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Extended Likelihood Inference in Reliability

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Notation

θ,	generic parameter of interest
ML,	maximum likelihood
x,	number of nonsurvivors in a binomial experiment
n,	number of trials in a binomial experiment or the number of items on lifetest
p,	probability of nonsurvival
$L(\theta;\underline{x})$,	likelihood function of θ
$R(\theta;\underline{x})$,	relative likelihood function of θ
$g(\theta;\underline{y})$,	prior distribution of θ
$\underline{y}$,	vector of prior parameter values
$\underline{x}$,	vector of observed sample random variables
$L(\theta;\underline{x},\underline{y})$,	extended likelihood function of θ
$R(\theta;\underline{x},\underline{y})$,	extended relative likelihood function of θ
$f(\underline{x};\theta)$,	joint distribution of the observed sample random variables
ELI,	extended likelihood interval
ELPI,	extended likelihood prediction interval
λ,	failure-rate of an exponential time-to-failure model
r,	number of failures for an exponential lifetest model
T,	total time on test
t^*,	test termination time
a,	gamma shape parameter
b,	gamma scale parameter
n_o,	beta parameter representing the pseudo-number of trials
x_o,	beta parameter representing the pseudo-number of nonsurvivors

$R_m(h)$,	second-order likelihood function of h
$R_m(h;\underline{x},\underline{y})$,	extended second-order likelihood function of h
v,	a future failure-time from an exponential distribution
s,	the number of failures that occur in a future lifetest experiment or the number of nonsurvivors in a future binomial experiment
m,	the number of items on test in a future lifetest or binomial experiment
τ,	the calendar time period in a future lifetest experiment

EXTENDED LIKELIHOOD INFERENCE IN RELIABIITY

by

H. F. Martz, Jr., R. J. Beckman, and R. A. Waller

ABSTRACT

Extended likelihood methods of inference are developed in which subjective information in the form of a prior distribution is combined with sampling results by means of an extended likelihood function. The extended likelihood function is standardized for use in obtaining extended likelihood intervals. Extended likelihood intervals are derived for the mean of a normal distribution with known variance, the failure-rate of an exponential distribution, and the parameter of a binomial distribution.

Extended second-order likelihood methods are developed and used to solve several prediction problems associated with the exponential and binomial distributions. In particular, such quantities as the next failure-time, the number of failures in a given time period, and the time required to observe a given number of failures are predicted for the exponential model with a gamma prior distribution on the failure-rate. In addition, six types of life testing experiments are considered. For the bionomial model with a beta prior distribution on the probability of nonsurvival, methods are obtained for predicting the number of nonsurvivors in a given sample size and for predicting the required sample size for observing a specified number of nonsurvivors. Examples illustrate each of the methods developed. Finally, comparisons are made with Bayesian intervals in those cases where these are known to exist.

INTRODUCTION

The method of maximum likelihood as proposed by Fisher[1] is a well-known and widely used technique for estimating unknown parameters of a statistical model. More recently, further developments in this approach have been undertaken by Barnard,[2] Barnard, Jenkins, and Winsten,[3] Sprott and Kalbfleisch[4,5]

Kalbfleisch and Sprott,[6-8] Kalbfleisch,[9] and Hudson.[10] The application of this approach in reliability has been considered by Singpurwalla,[11] Bar Lev,[12] and Reiser and Bar Lev.[13] Whitney[14] gives an excellent tutorial account of the current state of the art of the likelihood method with examples for the binomial, exponential, and Weibull models. A comprehensive philosophical discussion of the method is given by Edwards.[15]

In order to introduce the concept of likelihood, consider a specified statistical distribution with a single unknown parameter θ which is to be estimated. The method of maximum likelihood (ML) selects as the estimate that value of θ, denoted by $\hat{\theta}$, which maximizes the likelihood of obtaining the observed data. This approach can be extended to provide an interval estimate of θ and this has been the thrust of the most recent interest in the likelihood method. Rather than consider only a single value of θ which maximizes the likelihood of producing the observed data, an interval of θ values around the maximum is considered which makes the likelihood of obtaining the observed data relatively large. This range of θ values is viewed as being supported by the data, while those values outside this range are viewed as receiving little support from the data.

To further introduce and quantify the notions of likelihood, consider the following example which is similar to that used by Whitney.[14] Suppose it is desired to estimate the probability of failure-to-start of a diesel generator that is used as an emergency power supply for a nuclear powered reactor. Suppose that n trials are performed and the number of failures-to-start x is recorded. For the sake of illustration, let us assume that all trials are independent with the same probability of failure p on each trial. The probability of observing exactly x failures is given by $\binom{n}{x}p^x(1-p)^{n-x}$. Now consider two specified values of p, say p_1 and p_2. The so-called "likelihood ratio" of p_1 to p_2 is the ratio of the two probabilities given by $p_1^x(1-p_1)^{n-x}/p_2^x(1-p_2)^{n-x}$. Suppose that γ is the value of this ratio. This ratio says that when $p = p_1$, the observed data are γ times as likely as when $p = p_2$. For example, if 2 failures were observed in 200 trials, the likelihood ratio of $p_1 = 0.008$ to $p_2 = 0.04$ is $(0.008)^2(0.992)^{198}/(0.04)^2(0.96)^{198} = 24.33$. That is, when $p = 0.008$ the observed data are roughly 24 times more likely than when $p = 0.04$. Thus, $p = 0.008$ is a much more plausible value of p than $p = 0.04$ and we say that $p = 0.008$ is more strongly supported by the data than $p = 0.04$. The ML estimate of p is well known to be $\hat{p} = x/n$. Thus, in our example, $\hat{p} = 2/200 = 0.01$. All comparisons such as given above can be summarized by considering the likelihood ratio of p to $\hat{p}$ as a function of p. This function is referred to as the <u>relative likelihood function</u> of p given x, and hence

$$R(p;x) = p^{x}(1-p)^{n-x}/\hat{p}^{x}(1-\hat{p})^{n-x} .$$

In our example,

$$R(p;2) = p^{2}(1-p)^{198}/(0.01)^{2}(0.99)^{198}$$

$$= 73152.886\ p^{2}(1-p)^{198} .$$

This function is plotted in Fig. 1.

Values of p such that $R(p;x)$ is close to 1 (which is the maximum value of $R(p;x)$) are regarded as being plausible and supported by the data, while values of p such that $R(p;x)$ is close to 0 are regarded as being implausible and unsupported by the data. In this spirit, the interval of values of p for which $R(p;x) \geq \gamma$ is referred to as a $100\gamma\%$ <u>likelihood interval</u> for p. Such

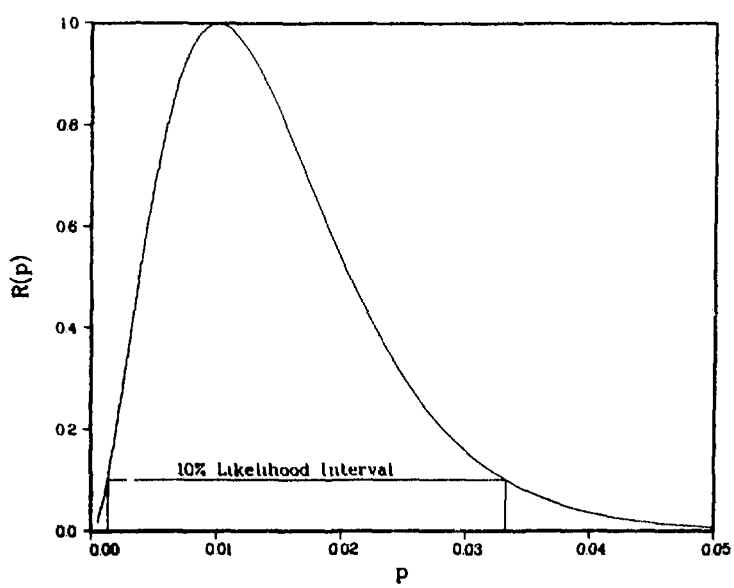

Fig. 1. A plot of the relative likelihood function R(p;2).

intervals have also been referred to as plausibility intervals (Singpurwalla[11]), credibility intervals (Jenkins and Watts[16]), and support intervals (Edwards[15]). For instance, in our example, values of p in the 10% likelihood interval (0.001, 0.033) have at least 10% relative likelihood and so are fairly plausible. On the other hand, values of p outside this interval are fairly implausible since there exist values of p (near $\hat{p}$) for which the observed data are at least 10 times more likely. The 50% likelihood interval can be interpreted as containing very plausible values of p (Whitney[14]). Here the 50% likelihood interval is (0.004, 0.021). The 10% likelihood interval is shown in Fig. 1.

