

## Problems with the Quenched Approximation in the Chiral Limit

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In the quenched approximation, loops of the light singlet meson (the  $\eta'$ ) give rise to a type of chiral logarithm absent in full QCD. These logarithms are singular in the chiral limit [1-3], throwing doubt upon the utility of the quenched approximation. In previous work, I summed a class of diagrams, leading to non-analytic power dependencies such as  $\langle \bar{\psi}\psi \rangle \propto m_q^{-\delta/(1+\delta)}$  [3]. I suggested, however, that these peculiar results could be redefined away. Here I give an alternative derivation of the results, based on the renormalization group, and argue that they cannot be redefined away. I discuss the evidence (or lack thereof) for such effects in numerical data.

### 1. INTRODUCTION

Many simulations of lattice QCD use the so-called "quenched" approximation, in which the fermion determinant is left out of the measure. Since this approximation is likely to be with us for a few years yet, it behooves us to attempt to understand all we can about the peculiarities of the quenched theory.

I focus here on the effects of the extra light pseudo-Goldstone boson (PGB) present in the spectrum of quenched theory. I assume degenerate light quarks ( $N_f$  each of mass  $m_q$ ) in which case the extra PGB is the flavor singlet state. By analogy with QCD, I call this the  $\eta'$ .

The  $\eta'$  is a PGB in the quenched theory because the series of diagrams shown below are absent. The first of these diagrams is, however, present



in loop graphs, and introduces a new dimensionful parameter. This changes the power counting rules of chiral perturbation theory, and leads to enhanced chiral logarithms in quenched QCD.

This effect, first pointed out in Ref. [1], was first calculated by Bernard and Golterman (BG)[2]. They developed a chiral Lagrangian for quenched QCD, containing ghost states to cancel the effects of the quark loops. They found, e.g.

$$\begin{aligned} m_\pi^2/2\mu m_q &= [1 - 2\delta \ln(m_{\eta'}/\Lambda) + O(m_q)], \quad (1) \\ \delta &= m_0^2/(N_f 16\pi^2 f_\pi^2), \quad (2) \end{aligned}$$

where  $\Lambda$  is the ultraviolet cutoff,  $f_\pi = 93\text{MeV}$  and  $\mu$  are parameters in the chiral Lagrangian,

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and  $m_0$  is the scale characterizing the  $\eta'$  two-point function. These parameters are discussed further below. In full QCD, the term proportional to  $\delta$  is absent, and so  $m_\pi^2 = 2\mu m_q$  in the chiral limit. Chiral logarithms, proportional to  $m_q \ln(m_q)$ , vanish in this limit. (These are represented by the shorthand  $O(m_q)$  in Eq. 1.) In quenched QCD, on the other hand, the logarithm diverges in the chiral limit.

Using the quark-line method I obtained, independently, the same results for the enhanced chiral logs from  $\eta'$  loops [3, 4]. (I also noted that there are a number of quantities (e.g.  $f_\pi$ ,  $B_K$ ) which, at one-loop, and for degenerate quarks, are not affected by  $\eta'$  loops.) In order to address the question of the approach to the chiral limit, when the term  $\delta \ln(m_{\eta'})$  gets large, I summed the leading logarithms proportional to powers of this quantity, with the result

$$m_\pi^2/2\mu m_q = (2\mu m_q/\Lambda^2)^{-\delta/(1+\delta)} + O(m_q). \quad (3)$$

The divergence has been enhanced from a logarithm to a power law.

This result appears to violate PCAC relations such as  $f_\pi^2 m_\pi^2 = -2m_q \langle \bar{\psi}\psi \rangle$ . This is not so, however, since there is a whole tower of equations similar to Eq. 3, e.g.

$$-\langle \bar{\psi}\psi \rangle / (\mu f_\pi^2) = (2\mu m_q/\Lambda^2)^{-\delta/(1+\delta)} + O(m_q). \quad (4)$$

These results are of the same form as those describing critical behavior close to a second order fixed point. At first sight this is puzzling, because no such fixed points (with non-trivial exponents) are known in four dimensions. The puzzle is resolved by recalling that this lore applies to critical theories which are unitary, and thus presumably not to the massless limit of quenched QCD. The similarity to critical behavior does, however, suggest that it should be possible to recast the results

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Eqs. 3 and 4 into a renormalization group (RG) framework.

