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ABSTRACT

The development of constitutive laws for large-strain plastic flow requires both an appropriate kinematic framework to characterize the deformation and a suitable set of physical relations between the selected measures of stress and strain rate. In this paper it is argued that deformation is best characterized by taking, as the measure of strain rate, the stretching (the symmetric part of the velocity gradient) and assuming that it can be represented as the sum of an elastic and a plastic part. Though this is a natural extension (or perhaps only a restatement) of the 1930 hypothesis of Reuss, its consequences differ from some more recent hypotheses based on modern theories of deformation. Since plastic flow laws are expressed in rate form it is necessary to have a suitable definition of stress rate. Though this has been a subject of much analysis and numerous hypotheses, Dienes has shown that a unique stress rate follows from the necessity of formulating the constitutive law in material axes, and that such a stress rate is frame invariant. The same paper shows the relation of rate of angular velocity (material rotation rate), deformation and spin (vorticity). In this paper this formulation is used in expressing constitutive relations for plastic flow, including both ideal plasticity and kinematic hardening, and the results are compared with those obtained using the Zaremba-Jaumann-Noll approximation.

INTRODUCTION

The theory of plasticity can be extended to account for finite deformation in a variety of ways. In particular, Lee [1969] and Mandel [1973, 1981] have proposed that the deformation gradient F can be represented as the product of elastic and plastic parts F^eF^p, but this hypothesis does not, by itself, lead to a unique representation. A comprehensive discussion of this and related points has been put forth by Nemat-Nasser [1979] in which he argues that the deformation rate D can be represented as the sum of elastic and plastic parts without raising the paradoxical problems arising from the product decomposition indicated above. In addition to the kinematic issues discussed by Nemat-Nasser, other arguments in favor of the additive hypothesis for strain rates can be put forth. First, for flows without shear components it is straightforward to show that the choice of stretching (deformation rate) as a measure of strain rate is equivalent to the choice of logarithmic strain as a measure of strain. For such simple flows the addition of elastic and plastic strain rates becomes equivalent to taking the total stretch as the product of elastic and plastic stretches, a natural decomposition rule. Second, the stretching $D = (d_{ij})$ appears naturally in the calculations of the rate of work done on a deformable body V with surface S,

$$\dot{W} = \int_S u_i n_j \sigma_{ij} \, ds = \int_V \sigma_{ij} d_{ij} \, dv \quad . \tag{1}$$

It seems reasonable to preserve d_{ij} as a measure of strain rate in the formulation of constitutive laws since it necessarily appears in the calculation of internal energy. The separation of strain rate into a sum of parts allows the energy associated with each part to be computed, and these energies are additive. (Such energy considerations are fundamental in mechanics. It has been shown, for example, by Dienes [1978] that the energy equation can be used to derive the momentum equations, and that the converse does not hold.) Third, in finite difference calculations it is natural and straightforward to compute the velocity gradient $u_{i,j}$, and thus convenient to select this as the basis for computing strain rate (the symmetric part of the velocity gradient). None of these arguments is completely compelling and, in fact, any constitutive law relating any measures of stress and strain can be used for calculation. One should not expect, however, that all will be equally useful, stable, and accurate. The simplest and most natural hypothesis seems to be to adopt the stretching as the basic measure of deformation. In this article the consequences of this choice are pursued for ideal plasticity and kinematic hardening. The results seem plausible and straightforward.

CONSTITUTIVE RELATION

If it is assumed that the strain rate is the sum of elastic and plastic parts, it is necessary to formulate a constitutive law for each part. The elastic part of strain rate can be set proportional to the stress rate, as in hypoelasticity, but then it becomes important that the stress rate be suitably chosen. It has been shown by Dienes [7] that stress rate should be defined using ($\Omega = \dot{R}R^T$) as the rate of material rotation, and that Ω can be expressed exactly as the vorticity plus an additional rate term. For small deformations this stress rate is equivalent to the Zaremba-Jaumann-Noll (ZJN) stress rate, but for large deformations the ZJN stress rate is unrealistic and can lead to an instability.