One final note concerns the likelihood function. It is customary to refer to the joint probability function of the sample random variables as the likelihood function, denoted by $L(\theta;x)$, when considered as a function of the unknown parameter. In the case of the binomial model

$$L(p;x) = \binom{n}{x} p^x (1-p)^{n-x} .$$

EXTENDED LIKELIHOOD INFERENCE

We can extend the likelihood method to directly incorporate a prior distribution on the parameter of interest. A similar approach has been used by Blumenthal and Sanathanan[17] for constructing maximum likelihood estimates from truncated data. The prior distribution is assigned in order to account for its inherent uncertainty. This inherent uncertainty may be due to such factors as environmental effects, plant-to-plant differences, maintenance effects, and different operational demands. Such an approach is in widespread use in reliability (Schafer[18]) and was used in WASH-1400.[19] The main advantage of incorporating a prior distribution is that all available information regarding the parameter is used. The prior distribution is frequently chosen based on the best available subjective information concerning the parameter.

We restrict our attention to the case of a single unknown parameter θ which is to be estimated. We suppose that the prior distribution of θ contains the subjective data about θ in the form of prior parameter values in a specified prior model. For example, if a two-parameter gamma distribution is used as a prior model for the failure-rate of an exponential distribution, then the subjective data about θ are contained in the choices for the prior shape and scale parameters. In such a case, both the observed data and subjective data are contained in the extended likelihood function according to the following:

Definition: If $f(\underline{x};\theta)$ is the joint distribution of the observed sample data and $g(\theta;\underline{y})$ is the prior distribution of θ, then the extended likelihood function of θ is given by

$$L(\theta;\underline{x},\underline{y}) = g(\theta;\underline{y})\ f(\underline{x};\theta), \tag{1}$$

where $\underline{x} = (x_1 \ldots x_n)$ denotes the n-vector of observed sample random variables and $\underline{y} = (y_1 \ldots y_m)$ denotes the m-vector of prior parameter values. Later, these prior parameter values will be referred to as subjective data. Here the prior distribution $g(\theta;\underline{y})$ is used to weight the values of θ in $f(\underline{x};\theta)$. This concept is not new and the foregoing definition is included only for the sake of completeness.

It is a straightforward extension of the likelihood method to consider the following:

Definition: The extended relative likelihood function of θ, denoted by $R(\theta;\underline{x},\underline{y})$, is defined as

$$R(\theta;\underline{x},\underline{y}) = L(\theta;\underline{x},\underline{y})/\sup_{\theta} L(\theta;\underline{x},\underline{y}). \tag{2}$$

The interpretation of $R(\theta;\underline{x},\underline{y})$ is analogous to the interpretation of the (unextended) likelihood function $R(\theta;\underline{x})$ given in the preceding section.

There is one important advantage that $R(\theta;\underline{x},\underline{y})$ has over $R(\theta;\underline{x})$. If no observed data are available, then $R(\theta;\underline{y})$ may still be used to compare the relative likelihood that θ produced the subjective data. In particular, $R(\theta;\underline{y})$ gives the likelihood that θ produced the subjective data $\underline{y}$ relative to the modal value of $g(\theta;\underline{y})$. This will be further illustrated in the succeeding sections.

The set of values of θ for which $R(\theta;\underline{x},\underline{y}) \geq \gamma$ will be called the $100\gamma\%$ extended likelihood interval (ELI) for θ. The interpretation of this interval is the same as the (unextended) likelihood interval.

To illustrate these concepts, consider the case of a normal distribution. Suppose that we observe a normally distributed random variable x with unknown mean θ and known variance σ^2. Further suppose that θ has a normal prior distribution with mean λ and variance Ψ^2. Thus $\underline{y} = (\lambda, \Psi)$. After some algebraic manipulations we obtain

$$R(\theta; x, \underline{y}) = K \exp\left[-\left(\sigma^2+\Psi^2\right)\left(\theta - \frac{\sigma^2\lambda+\Psi^2 x}{\sigma^2+\Psi^2}\right)^2 / 2\sigma^2\Psi^2\right], \tag{3}$$

where

$$K = \exp\left[-(\sigma^2\lambda + \Psi^2 x)(\sigma^2\lambda + \Psi^2 x - 1)/(2\sigma^2\Psi^2)(\sigma^2+\Psi^2)\right].$$

From Eq. (3), the $100\gamma\%$ extended likelihood interval on θ is given by

$$\left(\frac{\sigma^2\lambda + \Psi^2 x}{\sigma^2 + \Psi^2}\right) - \left\{\frac{-2\sigma^2\Psi^2 \ln[\gamma/K]}{\sigma^2 + \Psi^2}\right\}^{\frac{1}{2}} \leq \theta$$

$$\leq \left(\frac{\sigma^2\lambda + \Psi^2 x}{\sigma^2 + \Psi^2}\right) + \left\{\frac{-2\sigma^2\Psi^2 \ln[\gamma/K]}{\sigma^2 + \Psi^2}\right\}^{\frac{1}{2}}. \tag{4}$$

Further, if it is known that $\sigma^2 = 0.25$, $\lambda = 2$, $\Psi^2 = 1$, and $x = 1$ was observed, then $K = \exp(-1.2)$ and the 10% extended likelihood interval on θ is easily computed to be (0.54, 1.86). For comparison, a 90% Bayesian probability interval on θ is found to be (0.47, 1.94) (see Hines and Montgomery,[20] p. 459, for the Bayesian interval equation). This comparison is presented only to illustrate the numerical agreement of the two procedures in this special case. It must be stressed that they are based on entirely different philosophies. The Bayesian interval emphasizes the probability of coverage which would be expected in repeated sampling experiments. On the other hand, the extended likelihood approach emphasizes the particular set of observed and subjective data.

THE EXPONENTIAL MODEL

The exponential model is widely used in reliability as a statistical model for time-to-failure at the component/subsystem/system level of analysis. In this section the necessary equations for use of the ELI methods in the exponential model are developed. These will also be used in the section PREDICTION IN THE EXPONENTIAL MODEL for solving several existing important and unsolved prediction problems regarding the exponential model.

Epstein[21-23] considered several possible life testing experiments; namely,

(i) Testing is terminated after a prespecified number of failures have occurred; failures are replaced.

(ii) Testing is terminated after a prespecified number of failures have occurred; failures are not replaced.

(iii) Testing is terminated after a prespecified time has elapsed; failures are replaced.

(iv) Testing is terminated after a prespecified time has elapsed; failures are not replaced.

(v) Testing is terminated either after a prespecified number of failures have occurred or after a prespecified time has elapsed, whichever occurs first; failures are replaced.

(vi) As in (v) except that failures are not replaced.

It is noted that case (iii) denotes the situation encountered when data are collected and reported for field operational devices, e.g., reactor component failures reported on an annual basis. By assuming an exponential time-to-failure model, Epstein gives confidence intervals for the mean-time-to-failure θ and some related reliability quantities. In some cases only approximate solutions were given. Reiser and Bar Lev[13] and Bar Lev[12] consider likelihood intervals based on these same testing experiments. Cases (i) and (ii) are sometimes referred to as Type II or censored lifetests, while cases (iii) and (iv) are called Type I or truncated lifetests.

