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THE TILTING MODE IN FIELD-REVERSED CONFIGURATIONS

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Field Reversed Configurations (FRCs) experimentally have exhibited remarkable stability on the magnetohydrodynamic (MHD) timescale,[1-3] despite numerous MHD calculations showing FRCs to be unstable.[4-11] It is easy to believe that local modes are stabilized by finite Larmor radius (FLR) effects, but more puzzling is the apparent stability of FRCs against global modes, where one would expect FLR effects to be less important. In this paper we study the tilting mode, which MHD has shown to be a rapidly growing global mode. The tilting mode in FRCs is driven by the pressure gradient, and magnetic compression and field line bending are the stabilizing forces. A schematic of the evolution of the tilting mode is shown in Fig. 1. The tilting mode is considered dangerous, because it would lead to rapid tearing across the separatrix (see Fig. 1c). Unlike spheromaks, the tilting mode in FRCs has a separatrix that is fixed in space, so that the mode is strictly internal.

MHD Results

We have studied the MHD stability of the tilting mode with two independent codes[12]: a trial function code that computes eigenfrequencies, and a linear time dependent MHD simulation code. The principal conclusions from our linear MHD calculations are:

1.) All FRC equilibria with flux surfaces ranging from elliptical to highly racetrack are unstable to tilting with typical growth times (for FRX-B) $\gamma^{-1} \sim 1\mu s$, whereas experimental lifetimes are $\tau_{life} \sim 20\text{-}40\mu s$. (See Fig. 2a-c.)

2.) The displacement $\vec{\xi}$ in the (r,z)-plane is primarily axial, $\vec{\xi}_n + \vec{\xi}_\parallel = \xi_z\hat{z}$, where $\vec{\xi}_n$ and $\vec{\xi}_\parallel$ are the normal and parallel displacements of a fluid element from a flux surface (ξ_θ is determined from incompressibility). If ψ labels a flux surface and s is the arclength along a B-line, we find that $\xi_z(\psi=0,s) = 0$. That is, the axial displacement vanishes at the separatrix, $\psi = 0$. This means that the tilting mode is internal, so the calculation is carried out in ψ-space from $\psi = -1$ (O-point) to $\psi = 0$. (See Fig. 2d-f.)

3.) For elliptical equilibria each flux surface has a rigid axial displacement, $\xi_z(\psi,s) = \xi_z(\psi)$, and the maximum displacement occurs at the O-point. (See Fig. 2d.)

4.) For racetrack equilibria $\xi_z(\psi,s)$ is a strong function of s, with the axial displacement localized to the tips of the flux surfaces. (See Fig. 2e,f.)

Several reasons have been proposed for the observed stability of FRCs against tilting: 1) Though unstable the instability saturates at low amplitude (the 3-D MALICE code shows this is not true for elliptical equilibria); 2) Nonideal effects might stabilize the mode (Shestakov's resistive linear MHD code still shows instability); 3) FRC spin-up might stabilize the tilt (but $\tau_s >> \gamma^{-1}$); 4) Kinetic effects might be important even for this global mode. We feel that the last possibility is the most likely, particularly since there are "betatron" particles that have large radial orbits about the field null (see Fig. 3b).

