surface. For infinite, homogeneous turbulence L_o would vanish. However, for microturbulence characteristic of the finite tokamak geometry, $k_\theta \delta r = O(1)$, $L_o = O(L_s)$.

We postulate that the perturbations are turbulent, an assumption supported by observation. The amplitudes are thus random variables which we further assume are Gaussian. This is never strictly correct, especially if the turbulence arises from a strange attractor in the background phase space. One may justify the approximation pragmatically by noting that because of the stochasticity to be discussed shortly and the structure of Eq. (1), interesting non-Gaussian statistics are predicted for P even when absent from b. We can thus attempt to study the mechanism of stochastic transport in isolation from non-Gaussian complications of the background. This argument fails for the self-consistent problem, which we do not treat here. The subjects of turbulence statistics and attractors in plasma are worthy of much further investigation.

The instantaneous field line orbits

$$\frac{dy}{dz} = \frac{x}{L_s} \quad , \quad \frac{dx}{dz} = b(y,z) \qquad (2)$$

are the canonical equations for the Hamiltonian $H(x,y;z) = x^2/2L_s - \int^{y} d\bar{y}\, b(\bar{y},z)$ with x and y considered as conjugate momentum and coordinate. We assume that H is stochastic according to a mean Chirikov criterion: the mean separation between resonance is $\Lambda = \delta r/\xi_\mu \approx \delta r / \sum_m \delta(k_\parallel) dk_\parallel \approx (q/s)(\delta m\, k_\theta)^{-1}$; the typical resonance width

$w = 4(2L_s<|b_\mu|^2>^{1/2}\, k_\theta^{-1})^{1/2}$; and we assume that $S \equiv w/\Delta > 1$. Both numerical experiments and some theory tell us that for S not too close to the threshold $S = 1$, most of the volume is stochastic:[7] typical field lines separate exponentially for small separations[8] and wander throughout a substantial portion of the volume. For large but finite S, there is of course a small invariant set which is non stochastic. However, practical estimates based on microturbulence theory as well as some observation predict that the Chirikov criterion will be very well satisfied in many situations. We are then justified in assuming the measure of the nonstochastic component to be negligible.

From a practical point of view, the detailed structure of $P(x,y,z,V,t)$ is of little direct interest. We are concerned with the mean motion of an ensemble of particles distributed uniformly at $t = 0$ over an equilibrium flux surface r_o. We therefore treat P as a random variable and the perturbation b as a given random coefficient, statistically specified and independent of P. Equation (1) then becomes a stochastic differential equation of standard type.[9] As is well known, such equations suffer from the closure problem, which means that the equation for $<P(x,V,t)>$ (the average taken over both background turbulence and initial conditions) contains the unknown pair correlation $V<b\,\partial P/\partial x>$. One proceeds by expressing this correlation in times of <P> by a closure approximation.

The fluctuations described by Eq. (1) are both inhomogeneous and anisotropic. Their character is determined, in part, by the ratio $R_K \equiv L_o/L_K$, where the separation δ between typical adjacent lines is taken to be $\delta(z) = \delta_o \exp(z/L_K)$. For physical reasons attention has focussed to date on the regime $R_K < 1$. In some ways R_K is analogous to the Reynolds number for fluids. However, we will see that even in the limit $R_K < 1$ the nonlinear physics of the stochastic lines can be nontrivial and important.

Equation (1) can be analyzed by standard formal techniques for stochastic differential equations; we discuss this in Sec. 4. However, more insight can be gained from simple qualitative arguments, to which we now turn.

3. Qualitative Physics of Stochastic Transport

Equation (1) has a natural interpretation in terms of two distinct physical processes: parallel motion along, and perpendicular diffusion across the lines. To describe the parallel physics, we approximate true lines by unperturbed lines and set $D_\perp$ and b to zero; the resulting equation preserves x as constant and the shear term in x can be removed. By choosing C to be the model operator $\partial_V(\nu V + D_V \partial_V)$, one is left with a standard Fokker-Planck equation whose solution is well known. If one applies this solution to the quasilinear approximation to the particle diffusion coefficient $D(t;V_0)$ of particles with initial velocity V_0,

$$D(t;V_0) = \int_0^t d\tau \sum_\mu \langle b_\mu(\tau) b_{-\mu}(0)\rangle \langle V(\tau) V_0 \exp ik_\parallel z(\tau)\rangle ,$$