Consider the exponential distribution given by

$$f(t;\lambda) = \lambda \exp(-\lambda t), \qquad \lambda, t > 0, \tag{5}$$

where t represents time-to-failure and λ is the failure-rate. The likelihood function for all six experiments is given by

$$L(\lambda;\underline{t}) = \lambda^{r} \exp(-\lambda T), \tag{6}$$

where $\underline{t} = (t_1, \ldots, t_r)$ is the vector of ordered observed failure times, r is the number of failures, and T is the total time on test which is defined below. Let n represent the number of items on test and let t* represent the test truncation time. Then T is defined in each corresponding experiment as

$$\text{(i)} \quad T = nt_r$$

$$\text{(ii)} \quad T = \sum_{i=1}^{r} t_i + (n-r)t_r$$

$$\text{(iii)} \quad T = nt^*$$

$$\text{(iv)} \quad T = \sum_{i=1}^{r} t_i + (n-r)t^*$$

$$\text{(v)} \quad T = nt_r, \text{ if } t_r < t^*$$
$$= nt^*, \text{ if } t_r > t^*$$

$$\text{(vi)} \quad T = \sum_{i=1}^{r} t_i + (n-r)t_r, \text{ if } t_r < t^*$$

$$= \sum_{i=1}^{r} t_i + (n-r)t^*, \text{ if } t_r > t^*$$

The gamma distribution is widely used as a prior model for λ. Schafer[18] investigated data from 32 different equipments and found that in 29 cases a gamma prior distribution adequately fit the data. This distribution will also be used here. The gamma prior distribution of λ is given by

$$g(\lambda;a,b) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} , \qquad \lambda,a,b>0, \tag{7}$$

where a and b are the prior shape and scale parameters, respectively. Because the prior mean of Eq. (7) is a/b, the parameter a can be interpreted as the number of failures in a prior lifetest of duration b hours. Waller et al;[24] Martz and Waller;[25] Grohowski, Hausman and Lamberson;[26] and Schick and Drnas[27] present simple methods for translating subjective prior percentile information about λ into corresponding values of a and b. These should be consulted when fitting gamma prior distributions.

The extended likelihood function according to Eq. (1) thus becomes

$$L(\lambda;\underline{t},\underline{y}) = g(\lambda;a,b)L(\lambda;\underline{t})$$

$$= \frac{b}{\Gamma(a)} \lambda^{a+r-1} e^{-\lambda(b+T)} \quad , \tag{8}$$

where $\underline{y} = (a,b)$. Further,

$$\sup_{\lambda} L(\lambda;\underline{t},\underline{y}) = \frac{b^a}{\Gamma(a)} \left(\frac{a+r-1}{b+T}\right)^{a+r-1} e^{-(a+r-1)} \quad , \tag{9}$$

which occurs at $\tilde{\lambda} = (a+r-1)/(b+T)$. Hence, the extended relative likelihood function given in Eq. (2) becomes

$$R(\lambda;\underline{t},\underline{y}) = \left(\frac{\lambda b + \lambda T}{a+r-1}\right)^{a+r-1} e^{-\lambda b - \lambda T + a + r - 1} \quad . \tag{10}$$

The solution to $R(\lambda;\underline{t},\underline{y}) \geq \gamma$ yields the $100\gamma\%$ ELI for λ. Upon taking logarithms and requiring that $a+r>1$, the $100\gamma\%$ ELI for λ in any of six lifetest experiments is given by the solution to the nonlinear inequality

$$(a+r-1)\ln\lambda - (b+T)\lambda - (a+r-1)\ln\left(\frac{a+r-1}{b+T}\right) + a+r-1-\ln\gamma \geq 0 \quad . \tag{11}$$

Also, for comparison, the usual $100(1-\alpha)\%$ Bayesian interval estimate of λ is given by (see Martz and Waller,[28] p. 32)

$$\chi^2_{\alpha/2;2r+2a}/(2T+2b) \leq \lambda \leq \chi^2_{1-\alpha/2;2r+2a}/(2T+2b) \quad , \tag{12}$$

where $\chi^2_{\alpha;\nu}$ is the $100(\alpha)$th percentile of a chi-square distribution with ν degrees of freedom.

Example 1: Consider a Type (iii) lifetest experiment for a certain component in which six failures were observed in a total of 3 504 000 operating hours. Further, suppose that a gamma prior distribution with $a = 1.5$ and $b = 1.0 \times 10^6$ hours is appropriate for this component. Assuming a constant failure-rate model (the exponential failure-time distribution), construct a 10% ELI for λ. Also, compute a 90% Bayesian interval estimate of λ and compare the results.

Solving Eq. (11) with $a = 1.5$, $b = 1.0 \times 10^6$, $T = 3{,}504\,000$, $r = 6.0$, and $\gamma = 0.10$ by means of a nonlinear root finder yields $(0.54 \times 10^{-6}$ f/h,

3.02×10^{-6} f/h) as the required 10% ELI. The extended relative likelihood function is plotted in Fig. 2 and the 10% ELI is indicated. From (12) the 90% Bayesian interval estimate of λ is computed to be $(0.81 \times 10^{-6}$ f/h, 2.77×10^{-6} f/h). Although the 10% ELI agrees quite well with the 90% Bayesian interval estimate, the ELI is slightly wider than the Bayesian interval. Thus, it is slightly more conservative.

<u>Example 2</u>: Consider the preceding example in which no test data are yet available. Based on a gamma prior distribution for λ with $a = 1.5$ and $b = 1.0 \times 10^6$ hours, a 10% ELI for λ may be computed by setting $T = r = 0$ in Eq. (11). Again, solving Eq. (11) by means of a nonlinear root finder yields the 10% ELI for λ given by $(1.85 \times 10^{-9}$ f/h, 3.82×10^{-6} f/h). The extended relative likelihood function is plotted in Fig. 3 and the 10% ELI is indicated. For comparison, the 90% Bayesian interval is $(1.76 \times 10^{-7}$ f/h, 3.91×10^{-6} f/h). The ELI interval is significantly wider than the Bayesian interval. The interpretation of the ELI interval here is as follows: values outside the interval $(1.85 \times 10^{-9}, 3.82 \times 10^{-6})$ have less than a 10% likelihood of having produced the subjective data $a = 1.5$, $b = 1.0 \times 10^6$. In other words, there exist values of λ close to the mode of

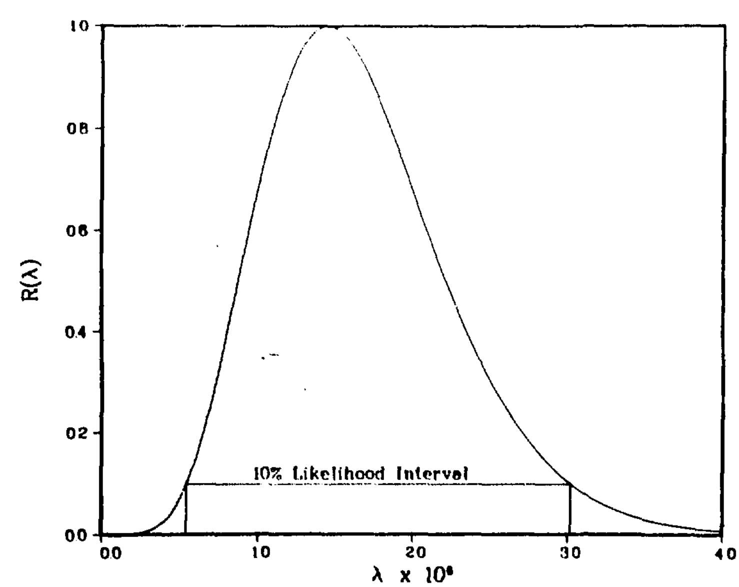

Fig. 2. A plot of the extended relative likelihood function in Example 1.