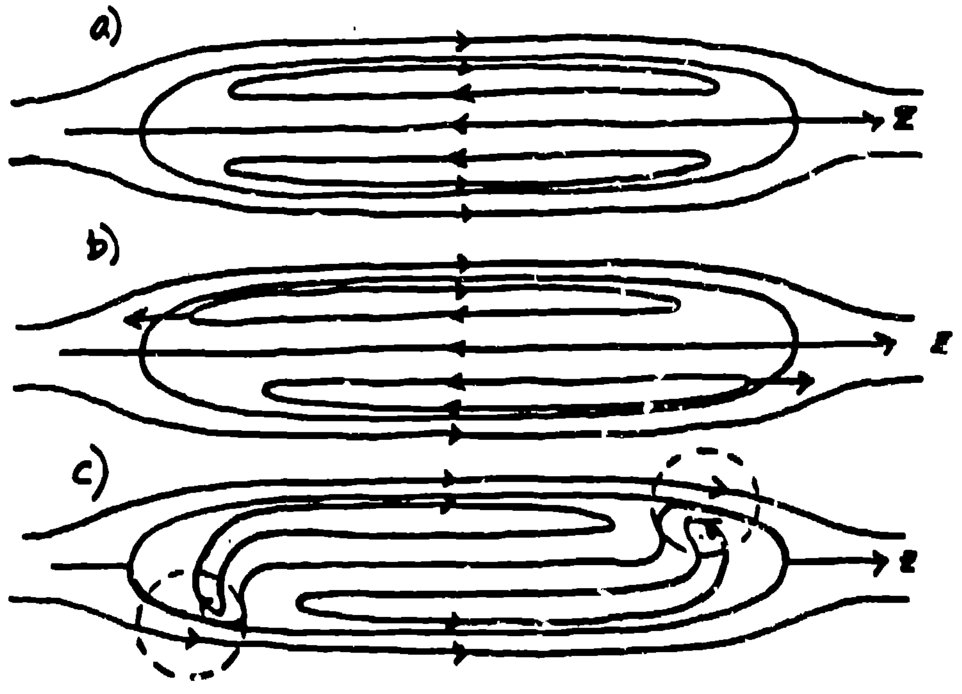

Figure 1. Schematic evolution of the tilting mode. a) Equilibrium state, b) Linear mode is an axial shift of flux surfaces, c) Nonlinear mode may lead to tearing in dashed circular regions.

