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ADIABATIC INVARIANTS FOR FIELD REVERSED CONFIGURATIONS

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Field reversed configurations (FRCs) are characterized by azimuthal symmetry, so two exact constants of the particle motion are the total particle energy E and the canonical angular momentum $p_\theta$. For many purposes it is desirable to construct a third (adiabatic) constant of the motion if this is possible. It is shown that for parameters characteristic of current FRCs that the magnetic moment $\mu$ is a poor adiabatic invariant, while the radial action J is conserved rather well.

The magnetic moment of a particle at position $\vec{r}$ is

$$\mu(\vec{r}) = \frac{m v_\perp{}^2(\vec{r})}{2B(\vec{r}_{gc})}, \qquad (1)$$

where $\vec{r}_{gc}$ is the position of the particle's guiding center. $B(\vec{r}_{gc})$ can be estimated from $B(\vec{r})$ by Taylor expanding B about $\vec{r}_{gc} = \vec{r}$. For $\mu$ to be conserved it is necessary that a particle see little change in B during a radial oscillation. In Fig. 1 we show trajectories of thermal particles in the fields of the FRX-B equilibrium as calculated by Spencer and Hewett.[1,2] This equilibrium has $T_i$ = 340eV, $n_{max}$ = $3\times10^{15}$cm$^{-3}$, $B_w$ = 6.5kG, $r_w$ = 12.5cm, $x_s$ = .46, and $P_{sep}/P_{max}$ = .44. For these parameters

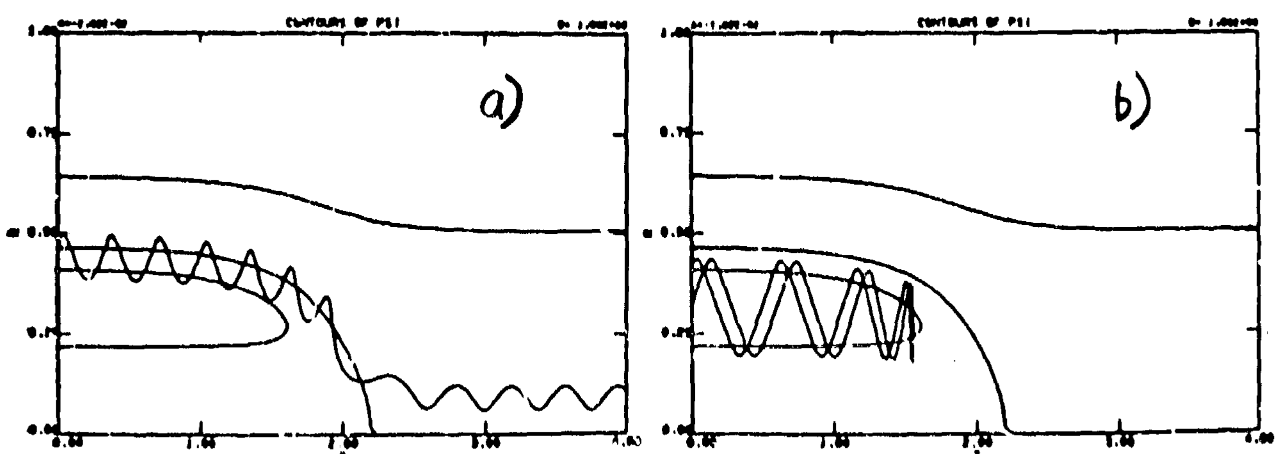


Figure 1. Particle trajectories and flux surfaces for a) cycloidal particle, and b) betatron particle, using FRX-B parameters.

the radial gradient in B in the midplane of the device is sufficiently steep that the particle sees a large variation in B during one radial oscillation. By the time the (cycloidal) particle has reached $z = 1$ in Fig. 1a it has experienced a variation in $\mu$ of $\Delta\mu \sim \pm 65\%$, which surely is unacceptable for $\mu$ to be considered conserved; for the betatron particle in Fig. 1b $\mu$ is meaningless. The conclusion is that $\mu$ is not a suitable invariant for any class of particles in the FRC. This suggests that not only is MHD an invalid model, but the guiding center model also is not applicable.

Fortunately there remains a small parameter of FRCs that can be exploited to derive a new adiabatic invariant to replace $\mu$. That small parameter is the elongation of the configuration, $\varepsilon$, which typically is in the range $.15 < \varepsilon < .25$. We now show that the radial action J is an invariant for elongated FRCs.