he can write

$$D(t;V_0) = V_0 D_m d(t) ,$$

where the quasilinear diffusion coefficient D_m of the lines is defined as

$$D_m \cdot \pi R \sum_m \int d k_\parallel \langle |b_\mu|^2(0)\rangle ,$$

and in the limit of static background fluctuations

$$d(t) = \int_0^{\langle z(t|V_0)\rangle} dz \int d\bar{z}\, C_\parallel(z - \bar{z}) P(\bar{z},t|V_0) ,$$

where $\langle z(t|V_0)\rangle \equiv [1 - \exp(-\nu t)](V_0/\nu)$ is the mean distance (averaged over the Langevin white noise fluctuations responsible for collisions) traveled in time t, $C_\parallel(z)$ is the Eulerian correlation function [of width $O(L_0)$] for b taken along the unperturbed lines, and the probability for fluctuations around the mean position is

$$P(z,t|V_o) = [2\pi\sigma_z^2(t)]^{-1/2}\exp[-z^2/2\sigma_z^2(t)] \quad ,$$

$$\sigma_z^2(t) \equiv (2\nu t - 3 + 4e^{-\nu t} - e^{-2\nu t})\ell^2 \quad ;$$

$\ell \equiv v_t/\nu$, where v_t is the thermal velocity, related to the parallel diffusion coefficient through $D_v = v_t^2/\nu$. One physically interesting limit is that of Eulerian correlation length small compared to a collisional mean free path ℓ: $R_\ell \equiv L_o/\ell < 1$. In Fig. 1 we plot $d(t)$ for the model function $C_\parallel(z) \equiv [H(z+L_o) - H(z-L_o)]/2L_o$ [$H(z)$ is the Heaviside function] for several values of R_ℓ, and $V_o = v_t$. For the smallest values of R_ℓ, three distinct regimes are apparent. In regime (a) the bulk of the particles remain within the domain L_o of magnetic correlations and $d(t) \approx V_o t/L_o = (\nu t)/R_\ell$. In regime (b) particles, still essentially collisionless, move on diffusing lines so asymptotically[5] $d(t) \approx 1$. This regime is quite small for $R_\ell \gtrsim 0.1$. In regime (c) most particles have collided at least once so a double diffusion law holds:[5] $\delta r^2 \sim D_m \delta z$, $\delta z^2 \sim D_\parallel t$, $\delta r^2 \sim D_m(D_\parallel t)^{1/2}$ so $d(t) \sim (\nu t)^{-1/2}$. Because energy confinement times in tokamaks are many collision times, this model taken naively would predict negligible stochastic transport in the limit $t \to \infty$.

This picture is incomplete, however, because it incorrectly assumes that particles remain always on their initial lines. If particles lose correlation with a given line in time τ_c, then the asymptotic diffusion coefficient is finite, $D(\infty) \approx V_o D_m d(\tau_c)$. Loss of correlations arises from two distinct effects. First, the background turbulence can have a finite correlation time τ_b. Because the properties of the background fluctuations are not well known, we will not discuss τ_b further here. (Let us note in passing that some interest attaches to perturbations arising from external coil asymmetries, in which case $\tau_b = \infty$.) Second, perpendicular collisional diffusion removes particles from lines. Let the perpendicular Eulerian correlation length of the background turbulence be $L_\perp [= O(k_\theta^{-1})]$. If one ignored the stochastic nature of the field, he would estimate the corresponding correlation time to be $\tau_\perp \sim L_\perp^2/D_\perp$. However, adjacent lines diverge exponentially and particles can cross the distance $L_\perp$ more rapidly by diffusing a small amount perpendicularly to a new line, moving rapidly along this new line, then repeating this process. The correlation time τ_δ for this process can be estimated as $\tau_\delta = \tau(L_\delta)$, where $\tau(z)$ is the time required to travel a parallel distance z and $L_\delta \approx L_K \ln[z(\tau_\perp)/L_K]$. This estimate is very rough because the lines become uncorrelated and diffuse independently for $(z) \sim L_\perp$. Finite shear introduces further correlations which we do not discuss here.