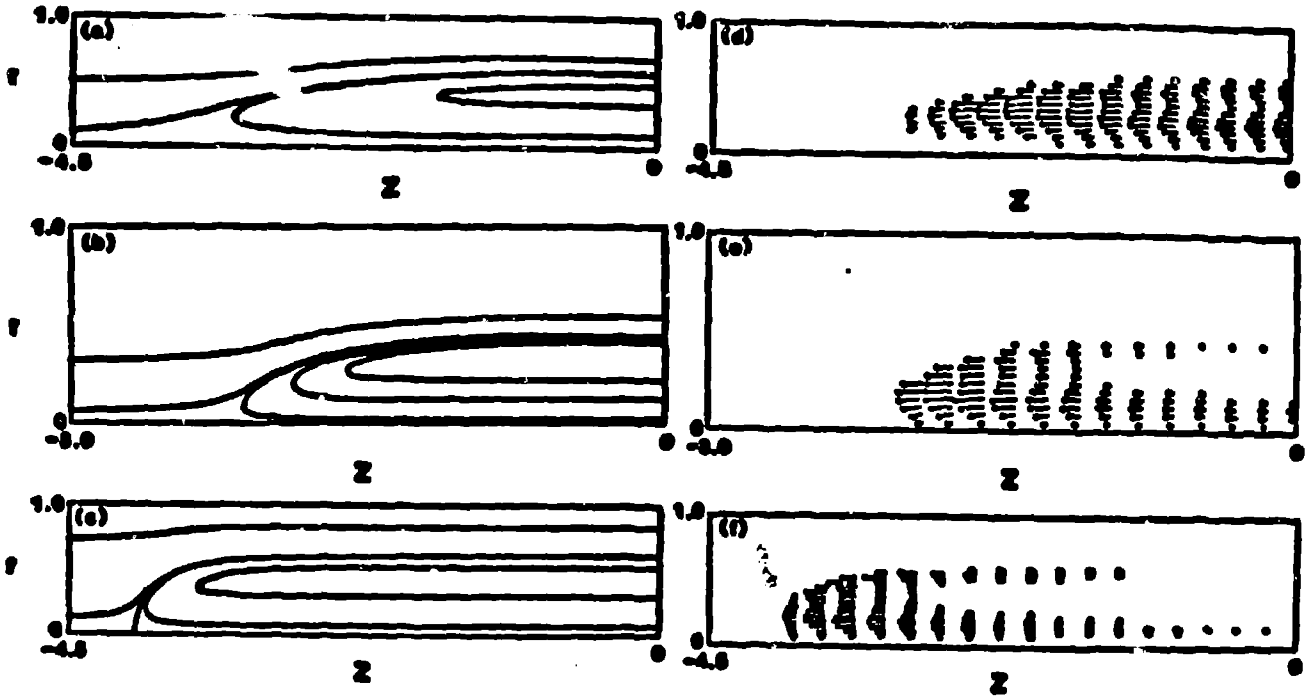

Fig. 2 Realistic numerical equilibria, (a) elliptical case, (b) FRX-B parameters, (c) highly racetrack case. For each equilibrium (a)-(c), in (d)-(f) are shown the corresponding projections of the displacement vector ξ in the (r,z)-plane from the initial value code.

Kinetic Treatment

An exact linear stability formalism for inhomogeneous Vlasov-Maxwell systems has been devised by Lewis, Symon, and Seyler.[13] Seyler and Barnes[14] used this formalism to study kinetic effects on the tilting mode with the Vlasov-fluid model (Vlasov ions; cold, massless electrons). Let $f_1(\vec{q},\vec{p};\omega)$ be the Laplace transform of the first order distribution function of the ions. Since θ is an ignorable coordinate of the equilibrium, all perturbation quantities have θ-dependence of $e^{in\theta}$, where n is a fixed integer for the problem. In general we can expand

$$f_1(\vec{q},\vec{p};\omega) = \sum_{r} \Gamma_r(\omega)\, w_r(\vec{q},\vec{p}). \qquad (1)$$

The functions w_rs are the Liouville eigenfunctions, and they satisfy

$$Lw_r = i\mu_r w_r, \qquad (2)$$

where L is the equilibrium Vlasov operator given by the Poisson bracket of the equilibrium Hamiltonian, $L = [\ ,H]$. μ_r is the Liouville eigenvalue, and it is a sum of terms containing integral multiples of the bounce frequencies of all the periodic equilibrium particle motions. The subscript r stands for the set of single particle invariants that labels a particular orbit, as well as a set of integers that labels harmonics of the equilibrium bounce motions. The sum over r in Eq. (1) means sum over all discrete labels and integrate over all continuous labels. In the case that there are three (exact or adiabatic) constants of the single particle motion in the equilibrium fields, then the Liouville eigenfunctions $w_r(\vec{q},\vec{p})$ can be found explicitly in terms of integrals of equilibrium quantities. When the form (1) is substituted into the linear Vlasov equation the coefficients $\Gamma_r(\omega)$ can be found in terms of the perturbation $\vec{\xi}(\vec{r};\omega)$. When that result for f_1 is substituted into the source terms of the linearized Maxwell's equations, one obtains a homogeneous integro-differential equation to be solved for $\vec{\xi}$. In the case of the tilting mode for elliptical FRCs, $\vec{\xi}(\vec{r};\omega) \rightarrow \xi_z(\psi;\omega)$, so the linearized equations of motion have the general form

$$D(\psi;\omega)\ \xi_z(\psi;\omega) = 0, \tag{3}$$

where D is the dispersion operator. Equation (3) can also be obtained from the variation of the dispersion functional Δ, defined by

$$\Delta(\xi_z^*,\xi_z) = (\xi_z,D\xi_z) \equiv \int d^3r\ \xi_z^*\ D\ \xi_z,$$

with respect to ξ_z^*. In the Vlasov-fluid model the dispersion functional has the form

$$\Delta = -2\delta W - \frac{i\omega e}{c} \int d^3r\ \vec{\xi}^*\cdot(\vec{\xi}\times\vec{B}_0) - V = 0, \tag{4}$$