Formulation of a constitutive law for ideally-plastic flow presents no real difficulty beyond defining the rates of stress and strain suitable, but a hardening plasticity theory is often needed for added realism, and such theories abound. They can be taken as either isotropic or anisotropic, but measurements of yield surfaces performed by Phillips and Tang [1972] tend to show that yield surfaces translate in stress space rather than expand isotropically. This behavior is represented in the kinematic hardening algorithm of Prager [1955], in which a back stress is defined by a rate equation. This formulation again raises the problem of defining stress rate in a suitable manner. It has been found by Nagtegaal and de Jong [1981] that use of the ZJN rate in connection with kinematic hardening leads to an unrealistic oscillatory behavior at large strains. As a result, Lee, Mallett and Wertheimer have proposed an alternative kinematics of plasticity [1983], while Dafalias [1983] has considered the finite deformation stress rate described by Green [1967] and studied in detail by Dienes [1979]. An alternative is proposed in this paper based on the finite deformation stress rate and the additive assumption for resolving strain rate into elastic and plastic parts. In an example, the consequences of these assumptions are compared with the ZJN theory for pure shear, and it is shown that the proposed approach leads to plausible results at large strain, whereas the ZJN theory leads to the unrealistic oscillation observed by Nagtegaal and de Jong, and no correction (as in Reuss' original theory [1930]) leads to paradoxical behavior. Similar results have been recently presented by Key [1983], but the emphasis in this paper is somewhat different, focussing on the definition of strain rate, and demonstrating how the current approach eliminates the unrealistic behavior at small strains obtained by Dafalias as a result of assuming rigid-plastic behavior.

THEORY

The classical approach to problems of plastic flow is to express the strain rate as the sum of elastic and plastic contributions

$$D = D^e + D^p \tag{2}$$

as suggested by Reuss (op. cit.) and to formulate separate constitutive laws for the terms on the right. Though a variety of definitions of strain rate have been proposed, many of which are described by Eringen [1967], the most straightforward is to follow the usage in hypoelasticity in which the velocity gradient G is taken as the sum of symmetric and antisymmetric parts, and the symmetric part, D, is selected to be the strain rate, so that

$$(u_{i,j}) = G = D + W \tag{3}$$

where W is the vorticity. This notation and much of the terminology used here follow Truesdell [1966]. Though the term "stretching" for D does not present any difficulty, when it is identified as the "strain rate" a semantic difficulty arises because this suggest that it represents the rate of a well-defined strain. Though this idea works for small deformations, for finite deformations no completely suitable strain has been defined in terms of the deformation, and, so far as I know, there is none whose rate is D. This is not a great loss, since an alternative quantity, the stretch, characterizes deformation adequately. Of course, one can define the strain matrix as the logarithm of the stretch matrix, and in the absence of shear this strain represents the logarithmic strain, but in general no useful result is forthcoming, and its rate of change is not D.

As the constitutive relation for the elastic component of strain rate D^e it is natural to take the stress rate to be linear in the stress rate. For small deformations the stress rate is adequately characterized by the approximate rate of Zaremba [1903], Jaumann [1911], and Noll [1955]

$$\check{\sigma} = \dot{\sigma} - W\sigma + \sigma W \tag{4}$$

but for large deformations it is necessary to use an exact formulation, as discussed by Dienes [1979], in which

$$\hat{\sigma} = \dot{\sigma} - \Omega\sigma + \sigma\Omega \tag{5}$$

where Ω denotes the rate of material rotation

$$\Omega = \dot{R}R^T \tag{6}$$

and R denotes material rotation. This expression for stress rate has been noted by Green and McInnis [1967], Storen and Rice [1975] and others, but the relation of Ω to vorticity, W, summarized in the subsequent paragraph, was derived by Dienes [1979].

The basis for the analysis of large deformation is the polar decomposition of the deformation gradient

$$F = \left(\frac{\partial x_i}{\partial \xi_j}\right) = VR \tag{7}$$

where x_i represents the coordinates of the point initially at ξ_i, V is the (positive definite) stretch and R is orthogonal. To obtain the relation of Ω and W define

$$Z = DV - VD \tag{8}$$

and

$$S = (I \text{ tr } V - V)^{-1} \tag{9}$$

where tr V denotes the trace of V and I denotes the identity matrix. The antisymmetric matrix Z can be represented as a vector with components z_i by means of the permutation symbol ε_{ijk} so that

$$Z_{ik} = \varepsilon_{ijk} z_j \quad . \tag{10}$$

Then, if one defines

$$A_{ik} = S_{j\ell} z_\ell \varepsilon_{ijk} \quad , \tag{11}$$

it can be shown by direct algebraic calculation that

$$\Omega = W + A \quad , \tag{12}$$

where A is the matrix of the A_{ij}. This algorithm has been successfully used in calculations of large deformation with SCRAM, a version of the finite difference SALE code developed by Amsden, Ruppel, and Hirt [1980], with the additional relation

$$\dot{V} = GV - V\Omega \quad , \tag{13}$$

used to update V in the course of the calculation. These results are kinematic and do not depend on material behavior.

