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MARGINAL DENSITIES OF RADIALLY SYMMETRIC DENSITIES IN TWO AND THREE DIMENSIONS

by

C. J. Everett and E. D. Cashwell

ABSTRACT

Necessary and sufficient conditions are given for a function p(x) on $0 < |x| < R$ to be the marginal density of a radially symmetric density f(r) in the case of two and three dimensions. The first case relies on an integral transform due to M. Bell, while for 3-space the theory is considerably simpler. The two cases appear to be quite different and no generalization to n-space is known to us.

I. THE CASE OF TWO DIMENSIONS

We give a summary of the theory for the plane, which was obtained in somewhat different form in an earlier paper.[1] This case is notably more difficult than that of 3-space, and requires a discussion of four sets of functions, Y, F, P, and Y*.

Y is the set of all finite, continuous, nonincreasing functions y(r) on [0,R], with y(0) = 1, y(R) = 0, having a finite continuous derivative $y'(r)$ on (0,R).

F is the set of all finite, continuous, nonnegative functions f(r) on (0,R), with

$$\int_0^R 2\pi r f(r)\, dr = 1. \tag{1}$$

Every such function f(r) defines a radially symmetric probability density $f(\sqrt{x^2+y^2})$ on the circle $C = \{(x,y);\ x^2+y^2 < R^2\}$ with

$$\int_C f(\sqrt{x^2+y^2})\ dx\ dy = 1.$$

P is the set of all marginal densities

$$p(x) = 2\int_0^{\sqrt{R^2-x^2}} f(\sqrt{x^2+y^2})\ dy;\ 0 < |x| < R \tag{2}$$

defined by functions f(r) in F.

Y* is the set of all functions y*(r) on [0,R] that arise as "Bell transforms" of the marginal densities p(x) in P, namely

$$y^*(r) = 2\int_0^{\sqrt{R^2-r^2}} p(\sqrt{r^2+x^2})\ dx,\ 0 \leqslant r < R$$
$$y^*(R) = 0\ . \tag{3}$$

This transform was discovered by M. Bell at CERN[2] and is essential for much that follows.

Theorem 1. The correspondence $y(r) \to f(r)$ defined by

$$f(r) = (-1/2\pi r)y'(r);\ 0 < r < R \tag{4}$$

is one-one on Y to all of F, its inverse $f(r) \to y(r)$ being given by

$$y(r) = \int_r^R 2\pi r f(r)\ dr;\ 0 \leqslant r \leqslant R,$$

i.e., y(r) is the (default) cumulative distribution function of the probability density $2\pi r f(r)$.

Proof. (a) The properties of y(r) in Y insure that f(r) in Eq. (4) is indeed in F. For example,

$$\int_0^R 2\pi r f(r)\ dr = \int_0^R 2\pi r(-1/2\pi r)y'(r)\ dr$$

$$\equiv -\lim \int_{\alpha}^{\beta} y'(r)\, dr = \lim\, (y(\alpha)-y(\beta)) = 1 - 0 \equiv 1, \text{ where } \alpha \to 0^+,\ \beta \to R^- .$$

(b) $y(r) \to f(r)$ is one-one, since $y_1'(r) = y_2'(r)$ on $(0,R)$ implies that the function $D(r) \equiv y_1(r) - y_2(r)$, which is continuous on $[0,R]$, and has derivative $D'(r) \equiv 0$ on $(0,R)$, has value $D(r) \equiv D(0) = 0$ on $(0,R)$. (c) To see that $y(r) \to f(r)$ is onto <u>all</u> of F, we take an arbitrary function $f(r)$ in F, and <u>define</u> a function

$$y(r) = \int_r^R 2\pi r f(r)\, dr;\ 0 \leqslant r \leqslant R .$$

One can then show that (1) $y(r)$ is finitely defined for all r on $[0,R]$ with $y(0)=1$, $y(R)=0$, (2) $y(r)$ is monotone nonincreasing on $[0,R]$, (3) $y(r)$ is continuous on $[0,R]$, and (4) $y'(r) = -2\pi r f(r)$ for every r on $(0,R)$, so that $y(r)$ as defined is indeed in Y, and finally, (5) $(-1/2\pi r)y'(r) \equiv f(r)$. (d) Since $y(r) \to f(r)$ is one-one on Y to all of F and

$$\int_r^R 2\pi r f(r)\, dr \to f(r) ,$$

the inverse $f(r) \to y(r)$ is clear.