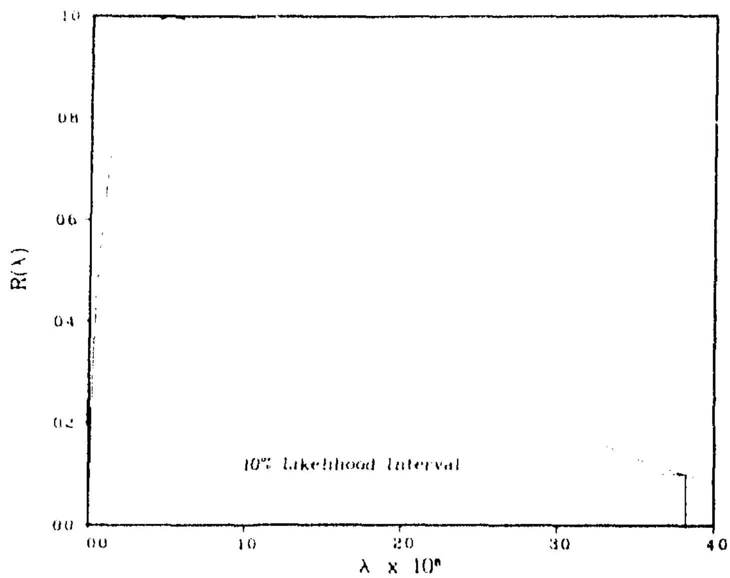

Fig. 3. A plot of the extended relative likelihood function in Example 2.

$g(\lambda; a,b)$ which are at least 10 times more likely to have produced a = 1.5, b = 1.0×10^6. The 90% Bayesian interval here may be interpreted as follows: the probability that λ lies within the interval (1.76×10^{-7}, 3.91×10^{-6}) is 90%. It is noted that the Bayesian interval emphasizes probability while the ELI emphasizes the particular set of subjective data used in the analysis. Since these are differing interpretations, no direct comparisons can be made between these intervals. Finally, by comparing the ELI with that in Example 1, it is observed that the use of objective test data shrinks the interval, as expected.

THE BINOMIAL MODEL

The binomial model is widely used in reliability as the appropriate model for use when only the survival/nonsurvival of a set of n items on lifetest is reported. It is also assumed that items fail independently of one another with the same probability of nonsurvival p. The necessary equations for constructing an ELI for p are developed and these will be further used in the section PREDICTION IN THE BINOMIAL MODEL where several important binomial prediction problems will be formulated and solved.

Consider the binomial distribution given by

$$f(x;p) = \frac{n!}{(n-x)!x!} p^x(1-p)^{n-x}, \qquad x = 0,1,\ldots,n, \qquad 0 \le p \le 1 \tag{13}$$

where x represents the number of nonsurvivors of a test of a specified duration in which n items are initially on test. The likelihood function $L(p;x)$ is also given by Eq. (13) when considered as a function of the probability of nonsurvival p.

The beta distribution is widely used as a prior model for p. It will also be used here and is given by

$$g(p;n_0,x_0) = \frac{\Gamma(n_0)}{\Gamma(x_0)\Gamma(n_0-x_0)} p^{x_0-1} (1-p)^{n_0-x_0-1}, \qquad n_0 > x_0 > 0, \tag{14}$$

where the prior parameter x_0 may be interpreted as the pseudo number of nonsurvivors in a prior test in which n_0 items are tested.

Weiler[29] and Waterman, Martz, and Waller[30] present a set of tables and graphs for use in translating subjective prior moment and percentile information about p into corresponding values of x_0 and n_0. These references should be consulted when fitting beta prior distributions.

The extended likelihood function from Eq. (1) becomes

$$L(p;x,\underline{y}) = \frac{\Gamma(n_0)n!}{\Gamma(x_0)\Gamma(n_0-x_0)(n-x)!x!} p^{x+x_0-1}(1-p)^{n+n_0-x-x_0-1}, \tag{15}$$

where $\underline{y} = (x_0,\ n_0)$. Further, it is easily determined that

$$\operatorname*{Sup}_{p} L(p;x,\underline{y}) = \frac{\Gamma(n_0)n!}{\Gamma(x_0)\Gamma(n_p-x_0)(n-x)!x!} \left[\frac{x+x_0-1}{n+n_0-2}\right]^{x+x_0-1} \left[\frac{n+n_0-x-x_0-1}{n+n_0-2}\right]^{n+n_0-x-x_0-1}, \tag{16}$$

which occurs at $\tilde{p} = (x+x_0-1)/(n+n_0-2)$. Thus, $R(p;x,\underline{y})$ upon simplification and collecting terms becomes

$$R(p;x,\underline{y}) = \left[\frac{p(n+n_0-2)}{x+x_0-1}\right]^{x+x_0-1} \left[\frac{(1-p)(n+n_0-2)}{n+n_0-x-x_0-1}\right]^{n+n_0-x-x_0-1} . \tag{17}$$

The solution to $R(p;x,\underline{y}) \geq \gamma$ yields the $100\gamma\%$ ELI for p. Upon taking logarithms and requiring that $n_0-x_0-1>0$, the $100\gamma\%$ ELI for p is given by the solution to the nonlinear inequality

$$\begin{aligned}(x+x_0-1)\ln p &+ (n+n_0-x-x_0-1)\ln(1-p) + (x+x_0-1)\ln(n+n_0-2) \\ &- (x+x_0-1)\ln(x+x_0-1) + (n+n_0-x-x_0-1)\ln(n+n_0-2) \\ &-(n+n_0-x-x_0-1)\ln(n+n_0-x-x_0-1) - \ln\gamma \geq 0 .\end{aligned} \tag{18}$$

For comparison, the usual $100(1-\alpha)\%$ Bayesian interval estimate of p is given by

$$\frac{x + x_0}{x+x_0+(n+n_0-x-x_0)F_{\alpha/2;2n+2n_0-2x-2x_0,2x+2x_0}} \leq p$$

$$\leq \frac{(x+x_0)F_{\alpha/2;2x+2x_0,2n+2n_0-2x-2x_0}}{n+n_0-x-x_0+(x+x_0)F_{\alpha/2;2x+2x_0,2n+2n_0-2x-2x_0}} , \tag{19}$$

where $F_{\alpha;\nu_1,\nu_2}$ is the $100(\alpha)$th percentage point of an F distribution with ν_1 numerator and ν_2 denominator degrees of freedom.

Example 3: Consider a certain component in typical use in light-water nuclear power reactors in the US. During a recent year, suppose that six failures of this component occurred out of 400 such components in use in a given plant during the year. Further suppose that a beta prior distribution with $x_0 = 1.4$ and $n_0 = 136.2$ is chosen based on best available published generic data for this component. Construct both a 10% ELI and a 90% Bayesian interval for the annual probability of nonsurvival p. Compare the results.

Solving Eq. (18) with $x = 6$, $n = 400$, $x_0 = 1.4$ $n_0 = 136.2$, and $\gamma = 0.10$ yields the required 10% ELI for p given by (0.0045, 0.0250). The extended relative likelihood function is plotted in Fig. 4 and the 10% ELI is indicated. From Eq. (19) the 90% Bayesian interval estimate of p becomes (0.0067, 0.0228) which is somewhat narrower than the ELI. However, they agree closely.

<u>Example 4</u>: Suppose that, in the preceding example, no test data are yet available, such as would be the case prior to startup of a new reactor. Thus $x = n = 0$. With the same prior distribution as in Example 3, the solution to Eq. (18) with $x = n = 0$, $x_0 = 1.4$, $n_0 = 136.2$, and $\gamma = 0.10$ yields the 10% ELI for p given by $(3.48 \times 10^{-6}, 2.64 \times 10^{-2})$. The extended relative likelihood function is plotted in Fig. 5 and the 10% ELI is shown. The corresponding 90% Bayesian interval for p becomes $(1.22 \times 10^{-3}, 2.64 \times 10^{-2})$. The Bayesian interval is observed to be significantly narrower than the ELI. The interpretation of these intervals is analogous to that given in Example 2. It is also noted that the incorporation of objective test data as in Example 3 significantly narrows the ELI, as expected.