The plastic stretching is taken proportional to the deviator stress, σ',

$$D^p = \lambda\sigma' = \lambda(\sigma - \bar{\sigma}I) \tag{14}$$

in ideal plasticity, and λ can be determined from the yield condition

$$\sigma'_{ij}\sigma'_{ij} = 2Y^2 \tag{15}$$

to be

$$\lambda = \sigma'_{ij}d^p_{ij}/2Y^2 \quad . \tag{16}$$

These equations can be combined to obtain an expression for the rate of change of deviator stress

$$\dot{\sigma}' = \Omega\sigma' + \sigma'\Omega + 2\mu(D' - \lambda\sigma') \tag{17}$$

that can be used conveniently in numerical calculations to update the stress. In impact and explosion problems it is necessary to supplement these equations with an equation of state for the mean stress $\bar{\sigma}$. In this paper, however, we consider only the simplest flows.

Before proceeding to a discussion of hardening behavior it will prove useful to consider first the consequences of these constitutive relations in the example of steady simple shear, for which

$$x_1 = \xi_1 + v(\xi_2)t \quad , \quad x_2 = \xi_2 \quad , \quad x_3 = \xi_3 \quad . \tag{18}$$

I showed in the previously cited paper that for such a flow

$$\Omega_{12} = \dot{\beta} \tag{19}$$

where β is defined by

$$e = \frac{dv}{d\xi_2}\, t = 2 \tan \beta \quad , \tag{20}$$

and that the vorticity is given by

$$W_{12} = \dot{e}/2 = \dot{\beta}/(1 + \beta^2) \quad . \tag{21}$$

Thus, Ω_{12} is well approximated by W_{12} for small strain, but the approximation fails for large strain. For such a flow the stretching is given by

$$D = \begin{pmatrix} 0 & \dot{e}/2 \\ \dot{e}/2 & 0 \end{pmatrix} \quad . \tag{22}$$

It follows that the mean stress satisfies

$$\bar{\sigma} = \bar{\sigma}^{o} e^{-\mu W/Y^2} \tag{23}$$

where $\bar{\sigma}^{o}$ is the initial value of the mean stress. Hence, if $\bar{\sigma}^{o}$ is initially zero, it is zero for all times. Since the flow stress remains on the yield surface after initial yielding, it follows that it depends on a single variable, which we select as θ, such that

$$\sigma_{11} = Y \sin \theta \quad , \quad \sigma_{12} = Y \cos \theta \quad . \tag{24}$$

With this constraint, each of the flow equations (17) reduces to

$$\frac{d\theta}{de} + \frac{\sin \theta}{2\varepsilon} = \frac{1}{1 + e^2/4} \tag{25}$$

where

$$\varepsilon = Y/2\mu \quad . \tag{26}$$

To determine the initial conditions we may use the elastic solution of Dienes [1979], which shows that yield occurs when $\theta = \varepsilon$, to first order, under the (realistic) assumption that ε is small. No closed form solution to (25) is known, but an approximate solution can be obtained by putting $\theta = \varepsilon\mu$.

$$\mu = 2 - \exp\,(1 - e/2\varepsilon) - e^2/2 + 2\varepsilon e \quad . \tag{27}$$

This expression for θ has a maximum of 1.99 for e = 0.14 when ε is set to 0.01. A numerical solution of (25) is shown in Figure 1. The maximum agrees with the analytic results, but the numerical solution shows in addition that θ goes to zero for large e. The figure also shows the behavior of θ when the effect of material rotation is ignored, which is tantamount to dropping the right side of (24). In that case θ drops rapidly to zero. If the ZJN approximation is used the right side of (25) becomes unity, and the resulting behavior of θ is also illustrated in the figure. In that case θ increases monotonically to .02 with increasing strain and remains at that value. In summary, it appears to be necessary to account for finite deformation in computing stress rate in plasticity as well as in elasticity if realistic material behavior is to be obtained at strains larger than 0.4.