Our goals, then, are to determine L_K, τ_δ, and therefore $D(\nu)$. Regarding L_K, Chirikov[7] has estimated a "typical" K-S entropy which in our units is $L_K \sim L_s(\bar{k}_\theta^2 D_m L_s)^{-1/3}$, where $\bar{k}_\theta$ is a typical azimuthal wavenumber. One can recover this result by requiring that the scale length for exponentiation agree with the scale length for loss of correlations by single line diffusion (this amounts to requiring continuity of Lagrangian magnetic correlations as a function of z). For diffusion of single lines, one estimates from the shear relation $\Delta y/\Delta z \sim \Delta r/L_s$ and the diffusion law $\Delta r^2 \sim D_m \Delta z$ for single lines that the mean square of the azimuthal phase fluctuation $\langle(k_\theta \delta y)^2\rangle$ is of order $(\Delta z/L_K)^3$. (This assumes that radial correlations do not play a role. If they dominate, L_K should presumably be replaced by $(k_r^2 D_m)^{-1}$, with D_m to be determined self-consistently from a strong turbulence type of calculation.) Alternatively, L_K emerges from a closure for $\langle\Delta y^2\rangle$ applied to the equations for relative separation

$$\frac{d}{dz}\Delta y = \frac{\Delta x}{L_s} \quad , \quad \frac{d}{dz}\Delta x = \frac{\partial b}{\partial y}\Delta y \quad . \tag{3}$$

If one expresses the three-point functions in the equations for $\langle\Delta x^2\rangle$, $\langle\Delta x \Delta y\rangle$, $\langle\Delta y^2\rangle$ in terms of four point ones by integration of (3), makes

Gaussian factorizations of the four-point functions, considers $z > L_0$ so that the Markovian approximation is valid, and neglects terms small in R_K, the equations reduce to

$$\frac{d^3}{dz^3} \langle \Delta y^2 \rangle = \left(\frac{2}{L_K}\right)^3 \langle \Delta y^2 \rangle \quad , \tag{4}$$

whose solution grows asymptotically as $\exp(2z/L_K)$, where $L_K \equiv L_s(\frac{1}{2} \bar{k}_\theta^2 D''_m L_s)^{-1/3}$ and $D''_m = \int_0^\infty d\zeta \langle b'(\zeta) b'(0) \rangle$, $b' = \partial b / \partial(\bar{k}_0 y)$. We have $D''_m = O(D_m)$. This scaling for L_K appears to require that the amplitudes b_ω be sufficiently random. Unfortunately, verification of this result by direct numerical integration of Eqs. (2) is extremely time-consuming and has not been done.

The physics of the asymptotic regimes (b) and (c) were discussed very qualitatively by Rechester and Rosenbluth.[5] It would appear, however, that quantitative determination of τ_δ and $D(\infty)$ as well as extension of these arguments to the self-consistent problem require a kinetic approach. In the next section we discuss aspects of our program in this direction.

4. Closure Approximations

We wish to derive information about D by applying statistical closure approximations to Eq. (1). Now it is well known that most workable closures can be characterized as formal expansions of Eulerian statistical functions around a Gaussian state.[10] (The recent Lagrangian schemes of Kraichnan[11] are an exception which we do not discuss here.) It is not immediately clear that such Eulerian-based schemes will succeed; the most prominant dynamical feature of the stochastic state, the exponential divergence of adjacent trajectories, involves structures, namely pairs of lines, and is therefore intrinsically non-Gaussian. However, as we will discuss, the exponential divergence does emerge from an appropriate (vertex-renormalized) Eulerian closure. This affords us in principle the first quantitative analytic description of the diffusion coefficient. Practically, the computational difficulties are severe and much further work is necessary.

It is convenient to adopt the following notation: for arbitrary functions A, B, let $A(\underset{\sim}{1}) \equiv A(x_1, y_1, z_1, V_1)$, $B(1) \equiv B(\underset{\sim}{1}, t_1)$. Define the stochastic response function $R(1;1')$ by the functional derivative of P with respect to a nonrandom source η added to the right-hand side of Eq. (1): $\tilde{R}(1,1') = \delta P(1)/\delta\eta(1')$; denote its mean by $R = \langle \tilde{R} \rangle$. The averaged response function $R(1,t;1',t')$ is the solution of Eq. (1) for $\langle P \rangle$ with initial condition $\delta(\underset{\sim}{1} - \underset{\sim}{1}')$ at $t = t'$. It can be interpreted as the probability density of a test particle at $(\underset{\sim}{1}, t)$ given that it was precisely at $(1', t')$.