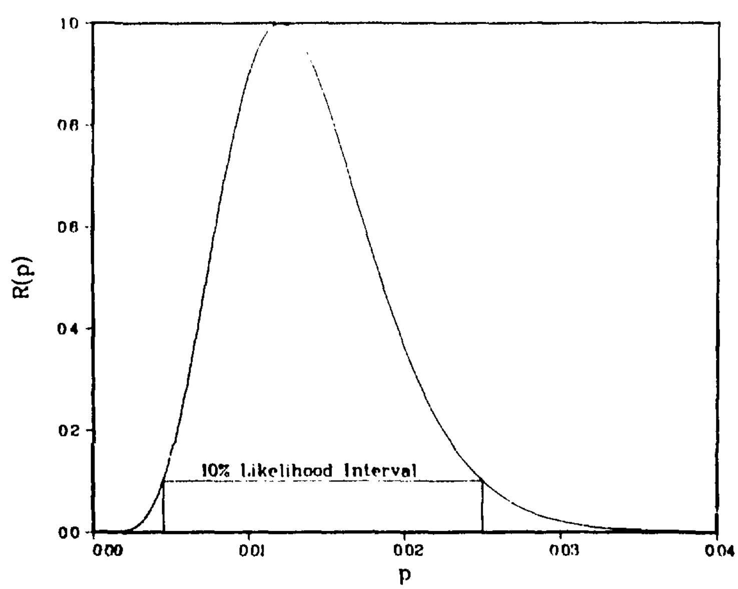

Fig. 4. A plot of the extended relative likelihood function in Example 3.

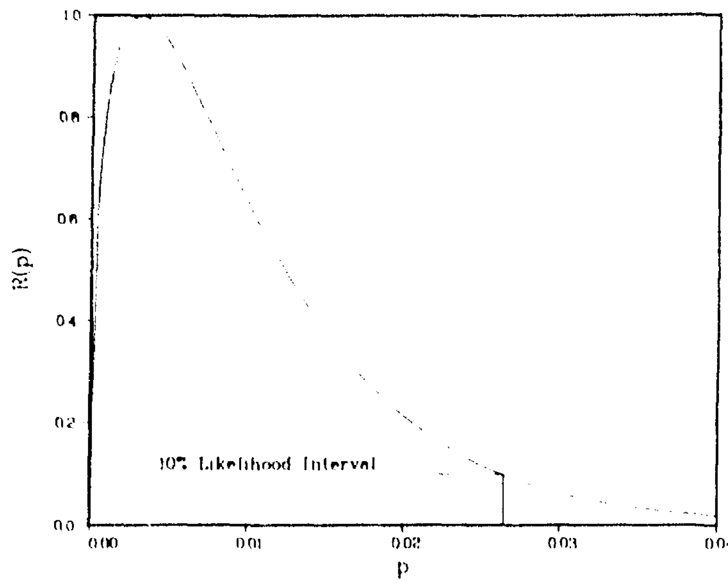

Fig. 5. A plot of the extended relative likelihood function in Example 4.

SECOND-ORDER EXTENDED LIKELIHOOD INFERENCE

Second-order likelihood methods for prediction have been developed by Kalbfleisch and Sprott,[6,7] and Kalbfleisch[9]. Fisher[31] also used this same argument to obtain the "likelihood" of a future observation. Bar Lev[12] and Reiser and Bar Lev[13] have used second-order likelihood methods for solving prediction problems related to the exponential distribution.

The notion of second-order likelihood will now be briefly summarized. Consider two independent samples $\underline{u} = (u_1 u_2 \ldots u_{n_1})$ and $\underline{v} = (v_1 v_2 \ldots v_{n_2})$ from specified distributions with underlying parameters θ_u and θ_v, respectively. Further, assume that the sample $\underline{v}$ is not entirely observable and further that a function of $\underline{v}$, say $h = h(\underline{v})$, is to be predicted. For example, in the case of a future sample from the same population it may be of interest to predict the number of sample observations that will fall in a given interval. Other examples will be found in the following two sections. Now, the joint information on θ_u and θ_v is summarized by the joint relative likelihood function given by

$$R(\theta_u, \theta_v; \underline{u}, h) = R_1(\theta_u; \underline{u}) R_2(\theta_v; h).$$

If $\underline{u}$ and h are given, then the plausibility that $\theta_u = \theta_v$ is measured by

$$R_M = \operatorname*{Sup}_{\theta_u=\theta_v} R(\theta_u,\theta_v;\underline{u},h) = \operatorname*{Sup}_{\theta} R_1(\theta;\underline{u})R_2(\theta;h).$$

Large (small) values of R_M provide support for (against) the claim that $\theta_u = \theta_v$. However, if it is <u>known</u> that $\theta_u = \theta_v$ and h is unknown, then it is desirable that $R_M = R_M(h)$ considered as a function of h be large. Thus, plausible values of h are those that make $R_M(h)$ large. The function $R_M(h)$ is referred to as the <u>second-order likelihood</u> of h. To summarize, the likelihood function rates the plausibility of θ according to how <u>likely</u> that value of θ makes a set of observed data that is known to have occurred. On the other hand, the second-order likelihood rates the plausibility of h according to how <u>likely</u> it makes the event $\theta_u = \theta_v$ which is known to be true.

Let us now extend the notion of second-order likelihood to directly incorporate a prior distribution on θ. A straightforward extension of the foregoing notions yields the following:

<u>Definition</u>: The <u>extended second-order likelihood</u> of h is defined as

$$R_M(h;\underline{x},\underline{y}) = \operatorname*{Sup}_{\theta} R_1(\theta;\underline{x},\underline{y})R_2(\theta;h) \ , \tag{20}$$

where $R_1(\theta;\underline{x},\underline{y})$ is the marginal extended relative likelihood function of θ corresponding to the first sample and $R_2(\theta;h)$ is the marginal relative likelihood function of θ in the second sample. It is noted that once θ has been realized according to the prior distribution $g(\theta;\underline{y})$, this value of θ is the same in both samples.

The set of values of g for which $R_M(h;\underline{x},\underline{y}) \geq \gamma$ will be called the $100\gamma\%$ <u>extended likelihood prediction interval</u> (ELPI) for h. The interpretation of this interval will be illustrated in Example 5 in the next section.

PREDICTION IN THE EXPONENTIAL MODEL

Three general prediction problems will be considered; namely, (i) based on either no observed data or data from any one of the six experiments described in the section THE EXPONENTIAL MODEL, prediction of the next failure-time; (ii) based on either no observed data or data from any one of the six experiments, prediction of the results in a future Type (iii) experiment; and (iii) based on either no observed data or data from any one of the six experiments, prediction of the results in a future Type (i) experiment. Examples illustrating each type of problem are given.

Prediction of the Next Failure Time

Consider a sample of observed data arising from any one of the six experiments described in the section THE EXPONENTIAL MODEL. Assuming a gamma prior distribution on λ, the extended relative likelihood function is given in Eq. (10). Suppose that we are interested in predicting the failure time v of another item taken from the same population in question. The likelihood of v in this second sample is

$$L(\lambda;v) = \lambda e^{-\lambda v} \ ,$$

with corresponding relative likelihood function

$$R_2(\lambda;v) = \lambda v \ e^{-\lambda v+1} \ .$$

Using Eq. (20), after some elementary algebraic manipulations, and simplification, the extended second-order likelihood of v becomes

$$R_M(v;\underline{t},\underline{y}) = v \left[\frac{(a+r)(b+T)}{(b+T+v)(a+r-1)}\right]^{a+r} \left[\frac{a+r-1}{b+T}\right] \ , \tag{21}$$

where $\underline{y} = (a,b)$. The $100\gamma\%$ ELPI for v is given by the solution to the nonlinear inequality

$$\ell n v - (a+r)\ell n(b+T+v) + (a+r)\ell n(a+r) + (a+r-1)\ell n(b+T) - (a+r-1)\ell n(a+r-1) - \ell n \gamma \geq 0 \ , \tag{22}$$

where we again require that $a+r-1 > 0$.

Example 5: Consider a Type (iii) lifetest experiment for a certain component in which six failures were observed in a total of 3 504 000 operating hours. Further, suppose that a gamma prior distribution with $a = 1.5$ and $b = 1.0 \times 10^6$ hours is appropriate for this component. Construct a 10% ELPI for another (future) failure-time observation from the same exponential distribution that generated the observed data.