<u>Theorem 2</u>. Every marginal density $p(x)$ in P is finite, continuous, nonnegative, and even on $0 < |x| < R$, with

$$\int_0^R p(x)\, dx = \tfrac{1}{2} .$$

Proof. From Eq. (2) it is obvious that $p(x)$ is nonnegative and even on $0 < |x| < R$, and easy to verify that

$$\int_0^R p(x)\, dx = \int_0^R \pi r f(r)\, dr = \tfrac{1}{2}, \text{ using polar coordinates.}$$

Transforming y to r in Eq. (2) by means of $x^2+y^2 = r^2$ one sees that p(x) may be written in the form

$$p(x) = \int_x^R 2f(r)\ r\ dr/(r^2-x^2)^{\frac{1}{2}};\ 0 < x < R \tag{5}$$

and from this one can deduce its finiteness and continuity. As an example, we include a proof that

$$\lim_{x \to x_o^-} p(x) = p(x_o) .$$

Fix $0 < A < x < x_o < B < R$ and let $M = \max f(r)$ on [A,B]. Writing

$$p(x) = \int_x^R 2f(r)\ r\,dr/(r^2-x^2)^{\frac{1}{2}} = \int_x^B + \int_B^R$$

$$p(x_o) = \int_{x_o}^R 2f(r)\ r\ dr/(r^2-x_o^{\ 2})^{\frac{1}{2}} = \int_{x_o}^B + \int_B^R ,$$

we see that $p(x) - p(x_o)$

$$= \left\{\int_x^B 2f(r)\ r\ dr/(r^2-x^2)^{\frac{1}{2}} - \int_{x_o}^B 2f(r)\ r\ dr/(r^2-x_o^{\ 2})^{\frac{1}{2}}\right\}$$

$$+ \int_B^R 2f(r)\ r\ dr\ \left(1/(r^2-x^2)^{\frac{1}{2}} - 1/(r^2-x_o^{\ 2})^{\frac{1}{2}}\right) .$$

The last integral approaches 0 as $x \to x_o$ since, for every $\varepsilon > 0$, there exists a $\delta > 0$, independent of r, such that

$$\left|1/(r^2-x^2)^{\frac{1}{2}} - 1/(r^2-x_o^{\ 2})^{\frac{1}{2}}\right| < \varepsilon\pi \text{ provided } |x-x_o| < \delta. \text{ Hence}$$

$$\left|\int_B^R\right| \leq \int_B^R 2f(r)\ r\ dr\ (\varepsilon\pi) \leq \varepsilon$$

As for the quantity in brackets, we have for $x < x_o$

$$\{\ \} = \int_x^{x_o} 2f(r)\ r\ dr/(r^2-x^2)^{\frac{1}{2}}$$

$$-\int_{x_o}^B 2f(r)\ r\ dr\ \left(1/(r^2-x_o^2)^{\frac{1}{2}} - 1/(r^2-x^2)^{\frac{1}{2}}\right),$$

each term of which approaches 0 as $x \to x_o^-$.

$$\text{For, } 0 \leq \int_x^{x_o} 2f(r)\ r\ dr/(r^2-x^2)^{\frac{1}{2}}$$

$$\leq 2M\ (x_o^2-x^2)^{\frac{1}{2}} \to 0$$

$$\text{and } 0 \leq \int_{x_o}^B 2f(r)\ r\ dr\ \left(1/(r^2-x_o^2)^{\frac{1}{2}} - 1/(r^2-x^2)^{\frac{1}{2}}\right)$$

$$\leq 2M\left((B^2-x_o^2)^{\frac{1}{2}} - (B^2-x^2)^{\frac{1}{2}} + (x_o^2-x^2)^{\frac{1}{2}}\right) \to 0\ .$$