One can write a formal Dyson equation for R:

$$\partial_{t_1} R(1,1') + [L(1,\bar{1}) - \Sigma(1,\bar{1})]R(\bar{1},1) = \delta(1-1') \quad ,$$

where $L(1,\bar{1})$ is the bracketed operator in Eq. (1) times $\delta(1-\bar{1})$ and where an integration convention over repeated arguments is assumed. In this notation the radial diffusion coefficient D can be expressed as

$$D = -\lim_{t-t'\to\infty} \int dV_1\, \phi(V_1) \int d(x_1 - x_1')(x_1 - x_1')^2\, \Sigma(1,\bar{1})R(\bar{1},1') \quad . \tag{5}$$

Several techniques are available for generating approximations to the renormalized collision operator Σ. The one which is clearest both logically and operationally is the functional scheme of Martin, Siggia, and Rose[11] although many of their results were anticipated by Kraichnan[12] who used the direct method of consolidating infinite perturbation series. In any case Σ can be expressed for Gaussian b in terms of a certain component of a joint probability matrix K which obeys the Bethe-Salpeter equation (BSE).[13] The theories actually provide coupled equations for R and the correlation function $C(1,1') \equiv \langle \delta P(1)\delta P(1') \rangle$ in terms of the covariance $F(1,1') \equiv \langle b(1)b(1') \rangle$. Denote fluctuations in P by "+", external perturbations by "-", and the random coefficient b by "o" and let R be the (+-) component of a two-point correlation matrix G_{ij} which also contains $C = G_{++}$ and $F = G_{oo}$. Further, define the bare vertex operator $U'(1,2,3) = V_1 \delta(1-2)\partial_{x_1}\delta(1-3)$. Then we have[11,13,6]

$$\Sigma(1,\bar{1}) = U'(1,2,3)\, K\begin{pmatrix} 2 & 3 & ; & \bar{2} & \bar{3} \\ 0 & + & ; & 0 & - \end{pmatrix} U'(\bar{2},\bar{3},\bar{1}) \quad ,$$

where in formal operator notation the BSE for K reads $K = \frac{1}{2}(GG + GG) + GGIK$. One can interpret $K_{o+o-}(1,2;1',2')$ as the probability that one will observe a field fluctuation at 1 <u>and</u> a particle at 2, given that one knew the field to be in the state $1'$ <u>and</u> the particle was in the state $2'$. The interaction term I describes the effects of field particle or more generally two-body correlations on K. If these are arbitrarily neglected, the Direct Interaction Approximation (DIA) emerges: $K \approx \frac{1}{2}(GG + GG)$; $K_{o+o-} \approx \frac{1}{2}RF$. This factorization into two-body functions does not describe the exponentiation. DIA is the "best Gaussian" approximation consistent with nonvanishing three-point functions; it does not retain phase information necessary to distinguish entities like pairs of lines. We conclude that DIA is inadequate for the present problem.

Higher order approximations can be generated by expanding I in powers of a generalized skewness operator[10] $\Gamma = G^{-1}G^{-1}K\gamma$, where γ is a symmetric matrix whose elements are U' of various arguments. The first nonvanishing term of the skewness expansion is $I(1,2;3,4) = \Gamma(1,5,3)G(5,\bar{5})\Gamma(2,\bar{5},4)$. To our knowledge, this renormalization was first proposed classically by Kraichnan[12] in connection with his model stochastic oscillator $\partial_t\psi = -i\omega\psi$ for Gaussian, time-independent ω. Our Eq. (1) is a generalization of this model to time-dependent, operator ω. [This similarity is most apparent when Eq. (1) is Fourier-transformed in y and z; a principal difference is that our model, but not Kraichnan's, is linearly dispersive.) Kraichnan showed that the solution of this approximation for the oscillator agreed very well with the exact R; this was particularly impressive since his model had infinite Reynolds number. In later work[14] Kraichnan studied the approximation for a model convection problem and concluded that it might describe the behavior of the two-time functions very accurately. The well-known problems of this and other Eulerian closures with random Galilean invariance do not appear to be relevant for the class of stochastic equation we consider here.

It is straightforward to write down the first vertex renormalization of Eq. (1). It is not straightforward, and perhaps impossible, to solve the resulting equations without further approximations. Simplications are possible in the limit $R_k < 1$. Space constraints prohibit a detailed discussion here; see Ref. 6. However, one may note the following points. For $R_K < 1$ the equation for $K(1,2;1',1')$ has approximately the form of a diffusion equation in the relative coordinate $\delta x \equiv x_1 - x_2$, with y-dependent diffusion coefficient $D_-(\delta y)$ such that $D_-(0) = 0$. For δy much less than $L_\perp$, $D_-(\delta y) \approx \frac{1}{2} \delta y^2 D''$ and the second moments of the resulting equation combine to give again Eq. (4). (The full solution of this short time solution for K is not Gaussian.) For $\delta y > L_\perp$, $D_- \to D$ and the DIA is recovered. Qualitatively, the equation has the behavior we expect physically; one can recover the results of Sec. 3 from expression (5). Quantitatively, very little is known about the detailed solution connecting the short and long time regimes for any continuous model of $D_-(\delta y)$ having proper limits at small and large δy.