Solving Eq. (22) with $a = 1.5$, $b = 1.0 \times 10^6$, $T = 3\,504\,000$, $r = 6$, and $\gamma = 0.10$ gives the required 10% ELPI as $(2.47 \times 10^4$ h, 4.58×10^6 h$)$. The extended second-order likelihood function is plotted in Fig. 6. The interpretation of this interval is as follows: a failure-time outside this interval has less than a 10% relative likelihood of ensuring that $\lambda_u = \lambda_v$ based on both the observed and subjective data. In other words, there exist values of v close to the supremum which ensure that the event $\lambda_u = \lambda_v$ is at least 10 times more likely to occur based on the observed and subjective data.

Example 6: In Example 5, suppose that no objective test data have been observed. Based on a gamma prior distribution for λ with $a = 1.5$ and $b = 1.0 \times 10^6$ hours, an ELPI for the first failure time v may be computed by setting $T = r = 0$ in Eq. (22). Solving Eq. (22) in this way yields the 10% ELPI for v given by $(4.09 \times 10^4$ h, 6.72×10^8 h$)$, which is considerably wider than the interval in Example 5 as expected. The extended second-order likelihood function for this example is plotted in Fig. 7.

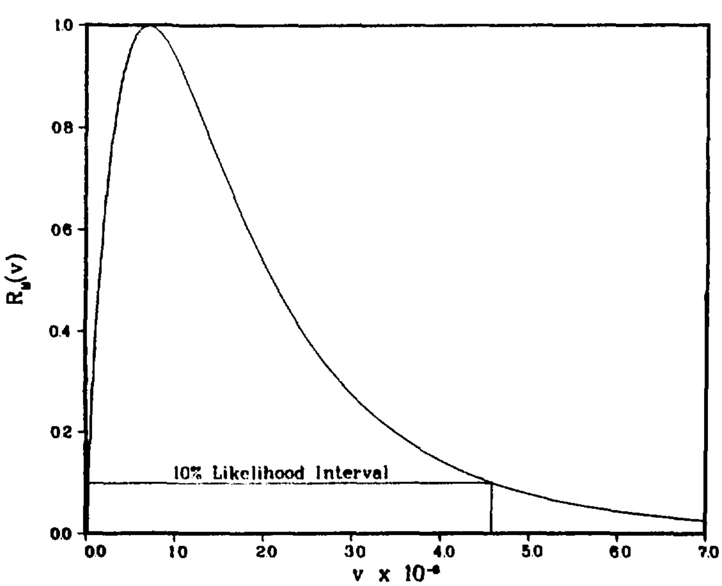

Fig. 6. A plot of the extended second-order likelihood in Example 5.

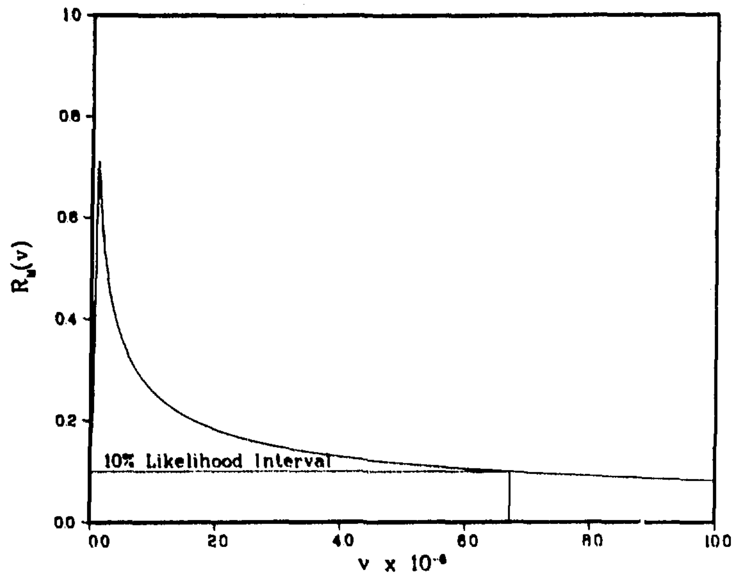

Fig. 7. A plot of the extended second-order likelihood in Example 6.

Finally, it is noted that classical prediction has been successfully accomplished for experiments (i) and (ii) by Lawless.[32,33] Classical prediction results for the remaining four cases have not been found. Also, (unextended) likelihood prediction results similar to those obtained here are given by Bar Lev[12] and Reiser and Bar Lev.[13]

Prediction in a Type (iii) Experiment

Consider a sampling experiment of Type (i) - (vi) and a gamma prior distribution on λ. From previous results, the extended relative likelihood of λ is given by Eq. (10). We shall consider the problem of predicting the number of failures s during a prespecified calendar time period τ in a future experiment in which there are m items on test and failures are to be replaced as they occur. This is a future Type (iii) experiment. The relative likelihood in this second (future) experiment is easily calculated to be

$$R_2(\lambda;s) = \begin{cases} \left(\frac{m\lambda\tau}{s}\right)^s e^{-m\lambda\tau + s} & , s \geq 1 \\ \\ e^{-m\lambda\tau} & , s = 0 \end{cases} \tag{23}$$

From Eq. (20), after some elementary algebraic manipulations and simplification, the extended second-order likelihood of s is given by

$$R_M(s;\underline{t},\underline{y}) = \begin{cases} \left[\dfrac{(a+r+s-1)(b+T)}{(b+T+m\tau)(a+r-1)}\right]^{a+r-1} \left[\dfrac{m\tau(a+r+s-1)}{s(b+T+m\tau)}\right]^{s} & , \; s > 0 \\ \\ \left[\dfrac{b+T}{b+T+m\tau}\right]^{a+r-1} & , \; s = 0 \end{cases} \tag{24}$$

where $\underline{y} = (a,b)$ as before. The $100\gamma\%$ ELPI for s is given by the solution to the nonlinear inequality

$$\begin{aligned}(a+r-1)\ell n(a+r+s-1) &+ s\ell n(m\tau) + s\ell n(a+r+s-1) - s\ell n(s) - s\ell n(b+T+m\tau) \\ &+ (a+r-1)\ell n(b+T) - (a+r-1)\ell n(b+T+m\tau) \\ &- (a+r-1)\ell n(a+r-1) - \ell n(\gamma) \geq 0, \qquad s > 0 \;,\end{aligned} \tag{25}$$

where we again require that $a+r-1 > 0$. For $s = 0$, the appropriate part of Eq. (24) would be used.

Example 7: Consider a Type (iii) experiment in which six failures were observed in a total of 3 504 000 operating hours during a recent one-year period. Further, suppose that a gamma prior distribution with $a = 1.5$ and $b = 1.0 \times 10^6$ hours is selected for the device under test. It is desired to predict the number of failures that will occur next year in which 400 devices will be in operation. This prediction is useful for determining the number of replacement devices that should be stocked as well as in determining the necessary maintenance policies.

Solving Eq. (24) with $a = 1.5$, $b = 1.0 \times 10^6$, $T = 3\ 504\ 000$, $r = 6$, $m = 400$, $\tau = 8760$, and $\gamma = 0.10$ yields the 10% ELPI for s given by (0.53, 13.54). Thus, conservatively, between 0 and 14 failures are anticipated next year with 10% likelihood. The 50% ELPI is (2.10, 9.19). The extended second-order likelihood function of s is plotted in Fig. 8 and the 10% ELPI is indicated.

<u>Example 8</u>: In Example 7, suppose that no objective test data are available. Based on the given prior distribution, it is desired to predict the number of failures that will occur in a one-year period in which 400 devices are in operation and failures will be replaced as they occur. Solving Eq. (24) with $a = 1.5$, $b = 1.0 \times 10^6$, $T = r = 0$, $m = 400$, $\tau = 8760$, and $\gamma = 0.10$, the 10% ELPI for s is found to be (0, 14.97). The extended second-order likelihood function of s is plotted in Fig. 9 and again the 10% ELPI is given.

Finally, it is worth noting that no classical prediction interval methods for predicting s are known to exist, although Epstein[22] does consider a confidence interval for the expected number of failures when predicting from a Type (i) to a Type (iii) experiment. Again, Bar Lev[12] and Reiser and Bar Lev[13] develop (unextended) likelihood prediction intervals analogous to those considered here.

