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AUTHOR(S): J. D. Louck, W. A. Beyer, L. C. Biedenharn, and P. R. Stein

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Los Alamos National Laboratory
Los Alamos, New Mexico 87545

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SYMMETRIES OF SOME HYPERGEOMETRIC SERIES: IMPLICATIONS FOR 3j- AND 6j-COEFFICIENTS *

J. D. Louck, W. A. Beyer
L. C. Biedenharn and P. R. Stein
Theoretical Division, Los Alamos National Laboratory
Los Alamos, New Mexico 87545

ABSTRACT. The occurrence of generalized hypergeometric series as factors in the Wigner-Clebsch-Gordan (3j) and Racah (6j) coefficients is well known. The recently discovered[1] S_5 symmetry of the Saalschützian ${}_4F_3$ series may be used to extend the symmetries of the 6j-coefficients to the much larger group generated by S_5 and the group of Regge symmetries. (A similar extension may be carried out for the 3j-coefficients.) The required extension of the domain of definition of the 6j-coefficients and the properties of its symmetry group is developed here.

1. INTRODUCTION

It was shown recently[1] that both the ${}_3F_2$ and Saalschützian ${}_4F_3$ generalized hypergeometric series possess the group S_5 of permutational symmetries. This is an extension of the trivial groups $S_3 \times S_2$ and $S_3 \times S_3$ symmetries corresponding to the separate permutations of numerator and denominator parameters that occur in the definitions of these series, respectively. These results are reviewed below. In this paper we apply this extended symmetry of the ${}_4F_3$ series to the Racah (Wigner 6j) coefficients of the quantal rotation group SU(2). (Similar results for the 3j-coefficients may also be obtained and will be published separately.)

The definition of the action of the group S_5 on the set of 6j-coefficients requires that the domain of definition of these coefficients (the standard triangle conditions) be extended. A principal result given here is an extension of the standard domain to an arbitrary point $\underset{\sim}{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, which is suitable for the definition of a group action of S_5. The symmetry group G of these extended 6j-coefficients is generated by the group S_5 and the well-known Regge group $\mathcal{R}$

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of symmetries, which is isomorphic to $S_4 \times S_3$. The group G is characterized more precisely in the sequel.

We review briefly the action of the Regge[2] group on the set of 6j-coefficients, which are conveniently defined in terms of the Bargmann[3] array. Next the 6j-coefficients are expressed in terms of the Saalschützian $_4F_3$ series. This is a key step, since it is this relation in its extended form that admits the successful 'joining' of the two group actions of S_5 and $\mathcal{R}$ to that of G, which at this point is realized as a group of 7 x 7 real matrices.

2. BASIC RELATIONS

The generalized hypergeometric series $_pF_q$ of unit argument is defined by

$$ {}_pF_q\begin{pmatrix} a_1,a_2,\ldots,a_p \\ b_1,b_2,\ldots,b_q \end{pmatrix} = \sum_k \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k k!} , \quad (1) $$

where $(x)_k$ denotes the rising factorial defined by $(x)_0 = 1$, $(x)_k = x(x+1)(x+2)\cdots(x+k-1)$ for $k = 1,2,\ldots$. We regard $_pF_q$ as a function of its numerator and denominator parameters $(a_1,a_2,\ldots,a_p)$ and $(b_1,b_2,\ldots,b_q)$. This function is clearly invariant under the group $S_p \times S_q$ of permutations of these parameters.

Under a suitable change of variables and redefinition of function, Thomae's identity[4] and the $S_3 \times S_2$ permutational symmetry of $_3F_2$ can be united into a single symmetry under S_5. These results are summarized by the following relations.