where δW is exactly the MHD incompressible potential energy, and the kinetic terms are contained in V, defined by

$$V(\omega) = \omega \sum_r f_0'(E)\ \frac{|(w_r,H_1)|^2}{\mu_r - \omega}, \tag{5}$$

where $f_0(E)$ is the equilibrium ion distribution function, and E is the total energy of a particle. In Eq. (5) H_1 is the single particle perturbation Hamiltonian

$$H_1 = e(\phi_1 - \frac{e}{c}\vec{v}\cdot\vec{A}_1) = \vec{\xi}\cdot(\vec{E}_0 + \vec{v}\times\vec{B}_0/c). \tag{6}$$

The second half of Eq. (6) follows, since in the Vlasov-fluid model one writes $\vec{A}_1 = \vec{\xi}\times\vec{B}_0$, and the gauge choice is $\phi_1 = \vec{\xi}\cdot\vec{E}_0$. $(w_r, H_1) \equiv \int d^3q d^3p \; w_r^* \; H_1$ is the orbit integral of the perturbation H_1 along the orbit r, and the particle resonances occur when $\mu_r - \omega = 0$. For FRCs the energy E and canonical angular momentum p_θ are exact invariants, and in the work of Seyler and Barnes it was assumed that the magnetic moment μ is an adiabtic invariant. In the limit of small Larmor radius this leads to a dispersion functional of the form

$$\Delta = -2\delta W + 2\omega^2 K + \omega F - R(\omega) = 0. \tag{7}$$

K is the MHD kinetic energy normalization, F contains the FLR terms, and R contains parallel kinetic effects and particle resonances. If one neglects R and solves Eq. (7) for ω, it is easy to show that the FLR term F has a stabilizing effect.

Neglecting the R term in Eq. (7) leads to a 2nd order ODE for $\xi_z(\psi)$. The first case to consider is the pure MHD limit, where F is neglected. One boundary condition is that $\xi_z(\psi=0) = 0$, and the other boundary condition is really a regularity condition that eliminates the singular (logarithmic) solution of $\xi_z(\psi)$ about the O-point. The regular solution has ξ_z going to a (nonzero) constant as $\psi \to -1$. However, when one adds the F term one finds that eliminating the singular solution of ξ_z at the O-point leads to a regular solution that vanishes at the O-point. That is, the addition of <u>any</u> amount of FLR terms, no matter how small, leads to a completely different behavior of the solution in the vicinity of the O-point. This result is unphysical and indicates that the small Larmor radius treatment of the ions in the vicinity of the field null is incorrect.

A correct kinetic treatment of the tilting mode in an FRC should recognize that there are large ion orbits, especially for the betatron particles. This introduces two significant complications into the problem. First, large ion Larmor radii mean that a local approximation is not valid, and hence the full integro-differential equation must be dealt with. Second, to even obtain expressions for the orbit integrals we need to have another adiabatic invariant to replace μ, which is not invariant for large orbit ions.

Adiabatic Invariants

In order for μ to be an adiabatic invariant it is necessary that B change little during one cycloidal period of the particle motion. In current FRCs the radial gradient of the B-field near the separatrix is sufficiently steep that a thermal ion feels a strong variation in B during a radial oscillation. Figure 3 shows particle trajectories for two different thermal ions in the (r,z)-plane for the Spencer-Hewett equilibrium corresponding to FRX-B. In Fig. 3a, by the time the (cycloidal) particle has reached z = 1 it has had a variation in μ of $\Delta\mu \sim \pm 65\%$, while $\Delta J \sim \pm 11\%$, where J is an adiabatic invariant, defined in Eq. (12), with which we replace μ. In Fig. 3b, the (betatron) particle has a maximum variation in J of $\Delta J \sim \pm 8\%$, and for this particle μ is meaningless. (For FRX-C parameters the variations in μ and J are about 50% of what they are for FRX-B parameters.) This means that for current FRCs it is never valid to assume that μ is an adiabatic invariant, which in turn means that not only is MHD an invalid model for FRCs, but a guiding center description also is invalid.