Predicting in a Type (i) Experiment

Again, consider a sampling experiment of Type (i) - (vi) and a gamma prior distribution on λ. The extended relative likelihood of λ is again given by Eq. (10). We now consider the problem of predicting the calendar time τ at

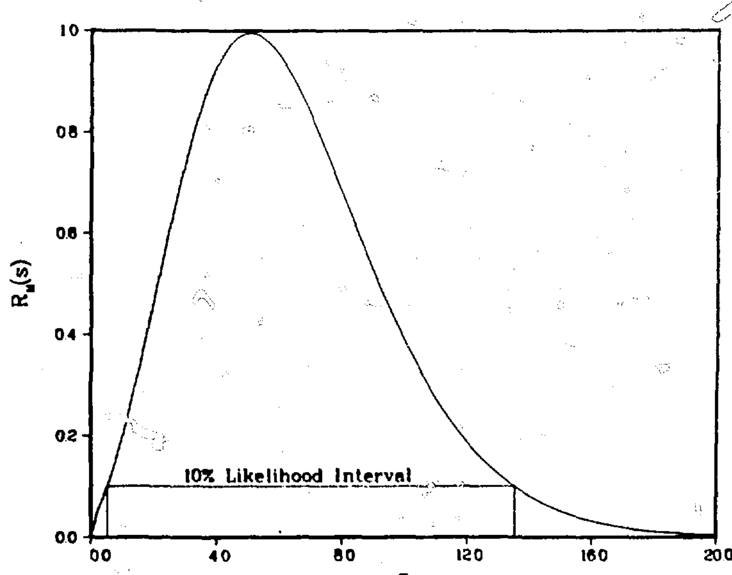

Fig. 8. A plot of the extended second-order likelihood in Example 7.

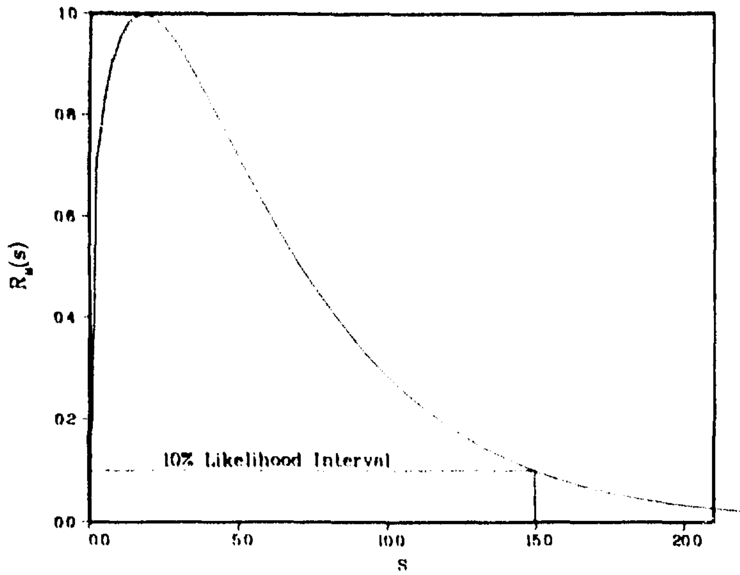

Fig. 9. A plot of the extended second-order likelihood in Example 8.

which the prespecified sth failure occurs in a future experiment in which there are m items on test and failures are to be replaced as they occur. This is a future Type (i) experiment. The relative likelihood in this second (future) experiment $R_2(\lambda;\tau)$ is given by Eq. (23) and the extended second-order likelihood of τ is given by the first portion of Eq. (24) ($s>0$) when considered as a function of τ instead of s. The $100\gamma\%$ ELPI for τ is obtained by solving the nonlinear inequality Eq. (25) for τ.

Example 9: Consider a Type (iii) experiment in which six failures were observed in a total of 3 504 000 operating hours during a recent one-year period. As before, consider a gamma prior distribution with $a = 1.5$ and $b = 1.0 \times 10^6$ hours. Suppose we wish to predict how long it will take in calendar time to observe four failures in a future experiment in which 400 devices will be in operation and failures will be replaced.

Solving Eq. (24) for τ with $a = 1.5$, $b = 1.0 \times 10^6$, $T = 3\,504\,000$, $r = 6$, $m = 400$, $s = 4$, and $\gamma = 0.10$ yields the 10% ELPI for τ given by $(1.55 \times 10^3$ h, 2.61×10^4 h). The extended second-order likelihood function of τ is plotted in Fig. 10 and the 10% ELPI for τ is shown.

Example 10: If no objective test data are available in Example 9, the 10% ELPI for τ is obtained by solving Eq. (24) with $a = 1.5$, $b = 1.0 \times 10^6$, $T = r = 0$, $m = 400$, $s = 4$, and $\gamma = 0.10$. The resulting 10% ELPI interval

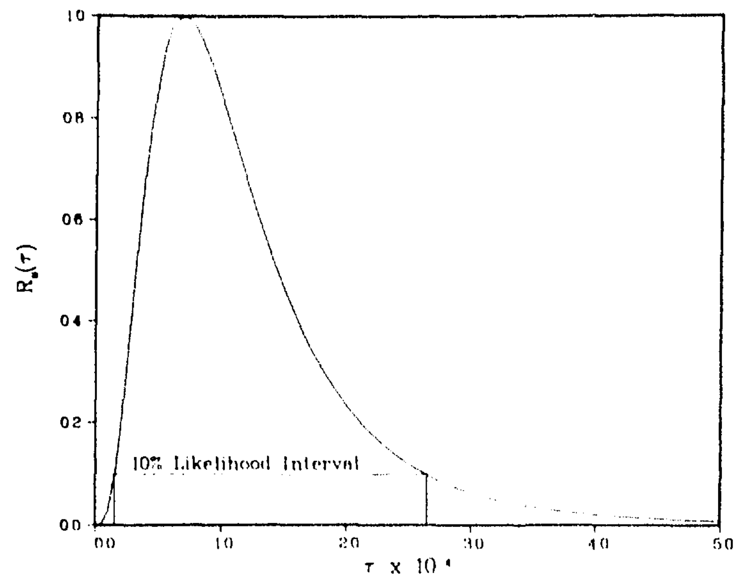

Fig. 10. A plot of the extended second-order likelihood in Example 9.

for τ is found to be $(1.70 \times 10^3$ h, 5.75×10^6 h), which is significantly wider than the interval given in Example 9.

PREDICTION IN THE BINOMIAL MODEL

Two general prediction problems will be considered; namely, (i) based on observed data from a binomial experiment, prediction of the number of nonsurvivors s in a future binomial experiment in which m items will be tested; and (ii) based on observed data from a binomial experiment, prediction of the required sample size to be tested in order to observe a specified number of nonsurvivors s.

Prediction of the Number of Nonsurvivors

Consider a binomial sampling experiment in which x nonsurvivors are observed among n items on test. Assuming a beta prior distribution on the probability of nonsurvival p, the extended relative likelihood function is given in Eq. (17). Suppose that we are interested in predicting the number of nonsurvivors s in a second (future) binomial experiment in which m items are to be tested. Now, the relative likelihood of p in the second experiment is given by

$$R_2(p;s) = \left(\frac{pm}{s}\right)^s \left[\frac{m(1-p)}{m-s}\right]^{m-s} . \tag{26}$$

Using Eqs. (17) and (26) in Eq. (20) yields the extended second-order likelihood of s given by

$$R_M(s;x,\underline{y}) = \left[\frac{n+n_0-2}{x+x_0-1}\right]^{x+x_0-1} \left[\frac{n+n_0-2}{n+n_0-x-x_0-1}\right]^{n+n_0-x-x_0-1}$$

$$\left[\frac{m}{s}\right]^s \left[\frac{m}{m-s}\right]^{m-s} \left[\frac{x+x_0+s-1}{n+n_0+m-2}\right]^{x+x_0+s-1} \left[\frac{n+n_0+m-x-x_0-s-1}{n+n_0+m-2}\right]^{n+n_0+m-x-x_0-s-1} \tag{27}$$

where $\underline{y} = (x_0,n_0)$. The $100\gamma\%$ ELPI for s is given by the solution to $R_M(s;x,\underline{y}) \geq \gamma$. The solution may be obtained upon taking logarithms and solving the resulting equation. It is also required that $n + n_0 > 2$ and that $x + x_0 > 1$ in order to solve the nonlinear inequality resulting from taking logarithms in Eq. (27).