Theorem 3. If $p(x)$ is any function which is finite, continuous, and non-negative on $(0,R)$, with

$$\int_0^R p(x)\ dx = \tfrac{1}{2},$$

then there exists a constant $C > 0$ such that $f(r) \equiv Cp(r)$ is a function of F.

Proof. Since $\int_\alpha^\beta xp(x)\ dx \leq R/2$, the integral

$$\int_0^R x\ p(x)\ dx \equiv K > 0 \text{ exists and hence}$$

$$\int_0^R 2\pi r(1/2\pi K)\ p(r)\ dr = 1$$

and $Cp(r) \equiv f(r)$ is in F, where $C = 1/2\pi K > 0$.

Theorem 4. If p(x) is any function which is finite, continuous, and non-negative on (0,R),

with $\int_0^R p(x)\ dx = \frac{1}{2}$, then its transform

$$y^*(r) = 2\int_0^{\sqrt{R^2-r^2}} p(\sqrt{r^2+x^2})\ dx;\ 0 \leqslant r < R$$

$$y^*(R) \equiv 0$$

is finite, continuous, and nonnegative on (0,R), with

$y^*(0) = 1$, $y^*(R) = 0$, and $\int_0^R y^*(r)\ dr = 1/2\ C$, where C is the constant of Theorem 3.

Proof. Let $f(r) = Cp(r)$ be in F, as in Theorem 3. By Theorem 2, its marginal density

$$\hat{p}(x) = 2\int_0^{\sqrt{R^2-x^2}} Cp(\sqrt{x^2+y^2})\ dy,\ 0 < |x| < R$$

is finite, continuous, nonnegative on (0,R), with

$$\int_0^R \hat{p}(x)\ dx = \frac{1}{2}.$$

Changing notation, this implies

$$\hat{p}(r) = 2\int_0^{\sqrt{R^2-r^2}} Cp(\sqrt{r^2+x^2})\ dx;\ 0 < r < R.$$

Thus $\hat{p}(r) = C\, y^*(r)$, where $y^*(r)$ is the above transform of $p(x)$, so $y^*(r)$ has the properties listed for $\hat{p}(r)$ on $(0,R)$. The rest of the theorem is trivial.

The transform $y^*(r)$ of a function $p(x)$ with the properties of Theorem 4 need not be continuous on the <u>closed</u> interval $[0,R]$. Thus $p(x) = 1/\pi\sqrt{(1-x^2)}$ on $(0,1)$ has $y^*(r) \equiv 1$ on $0 \leqslant r < 1$ and $y^*(1) = 0$.

Now consider the iterated mappings

$$y(r) \to f(r) \to p(x) \to y^*(r) \tag{C}$$

defined respectively by Eq. (4), (2), (3) for the sets Y, F, P, Y*. We then have the remarkable

<u>Theorem 5</u>. For every function $y(r)$ in Y, the final function $y^*(r)$ in (C) is $y(r)$ itself, and hence all three of these mappings are "one-one and onto all."

Proof. By Theorem 2, the marginal density $p(x)$ in P of a function $f(r)$ in F has the properties of Theorem 4. Consequently its transform is finite, continuous, and nonnegative on $(0,R)$, with

$$y^*(0) = 1 = y(0), \quad y^*(R) = 0 = y(R)\,.$$

Hence it suffices to prove that $y^*(r) = y(r)$ for $0 < r < R$. For such an r, we see that

$$\begin{aligned} y^*(r) &= 2\int_0^{\sqrt{R^2-r^2}} dx\; 2\int_0^{\sqrt{R^2-r^2-x^2}} f(\sqrt{r^2+x^2+y^2})\; dy \\ &= 4\int_0^{\pi/2} d\theta \int_0^{\sqrt{R^2-r^2}} f(\sqrt{r^2+\rho^2})\;\rho\, d\rho \\ &= \int_r^R 2\pi\, s\, f(s)\, ds = y(r) \end{aligned}$$

by Theorem 1. Here we have introduced polar coordinates (ρ,θ) in place of (x,y) and transformed ρ to s by means of $r^2+\rho^2 = s^2$.