Fortunately there remains one small parameter of current FRCs that can be exploited to provide an adiabatic invariant to replace μ. That small parameter is the elongation of the plasma, ε. Typically for an FRC ε is in the range $.15 < \varepsilon < .25$. We will see that the radial action J is an adiabatic invariant for elongated FRCs.

The equilibrium single particle Hamiltonian is (p_θ is a parameter throughout)

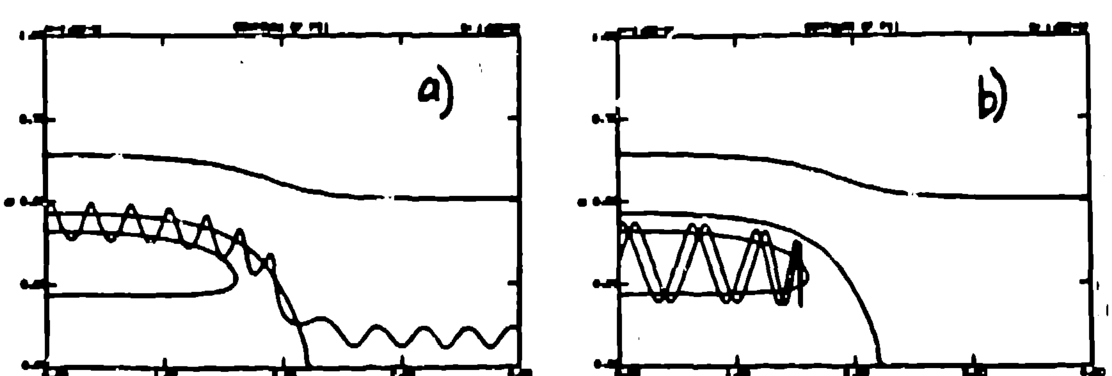

Figure 3. Particle trajectories and flux surfaces for a) cycloidal particle, and b) betatron particle, using FRX-B parameters.

$$H(r,p_r,z,p_z) = \frac{p_r^2}{2m} + \frac{p_z^2}{2m} + U(r,z), \tag{8}$$

where the two dimensional potential is

$$U(r,z) = \frac{[p_\theta - e\psi(r,z)/c]^2}{2mr^2} + e\Phi(\psi), \tag{9}$$

and Φ is the electric potential determined from ion pressure balance. The highly elongated nature of FRCs manifests itself in that the potential variation in z is much "slower" than it is in r. (This is true except for a highly racetrack equilibrium, where all the axial variation occurs at the tip of the flux surface on the same spatial scale length as the radial variation. See Fig. 2c.) Thus to do the perturbation theory for slow z variation we replace

$$U(r,z) \to U(r,\varepsilon z), \tag{10}$$

treat ε as small in the analysis, and then at the end let $\varepsilon \to 1$. The goal of this perturbation analysis is to produce the Hamiltonian that determines the radial and axial motion to lowest order in ε. A result of this procedure is that the radial action J, defined by

$$J(E_0,\varepsilon z) \equiv \frac{1}{2\pi} \oint dr\ p_r(r,E_0,\varepsilon z), \tag{12}$$

is an adiabatic invariant :

$$\frac{dJ}{dt} = 0 + O(\varepsilon^2). \tag{13}$$

The p_r in Eq. (12) is

$$p_r(r,E_0,\varepsilon z) = \{2m[E_0 - U(r,\varepsilon z)]\}^{1/2}, \tag{14}$$

where E_0 is a constant value of the radial Hamiltonian H_0

$$H_0(r,p_r,\varepsilon z) = \frac{p_r^2}{2m} + U(r,\varepsilon z) = \text{const.} \equiv E_0. \tag{15}$$