Example 11: Suppose that a lifetest has been conducted in which six nonsurvivors were observed among 400 devices tested. Further, suppose that the probability of nonsurvival p follows a beta distribution with parameters $x_0 = 1.4$ and $n_0 = 136.2$. In a proposed future lifetest experiment of the same size, it is desired to predict the number of nonsurvivors s by computing a 10% ELPI for this number.

Solving $R_M(s;x,\underline{y}) \geq 0.10$ with $x_0 = 1.4$, $n_0 = 136.2$, $x = 6$, $n = 400$, and $m = 400$ gives (0.46, 12.89) as the required interval. Thus, conservatively, as few as 0 or as many as 13 failures are anticipated among the 400 test devices with 10% likelihood. It is noted that, once p has been selected by nature according to the beta distribution, this value is assumed to be the same for both experiments. Figure 11 shows the extended second-order likelihood function of s, and the 10% ELPI is indicated.

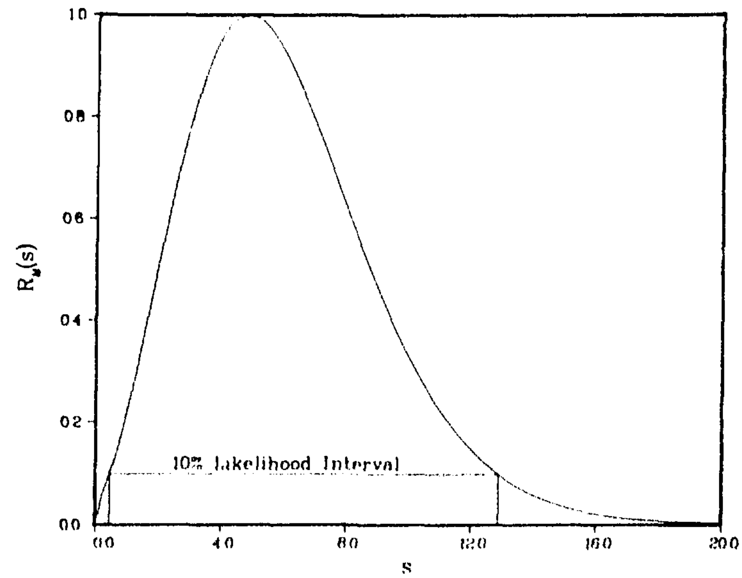

Fig. 11. A plot of the extended second-order likelihood in Example 11.

Prediction of the Sample Size

Based on the observed results of a binomial sampling experiment in which p has a beta distribution, suppose we are interested in predicting the sample size m required in a future binomial experiment in order to obtain s failures. The extended second-order likelihood $R_M(m;x,\underline{y})$ is given by Eq. (27) when considered as a function of m rather than s. The 100γ% ELPI for m is obtained by solving $R_M(m;x,\underline{y}) \geq \gamma$, which may again be effectively accomplished by taking logarithms. Again, we require that $n + n_0 > 2$, $x + x_0 > 1$, and $s > 0$.

Example 12: For the same objective and subjective data as in Example 11, how many items should be tested in a future experiment in order to obtain four nonsurvivors? Compute a 10% ELPI on this sample size.

Solving $R_M(m;x,\underline{y}) \geq 0.10$ with $x_0 = 1.4$, $n_0 = 136.2$, $x = 6$, $n = 400$, and $s = 4$ yields the 10% ELPI for m given by (76, 1279). Thus, as few as 76 or as many as 1279 items will ensure that four nonsurvivors occur with 10% likelihood. The second-order extended likelihood function is plotted in Fig. 12 and the 10% ELPI for m is indicated.

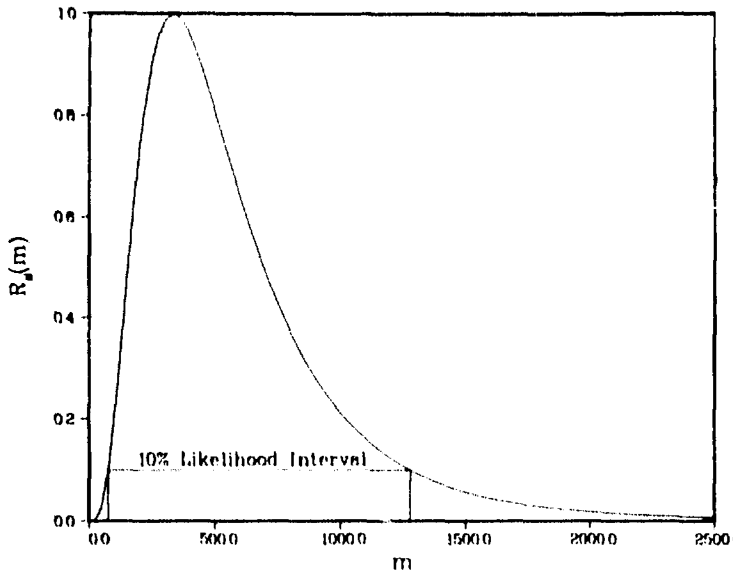

Fig. 12. A plot of the extended second-order likelihood in Example 12.

CONCLUSIONS

In this report, we have developed methods of extended likelihood inference which provide a basis for combining both subjective and objective information as an alternative to standard Bayesian methods of inference. Such an approach permits a unified treatment and solution to a number of problems associated with life testing. Several prediction problems associated with both the exponential and binomial models have been formulated and solved. In many of these problems, neither classical nor Bayesian solutions exist. The solutions presented here will be useful to reliability engineers engaged in such activities as test planning, maintenance scheduling, and spare-parts inventory control.

It is important to distinguish between Bayesian intervals and extended likelihood intervals. A Bayesian interval associates a probability that the interval contains the parameter of interest based on the observed and subjective data. On the other hand, an extended likelihood interval gives a range of values of the parameter of interest which makes the observed and subjective data fairly likely. Thus, non-probabilistic methods are used which emphasize the data on hand rather than probability arguments which imply repetition of the experiment under consideration. However, in those examples where Bayesian intervals were known to exist, the Bayesian intervals agreed quite closely with the extended likelihood intervals, although the extended likelihood intervals generally tended to be somewhat wider and thus more conservative. Evans[34] has recently advocated such an approach in reliability.

As pointed out by Barnard,[35] one of the main drawbacks of the Bayesian approach is that it does not always make clear the proportion of the final result that comes from the objective data and the proportion that is due to the subjective prior distribution. The use of extended likelihood methods remedies this situation. Since both (unextended) likelihood and extended likelihood have the same interpretation, it is possible to directly compare the resulting intervals for observing the reduction in the length of the interval due to the use of a prior distribution. For example, consider Example 3 in which $x = 6$, $n = 400$, $x_0 = 1.4$, and $n_0 = 136.2$. The 10% extended likelihood interval for p was found to be (0.0045, 0.0250), with a width of 0.0205. Now, using $x = 6$ and $n = 400$ ($\hat{p} = 0.015$), the (unextended) 10% likelihood interval for p is easily found to be (0.0054, 0.0320), with a width of 0.0266. Thus, the use of a prior distribution on p has the net effect of reducing the width of the 10% likelihood interval by 23%. There is an alternative way of determining the influence of the prior distribution. Consider the following question: How large would n have to be with $\hat{p}$ fixed at 0.015, in order for the width of the 10% (unextended) likelihood interval to be the same as the width of the 10% extended likelihood interval? By iteratively solving the relative likelihood function of p in the section INTRODUCTION, it is easily found that $n = 662$ yields a 10% likelihood interval whose width is 0.0205, the width of the 10% extended likelihood interval for p. This represents an effective 35% increase in the sample size, or 262 component tests. Thus, it may be concluded that the use of subjective data in the form of a prior beta distribution with $x_0 = 1.4$ and $n_0 = 136.2$ is equivalent to the use of 262 test units when computing 10% likelihood intervals.

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