Theorem 6. If a function $p(x)$ is indeed the marginal density of a function $f(r)$ in F, that is to say, $p(x)$ is in the set P, then its complete chain

$$y(r) \rightarrow f(r) \rightarrow p(x) \rightarrow y^*(r) = y(r)$$

shows that its unique radial source density is given by

$$f(r) = (-1/2\pi r) \frac{d}{dr} 2 \int_0^{\sqrt{R^2-r^2}} p(\sqrt{r^2+x^2})\, dx; \; 0 < r < R \tag{6}$$

explicitly in terms of $p(x)$.[2]

Proof. For, $f(r) = (-1/2\pi r)\, y'(r)$ by Theorem 1, and $y(r) = y^*(r)$ by Theorem 5.

Theorem 7. A finite, continuous, nonnegative, even function $p(x)$ on $0 < |x| < R$, with

$$\int_0^R p(x)\, dx = ½$$

is in fact the marginal density of some function $f(r)$ in F iff its transform

$$y^*(r) = 2 \int_0^{\sqrt{R^2-r^2}} p(\sqrt{r^2+x^2})\, dx, \; 0 \leqslant r < R$$

$$y^*(R) \equiv 0$$

is continuous nonincreasing on $[0,R]$ and has a continuous derivative on $(0,R)$, in which case its unique source density $f(r)$ is given by Eq. (6).

Proof. The necessity of these conditions is obvious since $Y^* = Y$. Conversely, if a function $p_o(x)$ has these properties, then by Theorem 4, its transform $y_o^*(r)$ is certainly some function $y(r)$ in Y, so we have

$$p_o(x) \rightarrow y_o^*(r) \equiv y(r) \in Y .$$

If $\quad y(r) \rightarrow f(r) \rightarrow p(x) \rightarrow y^*(r) = y(r)$

is the complete chain (C) for $y(r)$, then $p_o(x)$ and $p(x)$ have the same transform, so that

$$2\int_0^{\sqrt{R^2-r^2}} p_o(\sqrt{r^2+x^2})\,dx = 2\int_0^{\sqrt{R^2-r^2}} p(\sqrt{r^2+x^2})\,dx;\ 0 \leqslant r \leqslant R .$$

Changing notation, it follows that

$$2\int_0^{\sqrt{R^2-x^2}} p_o(\sqrt{x^2+y^2})\,dy = 2\int_0^{\sqrt{R^2-x^2}} p(\sqrt{x^2+y^2})\,dy;\ 0 \leqslant x \leqslant R .$$

By Theorem 3, there are constants C_o, C such that $f_o(r) \equiv C_o\, p_o(r)$ and $f(r) \equiv Cp(r)$ are in F, so that

$$2\int_0^{\sqrt{R^2-x^2}} f_o(\sqrt{x^2+y^2})\,dy = (C_o/C) \cdot 2\int_0^{\sqrt{R^2-x^2}} f(\sqrt{x^2+y^2})\,dy .$$

But then $m_o(x) = (C_o/C)\, m(x)$, where $m_o(x)$, $m(x)$ are the marginal densities of $f_o(r)$, $f(r)$. Since

$$\int_0^R m_o(x)\,dx = 1/2 = \int_0^R m(x)\,dx,$$

it follows that $(C_o/C) = 1$, and hence the functions $f_o(r)$, $f(r)$ of F have the same marginal density. Since the mapping $f(r) \to p(x)$ is one-one by Theorem 5, we have $f_o(r) = f(r)$ and hence $p_o(x) = p(x)$. Thus $p_o(x)$ is the marginal density of $f(r)$ as given in Eq. (6), with $p(x) = p_o(x)$.