For each z in Eq. (15), E_0 varies over a range of values. For each z and E_0 in Eq. (12), J has a certain value. The relation $J = J(E_0,\varepsilon z)$ can be inverted to give E_0 as a function of J and εz: $E_0 = K_0(J,\varepsilon z)$. By using multiple time scale analysis on z as a function of time, it can be shown that the Hamiltonian that produces the lowest order radial and axial motion is

$$K(J,z,p_z) = K_0(J,\varepsilon z) + \frac{{p_z}^2}{2m} + O(\varepsilon^2). \tag{16}$$

Essentially we have transformed coordinates $(r,p_r) \rightarrow (\phi,J)$, where ϕ is the angle variable conjugate to J, and we have eliminated ϕ from the Hamiltonian to order ε^2. The function $\phi = \phi(r,p_r)$ can be found in the usual way from the generating function $F(r,J)$, where $p_r = \partial F/\partial r$. By solving Eq. (16) for p_z and using the equation $dz/d\tau = p_z/m$, we can show that the (slow) axial time of a particle's position is (letting $\varepsilon \rightarrow 1$)

$$\tau(z) = \int_{z_1}^{z} \frac{dz'}{\{\frac{2}{m}[E - K_0(J,z')]\}^{1/2}}, \tag{17}$$

where E is a constant value of the total Hamiltonian K in Eq. (16). If z_1 and z_2 are the two turning points of the axial motion, then the axial period T_z is a function of E, J, and P_θ.

Of course, J (or μ for that matter) is not an adiabatic constant of the motion for particles that pass in the vicinity of the spindle point. In fact, particles with positive p_θ have orbits that are not confined axially (see Fig. 3a), so these particles are lost through the spindle point region in an axial transit time. Since the tilting mode has a displacement that vanishes at the separatrix, one can only hope that difficulties associated with the spindle point are not an essential part of the stabilization of the tilting mode.

Now that we have the Hamiltonian K that describes the lowest order radial and axial motion, we can return to Eq. (2) to find to lowest order in ε the Liouville eigenfunctions and eigenvalues. The details of this calculation are lengthy so we merely present the results. In terms of the canonical coordinates $(\phi,J;\tau,E;\theta,p_\theta)$ we have

$$w_r(\phi,J,\tau,E,\theta,p_\theta) = d_r\delta(E-E')\delta(J-J')\delta(p_\theta-p_\theta')e^{in\theta}\, u_r(\phi,\tau), \tag{18}$$

where

$$u_r(\phi,\tau) = \exp\{i[\ell\phi + \ell\Omega_o\tau_1(\tau) + p\Omega_z\tau + n\Omega_\theta\tau_2(\tau)]\}, \tag{19}$$

and d_r is a normalizing factor. E', J', and p_θ' are continuous indices (i.e., numbers), and ℓ and p are any integers; these quantities collectively denote the label r. The operator L commutes with multiplication by any function of the constants of the motion, and L also commutes with $\partial/\partial\theta$. Thus w_r can be chosen to be a simultaneous eigenfunction of these operators, and this explains the delta-functions and the $e^{in\theta}$ in Eq. (18). The dependence of u_r on ϕ and τ (corresponding to the two nonignorable directions of the equilibrium, r and z) is chosen here so that

$$u_r(\phi+2\pi,\tau+T_z) = u_r(\phi,\tau). \tag{20}$$

That is, Eqs. (18-19) are the w_r for particles that are trapped; a similar expression could be written for particles whose orbits are not confined spatially. The remaining quantities in Eq. (19) are given by