The approximate relative diffusion equation for K appears to contain the minimum allowable amount of complexity consistent with known qualitative physics. That it is still very complicated is disheartening, but at least one has a basis for further concrete calculations. However, more detailed solutions for K should undoubtedly be preceded by further elucidation of the physics of the background and the role of the self-consistent random currents which flow in the regime of destroyed tori. In the limit $R_K > 1$ a speculation is that the DIA may be adequate if proper account is taken of the radial eigenstructure of the background. However, this limit requires a large fluctuation level which may not be realized in practice.

5. Conclusion

The problem of plasma transport in stochastic magnetic fields is rich in interesting linear and nonlinear physics, and is of much practical importance for our understanding of plasma confinement. The present work has barely begun to formulate, let alone answer the relevant questions. The field is a large and challenging one for future work.

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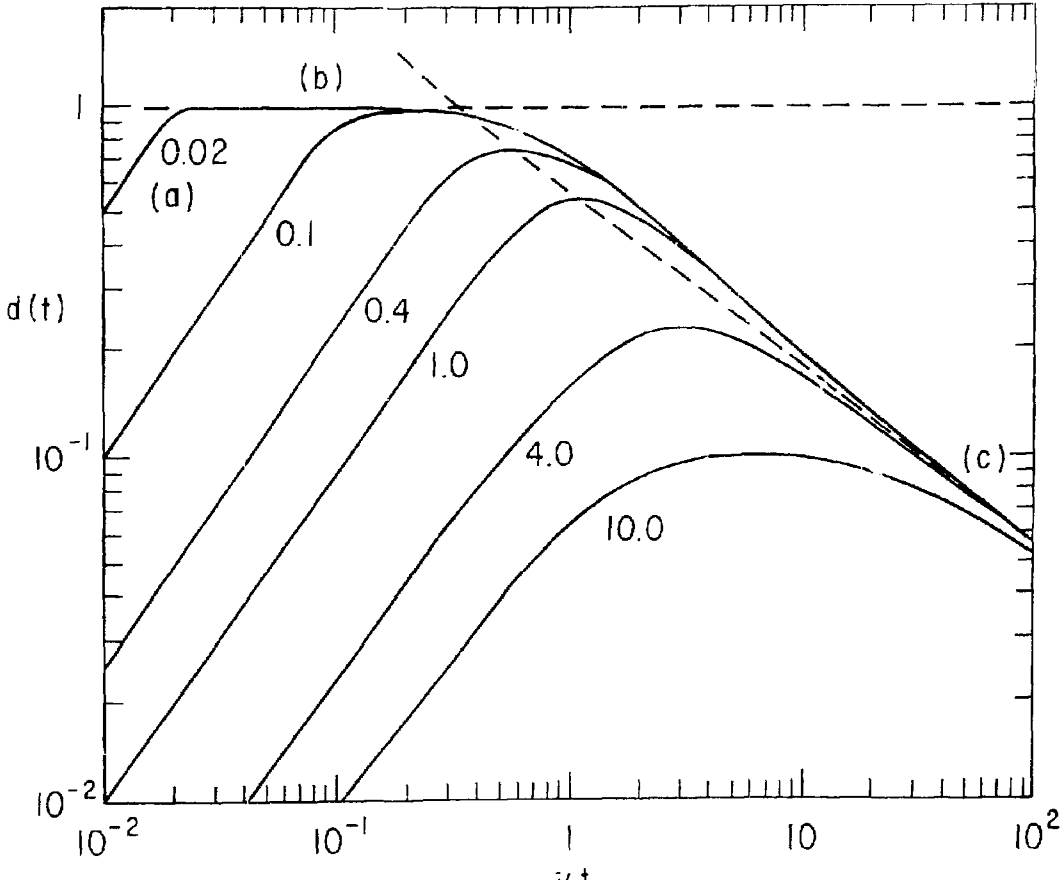

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[illegible] The parallel transport function d(t) [illegible] a model [illegible] tic cor- [illegible] function (see text) and various values [illegible] are discussed in [illegible]