As a nontrivial example we consider the function $p(x) = -\frac{1}{2}\log|x|$ on $0 < |x| < 1$. This is a marginal density according to Theorem 7. In fact, its transform is

$$y^*(r) = (1-r^2)^{\frac{1}{2}} - r\arctan\left((1-r^2)^{\frac{1}{2}}/r\right);\ 0 \leqslant r \leqslant 1$$

with $\frac{d}{dr}\, y^*(r) = -\arctan\left((1-r^2)^{\frac{1}{2}}/r\right)$ on $(0,1)$.

Hence it must be the marginal density of the unique function $f(r) = \frac{1}{2\pi r}$ arctan $\left((1-r^2)^{\frac{1}{2}}/r\right)$. It is a nice exercise to verify that the original p(x) is in fact the marginal density of the stated f(r).

We include some examples (for R=1) which illustrate various features of the chain (C).

Ex. 1. $y(r) = 1-r^2$, $f(r) = 1/\pi$, $p(x) = (2/\pi)(1-x^2)^{\frac{1}{2}}$, $y^*(r) = y(r)$, p(x) decreasing.

Ex. 2. $y(r) = (1-r^2)^{\frac{1}{2}}$, $f(r) = 1/2\pi\bullet\sqrt{1-r^2}$, $p(x) = \frac{1}{2}$, $y^*(r) = y(r)$, p(x) constant.

Ex. 3. $y(r) = 1-r^4$, $f(r) = (2/\pi)r^2$, $p(x) = (4/\pi)\,x\left(x^2(1-x^2)^{\frac{1}{2}} + (1/3)(1-x^2)^{3/2}\right)$, $y^*(r) = y(r)$, p(x) nonmonotone.

Two simple examples of functions p(x) which are finite, continuous, nonnegative, even on $0 < |x| < 1$, with

$$\int_0^1 p(x)\,dx = \tfrac{1}{2}, \text{ but \underline{not} marginal are}$$

Ex. 4. $p(x) = 1/\pi\sqrt{1-x^2}$, with $y^*(r) = 1$ on $[0,1)$, $y^*(1) = 0$.

Ex. 5. $p(x) = (3/2)\,x^2$, with $y^*(r) = 3r^2(1-r^2)^{\frac{1}{2}} + (1-r^2)^{3/2}$ nonmonotone.

With suitable changes the theory applies to the case $R = \infty$. We give two examples.

Ex. 6. The normal density $p(x) = (2\pi)^{-\frac{1}{2}}\,e^{-x^2/2}$ on $(-\infty,\infty)$ is the marginal density of $f(r) = (1/2\pi)\,e^{-r^2/2}$ on $(0,\infty)$, in accord with Eq. (6).

Ex. 7. The Cauchy density $p(x) = 1/\pi(1+x^2)$ on $(-\infty,\infty)$ is the marginal density of $f(r) = 1/2\pi(1+r^2)^{3/2}$ on $(0,\infty)$ by Eq. (6).

II. THE CASE OF THREE DIMENSIONS

For this case we need only introduce two sets of functions F and P.

F is the set of all finite continuous nonnegative functions f(r) on (0,R) for which

$$\int_0^R 4\pi r^2\, f(r)\,dr = 1. \tag{7}$$

Such a function f(r) defines a radially symmetric density $f(\sqrt{x^2+y^2+z^2})$ on the sphere

$$S = \{(x,y,z);\ x^2 + y^2 + z^2 < R^2\}$$

with $\int_S f(\sqrt{x^2+y^2+z^2})\ dx\ dy\ dz = 1.$

P is the set of all marginal densities

$$p(x) = \int_{C_x} f(\sqrt{x^2+y^2+z^2})\ dy\ dz;\ 0 < |x| < R, \tag{8}$$

where (y,z) ranges over the circle

$$C_x = \{(y,z);\ y^2+z^2 < R^2-x^2\}$$

and f(r) is in F.