KINEMATIC HARDENING

In their analysis of hardening behavior Nagtegaal and de Jong [1981] observed spurious oscillatory behavior of stress with increasing strain. Dafalias [1983] recently showed that this oscillatory behavior is a consequence of using the ZJN approximation, and disappears when the finite deformation theory is used. He was able to obtain an analytical solution for the stresses in a simple shear flow by assuming rigid-plastic behavior. Here we consider kinematic hardening with the finite deformation theory used to characterize both the elastic strain rate and the back stress. Though I described a general kinematic hardening theory suitable for high-pressure behavior in an earlier paper [Dienes, 1975] only the simplest kinematic hardening theory is

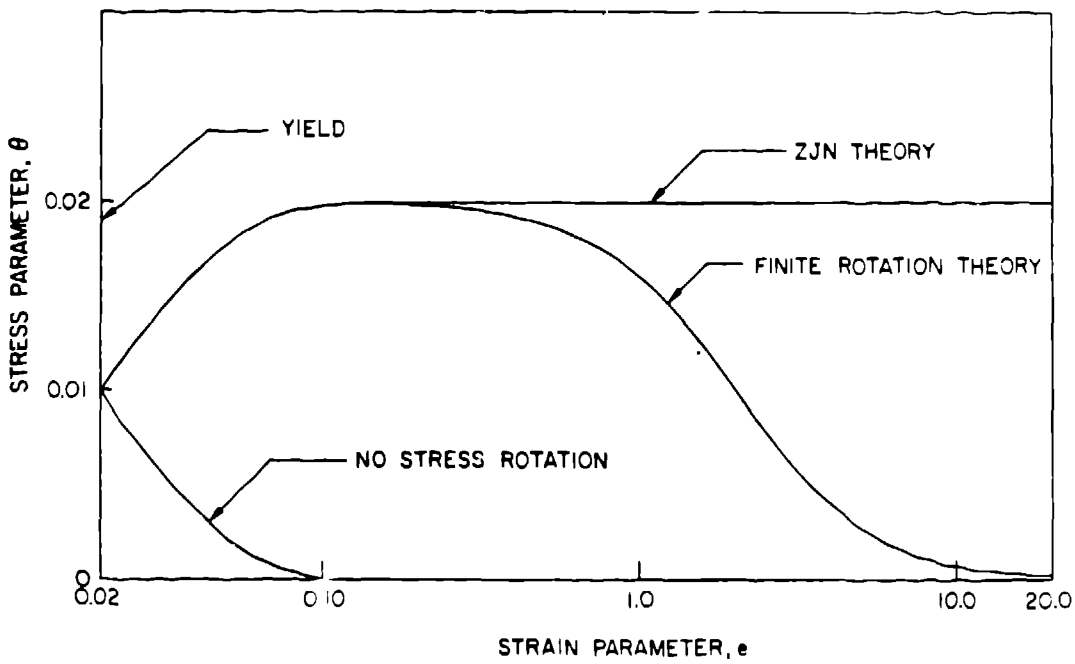

Fig. (1) - A comparison of the stress parameter θ ($\sigma_{12} = Y \sin \theta$) for simple shear; (1) in the absence of stress rotation (Reuss); (2) using the Zaremba-Jaumann-Noll (ZJN) approximation; and (3) using finite rotation theory. For e less than 0.02 the behavior is elastic. The ZJN approximation produces insignificant error for strains below 0.2.

considered here, with the object of illustrating the importance of an adequate stress rate theory.

The experimental results of Phillips and Tang [op. cit.] show that the subsequent yield surfaces observed after loading resemble those of kinematic rather than isotropic hardening theory. In the simplest representation of such behavior, the deviator stress is considered as the sum of a back stress α and a plastic stress $\bar{S}$

$$\sigma' = \alpha + \bar{S} \tag{28}$$

with $\bar{S}$ lying on a displaced yield surface, so that

$$\bar{S}_{ij}\bar{S}_{ij} = 2Y^2 \tag{29}$$

where Y is the yield stress in simple shear. Following Prager [1955], the back stress rate is assumed proportional to the plastic strain rate

$$\hat{\alpha} = \dot{\alpha} - \Omega\alpha + \alpha\Omega = bD^p \tag{30}$$

where b is the hardening modulus of the material, except that Ω rather than W is used for the rate of material rotation. The generalization to hardening plasticity is completed by the relations