## 2. RG DERIVATION

It is simplest to use the quenched chiral Lagrangian of Bernard and Golterman,  $\mathcal{L}_{\text{BG}}$ . The symmetry of  $\mathcal{L}_{\text{BG}}$  is the graded group  $U(3|3)_L \times U(3|3)_R$ , and there are extra pseudoscalar and fermionic PGBs. It turns out, however, that for amplitudes involving only the physical PGBs as external particles, and for external momenta  $p \ll 4\pi f_\pi$ , one need only consider part of  $\mathcal{L}_{\text{BG}}$ :

$$\mathcal{L}_{\text{BG}} = \text{Tr}[\partial_\mu \pi \partial_\mu \pi] - \frac{1}{2} \mu m_q f_\pi^2 \text{Tr}[U + U^\dagger] + \frac{1}{2} m_0^2 \eta'^2 + \mathcal{L}'_{\text{BG}}. \quad (5)$$

The pion field here includes the  $\eta'$ :

$$\pi = \sum_{a=0}^8 \pi_a T_a, \quad \text{Tr}[T_a T_b] = \frac{1}{2} \delta_{ab}, \quad U = \exp[2i\pi/f_\pi].$$

$\mathcal{L}'_{\text{BG}}$  includes various types of terms which can be neglected since they do not give rise to enhanced chiral logarithms. First, there are other leading order terms (i.e. of  $O(p^2, m_q)$ ) containing the additional PGB fields, which have the effect of canceling various diagrams involving the physical PGB's. These cancellations are discussed in detail in Refs. [2, 3]. Most important here are the restrictions on the insertions of the  $\eta'^2$  term. While this looks like a mass term, it can in fact only be inserted once on each internal  $\eta'$  line, because of cancellations with terms from  $\mathcal{L}'_{\text{BG}}$ . By contrast, in full QCD  $m_0^2$  does contribute to the  $\eta'$  mass,  $m_{\eta'}^2 = 2\mu m_q + m_0^2$ . A standard phenomenological analysis using the physical meson masses, and allowing for non-degenerate quarks, gives  $m_0 \approx 0.9\text{GeV}$  in full QCD.

Using this value as an estimate in the quenched approximation, one finds  $\delta \approx 0.2$ . This estimate is unreliable for two reasons. First, parameters in the quenched and full chiral Lagrangians need not be the same. Second, I have excluded a possible wavefunction renormalization term  $\frac{1}{2}(A-1)\partial_\mu \eta' \partial_\mu \eta'$ , which reduces the estimate of  $\delta$  by a factor of  $A$ . This term does not, however, contribute enhanced chiral logarithms, because it does not introduce a new scale. It simply makes the estimate of  $\delta$  less certain.

$\mathcal{L}'_{\text{BG}}$  also includes the remainder of the full kinetic term

$$\frac{1}{4} f_\pi^2 \text{Tr}[\partial_\mu U \partial_\mu U^\dagger] = \text{Tr}[\partial_\mu \pi \partial_\mu \pi] + O(\pi^4).$$

Only the  $O(\pi^2)$  term need be kept, because the higher order vertices do not contain the  $\eta'$  field, and do not contribute enhanced chiral logarithms. On the other hand, the full mass term must be kept, since the  $\eta'$  is contained in all terms.

Finally,  $\mathcal{L}'_{\text{BG}}$  contains  $O(p^4)$  terms which are suppressed in the chiral limit.

I now turn to the calculation. In full QCD, power counting shows that logarithms always come with a factor of the quark mass or the external momenta squared, e.g.

$$m_\pi^2/(2\mu m_q) = 1 + 2m_\pi^2 \ln(m_\pi/\Lambda)/(N_f 16\pi^2 f_\pi^2).$$

The logarithmic cut-off dependence can thus be absorbed by the coefficients of the  $O(p^4)$  part of the Lagrangian, while the leading order coefficients are unaffected.

If one uses a simple momentum cut-off, rather than, say, dimensional regularization, then there are also quadratic divergences. In full QCD, these do affect the leading order coefficients. One must introduce a scale dependence in the bare decay constant,  $f(\Lambda)$ , so that physical quantities are independent of the cut-off. These divergences are, however, absent in the quenched theory, so  $f_\pi$  remains independent of scale.