Introducing polar coordinates (ρ,θ) into C_x, we see that

$$p(x) = \int_{C_x} f(\sqrt{x^2+\rho^2})\ \rho\ d\rho\ d\theta = \int_0^{\sqrt{R^2-x^2}} 2\pi\rho f(\sqrt{x^2+\rho^2})\ d\rho;\ 0 < |x| < R. \tag{9}$$

For fixed x on (0,R), we make the ρ to r substitution $x^2+\rho^2 = r^2$ to obtain

$$p(x) = \int_x^R 2\pi r f(r)\ dr;\ 0 < x < R\ . \tag{10}$$

The latter exists, because

$\int_x^\beta$ is a nondecreasing function of β and bounded above by

$\int_x^\beta (1/2x)\ 4\pi r^2 f(r)\ dr \leqslant 1/2x.$

Theorem 8. Every marginal density p(x) in P is finite, continuous, non-negative, and even on $0 < |x| < R$ with

$$\int_0^R p(x)\ dx = 1/2.$$

Proof. Eq. (10) implies p(x) is finite and continuous on (0,R), while Eq. (9) shows that it is nonnegative and even on $0 < |x| < R$. Moreover, from Eq. (9) we see that

$$\int_0^R p(x)\ dx = \int_0^R dx \int_0^{\sqrt{R^2-x^2}} 2\pi\rho f(\sqrt{x^2+\rho^2})\ d\rho = \int_0^R 2\pi r^2 f(r)\ dr = 1/2.$$

Here we have employed polar coordinates (r,θ) in the transformation $x = r \cos \theta$, $\rho = r \sin \theta$.

Theorem 9. Every marginal density p(x) in P has a continuous derivative on $0 < |x| < R$, nonnegative on (-R,0) and nonpositive on (0,R). In fact,

$$p'(x) = -2\pi x f(x);\ 0 < x < R.$$

Proof. Fixing x_o on (0,R) we have from Eq. (10),

$$\lim_{x \to x_o} (x-x_o)^{-1} \left(p(x) - p(x_o)\right)$$

$$= \lim_{x \to x_o} (x-x_o)^{-1} \lim_{\beta \to R} \left(\int_x^\beta - \int_{x_o}^\beta\right)$$

$$= -\lim_{x \to x_o} (x-x_o)^{-1} \int_{x_o}^x 2\pi r f(r)\ dr$$

$$= -\lim_{x \to x_o} 2\pi\xi f(\xi) = -2\pi x_o f(x_o),$$

since f(r) is continuous on (0,R) and ξ is between x and x_o.

Theorem 10. Every marginal density p(x) in P has

$$\lim_{x \to R^-} p(x) = 0.$$

Proof. From Eq. (10) we obtain

$$\lim_{x \to R} \int_x^R 2\pi r f(r)\, dr = \lim_{x \to R} \lim_{\beta \to R} \left(-\int_{R/2}^{x} + \int_{R/2}^{\beta} \right)$$

$$= -p(R/2) + p(R/2) = 0.$$

A marginal density p(x) may well be unbounded as $x \to 0^+$. Thus the function $f(r) = (1/4\pi R)r^{-2}$ is in F, and from Eq. (10) we find $p(x) = (1/2R)(\log R - \log x)$ with

$$\lim_{x \to 0^+} p(x) = \infty.$$

However, p(x) does have the property of

Theorem 11. Every marginal density p(x) in P has

$$\lim_{x \to 0^+} x\, p(x) = 0.$$

Proof. From Theorems 8, 9, 10, it follows that

$$1 = \int_0^R 4\pi r^2 f(r)\, dr = -\int_0^R 4\pi r^2 (1/2\pi r)\, p'(r)\, dr$$

$$= -2\int_0^R r\, p'(r)\, dr = -2\, r\, p(r)\Big]_0^R + 2\int_0^R p(r)\, dr$$

$$= -0 + 2 \lim_{r \to 0} r\, p(r) + 1,$$

and the result follows formally. (See however the note after Theorem 13.)