$$D^e = \hat{S}/2\mu + I\dot{\sigma}/3k \tag{31}$$

where σ denotes the mean stress; k, bulk modulus; and

$$D^p = (\lambda/2Y)\tilde{S} \quad . \tag{32}$$

With these assumptions it is straightforward to show that

$$\lambda = \gamma\tilde{S}_{ij}d_{ij} = \gamma\dot{W} \tag{33}$$

where

$$1/\gamma = (1 + b/2\mu)Y \quad . \tag{34}$$

Using the same procedure as in the analysis of ideally plastic behavior, it is straightforward to show that, if $Z = \alpha_{11} + \alpha_{22}$ vanishes initially, then it remains zero throughout the deformation. As before, it proves natural to set

$$\tilde{S}_{11} = Y \sin\theta \quad , \quad \tilde{S}_{12} = Y \cos\theta \quad . \tag{35}$$

Direct calculation shows that θ is still governed by (25) in the presence of hardening.

If we define the dimensionless quantities

$$a_1 = \alpha_{11}/b \quad , \quad a_2 = \alpha_{12}/b \quad , \quad 1/q = 2 + b/\mu \tag{36}$$

then these equations reduce to the pair

$$\frac{da_1}{de} = q \cos\theta \sin\theta + \frac{a_2}{1 + e^2/4} \tag{37}$$

$$\frac{da_2}{de} = q \cos^2\theta - \frac{a_1}{1 + e^2/4} \quad . \tag{38}$$

The results of numerical integration of these equations are are illustrated in Figures 2 and 3, and, as in Figure 1, comparisons are made with the ZJN approximation and with the behavior calculated in the absence of a correction for rotation. These results illustrate that it is important to account for material rotation exactly if reasonable results are to be obtained for large deformation. The results are similar to those of Dafalias except for small strains, since here the rigid-plastic assumption is relaxed. (This makes it necessary to perform the integrations numerically, whereas Dafalias obtains an analytic solution.) In general, however, numerical solutions are required, and the algorithm of equations (8-13) seems well suited for that purpose. In referring to this algorithm both Lee et al [1983] and Dafalias [1983] suggest

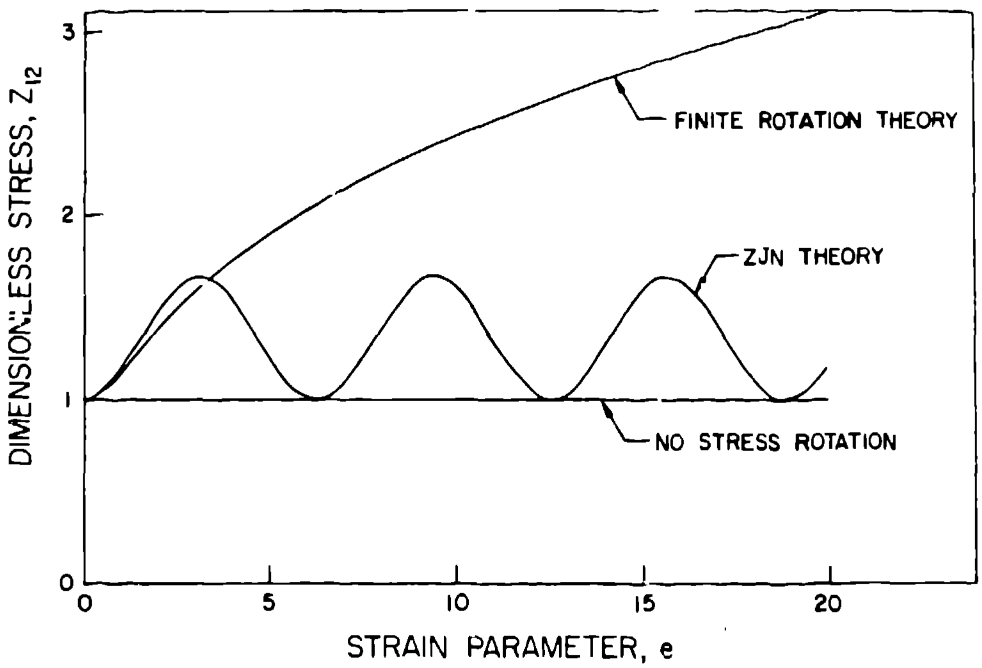

Fig. (2) - Plot of the dimensionless shear stress as a function of the strain parameter e, comparing solutions: (1) in the absence of stress rotation (Reuss); (2) in the ZJN approximation; and; (3) using finite rotation theory.