In quenched chiral perturbation theory power counting is changed because of the additional dimensionful parameter  $m_0$ . Insertions of the  $\eta'^2$  vertex into diagrams that were previously quadratically divergent produces logarithmically divergent integrals, giving factors of  $\delta \ln(m_{\eta'}/\Lambda)$ . For example, Fig. 1b leads to Eq. 1. Since  $\delta$  does not vanish in the chiral limit, the scale dependence must be absorbed into the leading order coupling:  $\mu \rightarrow \mu(\Lambda)$ . There are also divergent corrections to  $m_0^2$ , but these can be shown to be suppressed by powers of  $m_\pi^2/(4\pi f_\pi)^2$ .

In Ref. [3] I summed all diagrams containing leading logarithms, i.e. those proportional to  $[\delta \ln(m_{\eta'})]^n$  with no additional factors of  $m_\pi^2$  or  $\delta$ . Two crucial points allowed this. First, in order to get a leading logarithm one must insert each  $\eta'^2$  vertex in an initially quadratically divergent loop. Furthermore, every loop in the initial diagram must receive such an insertion, otherwise the loop will give factors of  $m_\pi^2$ . Simple power

counting shows that, for degenerate quarks, the diagrams which remain are those with any number of  $\eta'$  loops coming out of a single mass-term vertex, as illustrated by Figs. 1b and c. These are easily summed. Second, other diagrams involving more than one  $\mu$  vertex can be shown to lead to non-leading logarithms using Weinberg's theorem [5].

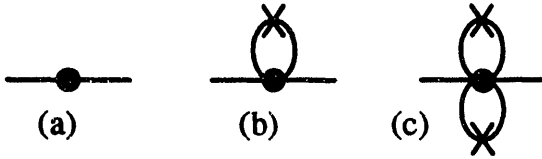


Figure 1. Contributions to  $m_\pi$ . The cross represents the  $\eta'^2$  vertex, the circle the  $\mu$  term.

I wish to recast the diagram summation into a RG framework. Start at a scale  $\Lambda_0 \approx 1\text{GeV}$  at which the coupling is  $\mu(\Lambda_0) = \mu_0$ . Reduce  $\Lambda$ , varying  $\mu(\Lambda)$  so as not to change amplitudes with small external momenta. When  $\Lambda$  drops to  $m_{\eta'}$ , the loop diagrams cease to generate logarithms, and  $\mu(\Lambda)$  no longer "runs". At this point use the tree level formula to determine  $m_{\eta'}$ :

$$m_\pi^2 = m_{\eta'}^2 = 2\mu(m_{\eta'})m_q . \quad (6)$$

A crucial check on this program is that the RG equation does not depend on the choice of physical amplitude used to determine it. The two-point, four-point, six-point, etc. amplitudes all yield the same result. This ensures that  $\mathcal{L}_{\text{BG}}$  retains its chiral invariant form.

To obtain the RG equation, consider the two point amplitude. As already noted, the diagrams which contribute at leading order are those of the type shown in Fig. 1, but with any number of  $\eta'$  loops. Let  $A_n$  be the contribution of the diagram with  $n$   $\eta'^2$  vertices. Each loop gives a factor of  $\delta \ln(\Lambda^2/m_{\eta'}^2)$ , and there is a combinatoric factor of  $1/n!$ , so

$$A_n = 2\mu(\Lambda)m_q[\delta \ln(\Lambda^2/m_{\eta'}^2)]^n/n! .$$

Thus if we change  $\ln(\Lambda^2)$  by  $d \ln(\Lambda^2)$ ,

$$dA_n = \delta A_{n-1} d \ln(\Lambda^2) + A_n d \ln \mu ; \quad A_{-1} = 0 .$$

Summing over  $n$  we see that the total amplitude will be unchanged if  $\mu(\Lambda)$  changes according to

$$d \ln \mu = -\delta d \ln(\Lambda^2) \Rightarrow \mu(\Lambda) = \mu_0(\Lambda_0^2/\Lambda^2)^\delta . \quad (7)$$

The self-consistent equation, Eq. 6 becomes

$$m_\pi^2 = m_{\eta'}^2 = 2\mu_0 m_q (\Lambda_0^2/m_{\eta'}^2)^\delta . \quad (8)$$

The solution is Eq. 3, except that we now know that  $\mu$  is to be evaluated at the scale  $\Lambda$ .

More generally, all amplitudes coming from the mass term in  $\mathcal{L}_{\text{BG}}$ , and involving momenta less than  $m_{\eta'}$ , will come with an overall factor of

$$\mu(m_{\eta'}) = \mu_0(\Lambda_0^2/m_{\eta'}^2)^\delta . \quad (9)$$

Eq. 4 follows from this result.