Clearly, Theorem 9 implies that the correspondence $f(r) \to p(x)$ on F to P defined by Eq. (8) is one-one, so we may state

Theorem 12. The unique function f(r) of F with a given marginal density p(x) is

$$f(r) = (-1/2\pi r)\ p'(r);\ 0 < r < R. \qquad (11)$$

The properties of these theorems are characteristic of a marginal density, as specified in

<u>Theorem 13</u>. A function $p(x)$ on $0 < |x| < R$ is the marginal density of a function $f(r)$ in F iff

(A) $p(x)$ is finite, continuous, nonnegative, and even on

$$0 < |x| < R, \text{ with } \int_0^R p(x)\ dx = 1/2,$$

(B) $p(x)$ has a continuous derivative $p'(x) \leq 0$ on $(0,R)$,

(C) $\lim_{x \to R^-} p(x) = 0$, and

(D) $\lim_{x \to 0^+} x\ p(x) = 0.$

In fact, a function $p(x)$ with these properties is the marginal density of the unique function $f(r) = (-1/2\pi r)\ p'(r)$ in F.

Proof. We have already seen the necessity of these conditions. Conversely, given a function $p(x)$ on $0 < |x| < R$ satisfying (A) - (D) we <u>define</u> a function

$$f(r) = (-1/2\pi r)\ p'(r);\ 0 < r < R \qquad (12)$$

in terms of $p(x)$. (a) We first verify that this $f(r)$ is indeed in the set F. For, $f(r)$ is continuous and nonnegative by (B), and from (A-D) we see that

$$\int_0^R 4\pi r^2(-1/2\pi r)\ p'(r)\ dr = -2\int_0^R r\ p'(r)\ dr$$

$$= -2\ r\ p(r)\Big]_0^R + 2\int_0^R p(r)\ dr$$

$$= -0 + 0 + 2(\tfrac{1}{2}) = 1.$$

Hence the f(r) of Eq. (12) is in F. (b) It only remains to show that this f(r) of F has the given function p(x) as its marginal density. To see this we evaluate the integral in Eq. (10) for $0 < x < R$:

$$\int_x^R 2\pi r(-1/2\pi r)\, p'(r)\, dr$$

$$= -\lim_{\beta \to R} \int_x^\beta p'(r)\, dr = \lim_{\beta \to R} (p(x) - p(\beta))$$

$$= p(x)$$

by property (C), and the Theorem follows from (A).

Note. The property (D) is in fact redundant since any function p(x) with the properties in (A), having a derivative $p'(x) \leq 0$ on $(0,R)$ necessarily has

$$\lim_{x \to 0^+} xp(x) = 0,$$

but we have let the statement of Theorem 13, and the proof of Theorem 11 stand. It is possible (but not easy) to show that (A) alone does not imply (D). (See Part III.)

We conclude with some examples of the three dimensional case.

Ex. 8. $p(x) = (5/8)\,(1-x^4)$, $-1 < x < 1$ is the marginal density of $f(r) = (5/4\pi)r^2$ on $0 < r < 1$.

Less trivially, we have

Ex. 9. $p(x) = |x|^{m-1}\,(1-|x|)^{n-1}/2B(m,n)$, $m > 0$, $n > 0$, $0 < |x| < 1$

is a marginal density iff $m \leq 1$, $n > 1$, $m+n \geq 2$, or $n > 1$, $m+n < 2$. Such a function is the marginal density of

$$f(r) = r^{m-3}\,(1-r)^{n-2}\,((m+n-2)\,r + (1-m))/4\pi B(m,n),\ 0 < r < 1.$$

A natural candidate for $f(r)$ is given in

Ex. 10. $f(r) = r^{m-1}(1-r)^{n-1}/4\pi B(m+2,n)$, $0 < r < 1$, $m > -2$, $n > 0$

is in F, and has as marginal density on (0,1)

$$p(x) = \int_x^1 r^m(1-r)^{n-1}\, dr/2B(m+2,n).$$

With some obvious modifications, everything applies to the interval $(0,\infty)$ in place of (0,R). A few examples of this kind are:

Ex. 11. $p(x) = |x|^{m-1} e^{-|x|}/2\Gamma(m)$, $m > 0$, $0 < |x| < \infty$, is a marginal density iff $m \leqslant 1$. Such a function is the marginal density of

$$f(r) = r^{m-3}(r+(1-m))\, e^{-r}/4\pi\Gamma(m),\quad 0 < r < \infty.$$

Ex. 12. $f(r) = r^{m-1} e^{-r}/4\pi\Gamma(m+2)$, $0 < r < \infty$, $m > -2$ is in F, with marginal density on $(0,\infty)$

$$p(x) = \int_x^\infty r^m e^{-r}\, dr/2\Gamma(m+2).$$

Ex. 13. The normal den :ty

$$f(r) = (2\pi\sigma^2)^{-3/2} \exp(-r^2/2\sigma^2);\ 0 < r < \infty,\ r = \sqrt{x^2+y^2+z^2},$$

has marginal density

$$p(x) = (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2);\ -\infty < x < \infty$$

in agreement with Eq. (9), and one easily verifies that $f(r) = (-1/2\pi r)\, p'(r)$; $0 < r < \infty$ in agreement with Eq. (11).

III. TWO NOTES ON "REAL VARIABLES"

In connection with Theorems 11 and 13, we include here an example and a theorem which are perhaps not as well known as they might be.

Example. We give an example of a function p(x) which is finite, continuous and nonnegative on (0,1], with

$$\int_0^1 p(x)\,dx$$

finite, for which

$$\lim_{x \to 0} x\,p(x)$$

does not exist.

Let p(x) = q(x)/x, where q(x) is a "saw tooth" function which, on the interval $[1/2^{j+1}, 1/2^j]$, j = 0, 1, 2, ... has a "tooth" defined by an isosceles triangle of unit height, and base of length $1/2^{2j+1}$, the base terminating at the right hand end point $x = 1/2^j$, q(x) being 0 elsewhere on the interval.

Then p(x) = q(x)/x has the properties stated.

$$\text{In fact, } \int_0^1 p(x)\,dx = \sum_0^\infty \int_{1/2^{j+1}}^{1/2^j} q(x)\,dx/x$$

$$< \sum_0^\infty \left(2^{j+1}\right)\left(1/2\right)\left(1/2^{2j+1}\right)\left(1\right) = \sum_0^\infty 1/2^{j+1} = 1,$$

whereas x p(x) = q(x) has no limit as $x \to 0$.

The redundance of condition (D) in Theorem 13 is seen from the following

Theorem 14. If p(x) is a function which is finite, continuous, nonnegative, and nonincreasing on (0,R), with a finite integral

$$\int_0^R p(x)\,dx, \text{ then}$$

$$\lim_{x \to 0} xp(x) = 0 \text{ necessarily.}$$

Proof. Writing $p(x) = q(x)/x$, we observe that the falsity of the conclusion would imply the existence of a null sequence x_n with $q(x_n) \geqslant \varepsilon > 0$, and hence $p(x_n) = q(x_n)/x_n \geqslant \varepsilon/x_n$. Since $p(x)$ is nonincreasing, we then have

$$\int_0^{x_n} p(x)\, dx \geqslant p(x_n) \int_0^{x_n} dx \geqq (\varepsilon/x_n)\, x_n = \varepsilon > 0, \text{ whereas}$$

$$\int_0^{x_n} p(x)\, dx \to 0 \text{ as } n \to \infty.$$

All the above assertions involving improper integrals are easily justified from their definitions as limits.

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