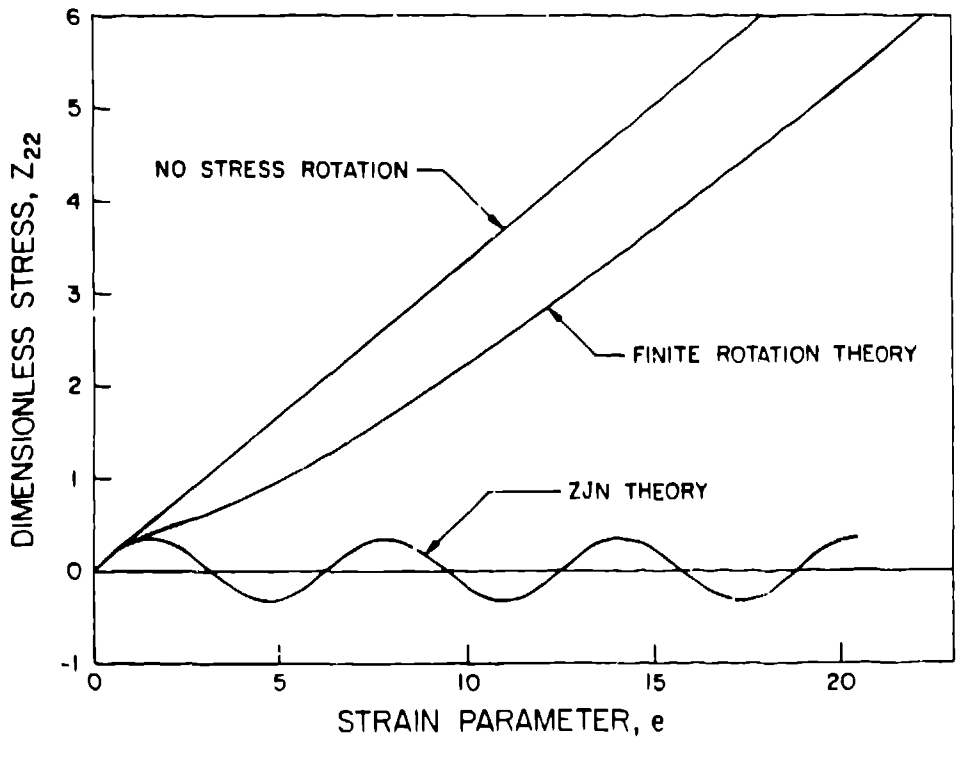

Fig. (3) - Plot of the dimensionless normal stress as a function of the strain parameter e, comparing solutions: (1) in the absence of stress rotation (Reuss); (2) in the ZJN approximation; and; (3) using finite rotation theory.

that it is valid only for hypoelastic deformations. This misinterpretation appears to arise from considering only the example I selected, which involves the simplest material for which a rate law formulation is appropriate. The underlying theory, however, is entirely general, involving only kinematics, and makes no material assumptions beyond those usual in continuum mechanics and the polar decomposition theorem.

In summary, the approach to plasticity outlined here maintains Reuss' classical superposition of elastic and plastic strain rates, but requires that the rate of material rotation Ω, given by (12), be used in computing stress rate, rather than its approximate representation, the spin (vorticity). In an example, it is shown herein that this approach provides reasonable results and can be readily implemented.

Professor Bell has performed an important service in organizing and extending experimental results concerning plastic flow [1979, 1981]. Their full understanding will require significant theoretical advances, including a better treatment of the role of microstructure. At large deformations it is necessary to account for the kinematics of deformation precisely, and continuum mechanics, especially the consequences of polar decomposition, provides a useful approach. The kinematics and physics of large deformation are particularly important in the analysis of material behavior on the microstructural level, where crystal anisotropy and mechanical instabilities play important roles. For example, microstructural instabilities probably underlie the second-order transitions observed by Bell. The behavior of microstructure is even more important in its role in the nucleation of fracture. Our understanding of the relationship of instabilities at the microstructural level and material strength has only just begun. To make significant progress it appears that it will be necessary to combine careful experiments, computer simulation, more precise kinematics, and condensed-matter physics, a process which has only recently started.

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