This derivation is unusual in that one must work to all orders to obtain the RG equation. No extra diagrams are summed up when solving the equation. Furthermore, the external momentum does not act as an infrared cut-off on the diagram since it does not flow through the  $\eta'$  loops. Nevertheless, I think it is useful to see the result emerge in this way. For one thing it may simplify the generalization to non-degenerate quarks. It also emphasizes that the non-analytic dependence on  $m_{\eta'}$  comes from the infrared part of the integrals.

In Ref. [3] I suggested removing the non-analyticities by assuming a non-analytic relation between the quark mass in the quenched chiral Lagrangian and that in quenched QCD. I no longer think that this is an option. The two Lagrangians are matched at  $\Lambda \sim 1\text{GeV}$ , in such a way that their infrared behaviors agree. Thus infrared divergences leading to non-analyticities cannot enter into the relationship between quark masses.

### 3. COMPARISON TO DATA

Present quenched data show no evidence for the divergences of Eqs. 3 and 4. This is illustrated with the data of Ref. [6] in Fig. 2. Eq. 3 with  $\delta = 0.2$  gives the dashed line. ( $\Lambda$  is chosen to match the data at  $m_q \approx 0.03$ .) One is free, however, to add terms linear and quadratic in  $m_q$ . With coefficients of typical size, it is possible to make the theoretical curve quite flat, as shown by the solid line. Thus this data set probably cannot rule out  $\delta \approx 0.2$ . However, new results on the spectrum presented at this conference are likely to give a much smaller bound on  $\delta$ .

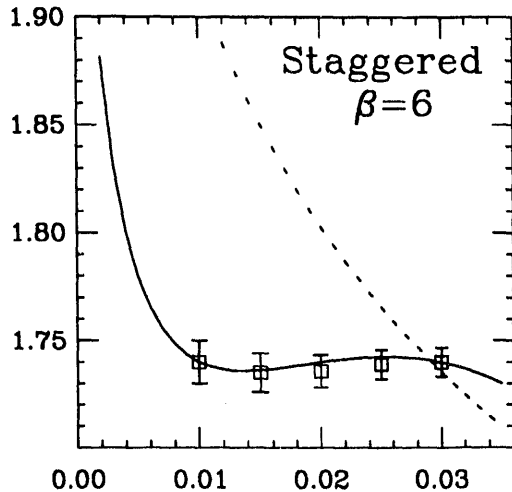


Figure 2.  $\ln(m_\pi^2/m_q)$  vs.  $m_q$  (in lattice units).

#### 4. CONCLUSIONS

I can think of three explanations for the lack of numerical evidence for the divergent terms. (1) The effect could be present (with  $\delta \approx 0.2$ ) but hidden by a conspiracy of higher order terms. This I think is unlikely given the latest data, but I await with interest the results. (2) The theoretical analysis could be wrong. This seems unlikely, for the essential physics is in the 1-loop calculation, and this requires only the presence of the light  $\eta'$  and of the  $\eta'^2$  vertex. See also the talk of Golterman [7]. (3) The effect is present but  $\delta \ll 0.2$ . As pointed out to me by Fukugita, Itoh *et al.* attempted to directly measure  $m_0^2$  with Wilson fermions [8]. They found, with poor statistics, a number  $\sim 6$  times smaller than  $(900\text{MeV})^2$ .

I presently favor the third option. The only doubt I have stems from the following fact: to calculate  $m_0^2$  one looks at the disconnected part of the pseudoscalar two point function, and finds the residue of the double pole at  $p^2 = -m_\pi^2$ . The same two point function at  $p = 0$  is proportional to the topological susceptibility,  $\chi$  [9]. If the double pole gives the dominant contribution to the correlator at  $p = 0$  (with no momentum dependence in the residue), one can show that  $f_\pi^2 m_0^2 = 6\chi$ . Present numerical values of  $\chi$  then imply  $m_0 \approx 900$  MeV. For  $m_0$  to be much smaller than this the residue of the pole must have considerable momentum dependence. In the effective

Lagrangian this would require a large value of the coefficient  $A$  mentioned above.

If  $\delta$  is very small, then the problems with the quenched approximation are pushed to small masses. This may mean that they can be ignored for practical purposes. A direct calculation of  $\delta$  would be very useful; another method for doing this is to study finite volume effects [2, 3].

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