$$\begin{aligned}
\omega_o(\tau) &= \frac{\partial K_o(J,p_\theta,z(\tau))}{\partial J} = 2\pi \left(\oint \frac{m\,dr}{p_r(r,J,p_\theta,z(\tau))}\right)^{-1} \\
\Omega_o &= \frac{1}{T_z}\oint d\tau\ \omega_o(\tau) \\
\tau_1(\tau) &= \tau - \frac{1}{\Omega_o}\int_0^\tau d\tau'\ \omega_o(\tau') \\
\Omega_z &= 2\pi/T_z \\
\Omega_\theta &= \frac{1}{T_z}\oint d\tau\ \omega_\theta(\tau) \\
\omega_\theta(\tau) &= \frac{\Omega_o}{2\pi}\oint \frac{m\,dr}{p_r(r,J,p_\theta,z(\tau))}\,\frac{v_\theta^{\,2}(r,z(\tau))}{r} \\
v_\theta(r,z) &= [p_\theta - e\psi(r,z)/c]/m \\
\tau_2(\tau) &= \tau - \frac{1}{\Omega_\theta}\int_0^\tau d\tau'\ \omega_\theta(\tau')
\end{aligned} \tag{21}$$

All quantities in (21) can be calculated from the equilibrium, and they are functions of the constants E, J, and p_θ. The Liouville eigenvalue is

$$\mu_r = \ell\Omega_o + p\Omega_z + n\Omega_\theta, \tag{22}$$

and clearly $\mu_r - \omega = 0$ describes resonances between the wave and harmonics of the radial, axial, and azimuthal drift motions of the particles.

With the Liouville eigenfunctions defined by (18) we can simplify the orbit integrals as

$$(w_r, H_1) \to d_r \oint d\phi \oint d\tau \, u_r(\phi,\tau) \, H_1(r,z,\xi_z(\psi)). \tag{23}$$

$\psi(r,z)$ is known from the equilibrium, and we can expand

$$\xi_z(\psi;\omega) = \sum_{n=1}^{N} a_n(\omega) \, \eta_n(\psi), \tag{24}$$

where $\{\eta_n\}$ is a chosen set of expansion functions, and the coefficients a_n are to be found. Thus $H_1(r,z,\xi_z)$ can be written as a function of r and z. By using the transformations $r = r(\phi,J,p_\theta,z)$ and $z = z(\tau,E,J,p_\theta)$, we can express $H_1(r,z,\xi_z) = f(\phi,\tau;\{a_n\})$. Finally, the sum over r in the kinetic term V of Eq. (5) means that

$$\sum_r \to \int dE' \int dJ' \int dp_\theta' \sum_{\ell,p} . \tag{25}$$

A dispersion matrix $D_{n,n'}(\omega)$ is constructed from the dispersion functional as

$$D_{n,n'}(\omega) \equiv \frac{\partial\Delta}{\partial a_n^{*} \partial a_{n'}},$$

and the eigenfrequency ω is determined by requiring that

$$\det D(\omega) = 0.$$

The procedure just outlined is a completely general formulation of the kinetic tilting calculation. The numerical computations involved are formidable, but we hope feasible. It may be necessary to approximate various aspects of the kinetic term V of Eq. (5) before numerical evaluation begins. For instance, the highly elongated nature of FRCs implies that the time scales of the radial, azimuthal, and axial motions are quite separated from one another, with

$$\Omega_0 \gg \Omega_\theta \gg \Omega_z. \qquad (26)$$

This means that $1/(\mu_r - \omega)$ is small, except for $\ell = 0$. Additional approximations in the orbit integrals may also be possible.

In summary, the MHD analysis of the tilting mode in FRCs has been completed, with the conclusion that all FRC equilibria should be very unstable to tilting. Of the possible reasons for explaining the observed stability of the experiments, kinetic effects appear to us to be the most likely stabilizing mechanism. This problem is very difficult, because the equilibrium is two-dimensional and the mode is global. The magnetic moment μ is not conserved for current FRC parameters, but the radial action J is a suitable adiabatic invariant for elongated equilibria. We have obtained a general expression for the dispersion functional for the kinetic tilting calculation which is in a form suitable for numerical evaluation. Possible analytical approximations of the problem have been indicated.

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