The equilibrium single particle Hamiltonian is ($p_\theta$ is a parameter throughout)

$$H(r,p_r,z,p_z) = \frac{p_r^2}{2m} + \frac{p_z^2}{2m} + U(r,z), \tag{2}$$

where the two dimensional potential is

$$U(r,z) = \frac{[p_\theta - e\psi(r,z)/c]^2}{2mr^2} + e\Phi(\psi), \tag{3}$$

and $\Phi$ is the electric potential determined from ion pressure balance. The highly elongated nature of FRCs manifests itself in that the potential variation in z is much "slower" than it is in r. (This is true except for a highly racetrack equilibrium, where all the axial variation occurs at the tip of the flux surface on the same spatial scale length as the radial variation.) Thus to do the perturbation theory for slow z variation we replace

$$U(r,z) \to U(r,\varepsilon z), \tag{4}$$

treat $\varepsilon$ as small in the analysis, and then at the end let $\varepsilon \to 1$. Mynick[3] used the slow-z approximation (4) and made some analytical approximations to obtain a Hamiltonian as an ›licit function of J that determines the radial and axial motions of a pa icle through order $\varepsilon^2$. Here we present a simple derivation of the lowest order Hamiltonian, without seeking to make approximations of the integrals involved, and we present numerical tests of the constancy of J for realistic equilibria. Define the radial action J as

$$J(E_0,\varepsilon z) \equiv \frac{1}{2\pi} \oint dr\ p_r(r,E_0,\varepsilon z). \qquad (5)$$

The $p_r$ in Eq. (5) is

$$p_r(r,E_0,\varepsilon z) = \{2m[E_0 - U(r,\varepsilon z)]\}^{1/2}, \qquad (6)$$

where $E_0$ is a constant value of the radial Hamiltonian $H_0$

$$H_0(r,p_r,\varepsilon z) = \frac{p_r^{\ 2}}{2m} + U(r,\varepsilon z) = \text{constant} \equiv E_0. \qquad (7)$$

For each z in Eq. (7), $E_0$ varies over a range of values. For each z and $E_0$ in Eq. (5), J has a certain value. The relation $J = J(E_0,\varepsilon z)$ can be inverted to give $E_0$ as a function of J and $\varepsilon z$:

$$E_0 = H_0(r,p_r,\varepsilon z) = K_0(J,\varepsilon z). \qquad (8)$$

That is, we have transformed from $(r,p_r)$ to action-angle variables $(\phi,J)$.

Since $\varepsilon$ is a parameter in the potential U, a perturbation solution for z(t) will depend on $\varepsilon$,

$$z(t;\varepsilon) = z_0(t) + \varepsilon z_1(t) + \varepsilon^2 z_2(t) + \ldots. \qquad (9)$$

(It is well known that a straightforward perturbation expansion as in (9) leads to secular growth of z(t), so that the solution soon becomes invalid as t becomes large.[4] These secularities can be removed from the solution by allowing z(t) to depend on $\varepsilon$ through various time scales $T_0$, $T_1$, ..., $T_n$, defined by $T_n = \varepsilon^n t$, n = 0,1,2,.... However, in this calculation we do not actually need the multiple time scale formalism.) When the expansion (9) is substituted into $U(r,\varepsilon z)$ we have

$$U(r,\varepsilon z) = U(r,\varepsilon(z_0+\varepsilon z_1+\ldots))$$

$$= U(r,\varepsilon z_0+\varepsilon^2 z_1+\ldots)$$

$$= U(r,\varepsilon z_0) + \varepsilon^2 z_1 \frac{\partial U(r,\varepsilon z_0)}{\partial(\varepsilon z_0)} + \ldots = U(r,\varepsilon z_0) + O(\varepsilon^2). \quad (10)$$

Therefore the Hamiltonian that determines to lowest order in $\varepsilon$ the radial and axial motion of a particle is obtained by substituting Eqs. (10,7,8) into Eq. (2) (letting $z_0 \to z$):

$$K(J,\varepsilon z,p_z) = K_0(J,\varepsilon z) + \frac{p_z^{\,2}}{2m} + O(\varepsilon^2). \quad (11)$$

From Eq. (11) we conclude that J is an adiabatic invariant:

$$\frac{dJ}{dt} = -\frac{\partial K}{\partial \phi} = 0 + O(\varepsilon^2).$$

For the cylcoidal particle in Fig. 1a, by the time the particle reaches z = 1 the variation in J has been $\Delta J \sim \pm 11\%$. For the betatron particle in Fig. 1b the variation in J is $\Delta J \sim \pm 8\%$ during an axial bounce time. (For FRX-C parameters the variations in $\mu$ and J are about 50% of what they are for FRX-B parameters.) By solving Eq. (11) for $p_z$ and using the equation $dz/dt = p_z/m$, the axial time of a particle's position is (letting $\varepsilon \to 1$)

$$\tau(z) = \int_{z_1}^{z} \frac{dz'}{\{\frac{2}{m}[E - K_0(J,z')]\}^{1/2}}, \quad (12)$$

where E is a constant value of the total Hamiltonian K in Eq. (11), and $z_1$ is a turning point of the axial motion. The error in using formula (12) compared to the exact particle trajectory data is .34% at z = 2 in Fig. 1a and .97% at the z-turning point in Fig. 1b.

Of course, J (or $\mu$ for that matter) is not an adiabatic constant of the motion for particles that pass in the vicinity of the spindle point.

In fact, particles with positive $p_\theta$ have orbits that are not confined axially (see Fig. 1a), so these particles are lost through the spindle point region in an axial transit time. In a similar vein Kim and Cary[5] studied particle orbits for an elliptical z-pinch, and found two regions of the equilibrium where either J or μ was conserved, with a stochastic region in between. What we have shown is that, where it is possible to define an adiabatic invariant in current FRC equilibria, the radial action is always conserved much better than the magnetic moment.

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