<u>Thomae's identity</u>:

$$ {}_3F_2\begin{pmatrix} a,b,c \\ d,e \end{pmatrix} = \frac{\Gamma(a')\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(d')\Gamma(e')} \, {}_3F_2\begin{pmatrix} a',b',c' \\ d',e' \end{pmatrix} , \quad (2a) $$

where $\underline{a} = (a,b,c,d,e)$ and $\underline{a}' = (a',b',c',d',e')$ are related by the column matrix multiplication

$$ \underline{a}' = t\underline{a} , \quad t = \begin{bmatrix} -1 & -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & 0 & -1 & 1 & 1 \end{bmatrix} . \quad (2b) $$

change of variables: $\underline{x} = (x,y,z,u,v)$

$$\underline{a} = A_1\underline{x}, \quad A_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \tag{3}$$

new function:

$$F(x,y,z,u,v) = {}_3F_2\left(\begin{matrix} x+u+v, & y+u+v, & x+u+v \\ x+y+z+2u+v, & x+y+z+u+2v & \end{matrix}\right) \Big/ D, \tag{4a}$$

where the denominator D is defined by

$$D = \Gamma(x+y+z+2u+v)\Gamma(x+y+z+u+2v)\Gamma(x+y+z). \tag{4c}$$

The symmetry of $F(x)$ is given by

$$F(p\underline{x}) = F(\underline{x}), \quad p \in S_5, \ \underline{x} \in \mathbb{R}^5. \tag{5}$$

Similarly, by a change of variables and redefinition of function, Bailey's[4] identity and the $S_3 \times S_3$ permutational symmetry of the Saalschützian ${}_4F_3$ series can be united into a single symmetry under S_5 [1]:

Bailey's identity:

$${}_4F_3\left(\begin{matrix} a,b,c,-n \\ d,e,f \end{matrix}\right) = \frac{(d')_n(e')_n(f')_n}{(d)_n\ (e)_n\ (f)_n}\ {}_4F_3\left(\begin{matrix} a',b',c',-n \\ d',e',f' \end{matrix}\right), \tag{6a}$$

where $\underline{a} = (a,b,c,d,e,f)$ and $\underline{a}' = (a',b',c',d',e',f')$ are related by

$$\underline{a}' = m\underline{a}, \qquad m = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6b}$$

with Saalschütz condition

$$a + b + c - d - e - f - n + 1 = 0. \tag{6c}$$

Here n is a nonnegative integer.

change of variables: $\underline{x} = (x,y,z,u,v,w)$

$$\underline{a} = A_2\underline{x}, \quad A_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

new function

$$\mathcal{P}_n(\underline{x}) = (x_1+x_2+x_3+x_4)_n(x_1+x_2+x_3+x_5)_n(x_4+x_5)_n$$

$$\cdot (-1)^n \cdot {}_4F_3\left(\begin{matrix} x_1+x_2, & x_2+x_3, & x_3+x_1, & -n \\ x_1+x_2+x_3+x_4, & x_1+x_2+x_3+x_5, & x_4+x_5 \end{matrix}\right)$$

$$= (-1)^n \sum_{k=0}^{n} \binom{n}{k}(x_1+x_2+x_3+x_4+k)_{n-k}(x_1+x_2+x_3+x_5+k)_{n-k}(x_4+x_5)_{n-k}$$

$$\cdot (x_1+x_2)_k(x_2+x_3)_k(x_3+x_1)_k \quad . \tag{8}$$

The symmetry of the polynomial $\mathcal{P}_n$ is given by

$$\mathcal{P}_n(p\underline{x}) = \mathcal{P}_n(\underline{x}),\ p \in S_5,\ \underline{x} \in \mathbb{R}^5 \ . \tag{9}$$

Relations (5) and (9) are the ones required for extending the symmetries of the 3j- and 6j-coefficients, respectively. For this the relation of these coefficients to the ${}_3F_2$ and ${}_4F_3$ series are needed. These results are well known (see Biedenharn and Louck[5] for references to the literature and for a discussion of the subtleties involved). We consider here only the 6j-coefficients, since many results for the 3j-coefficient can then be obtained from the well-known limit of the 6j-coefficients into the 3j-coefficients.[5] These results will be given elsewhere.

Let us recall the definition of the 6j-coefficient as given by Bargmann[5]. First consider the domain of definition. A Bargmann array B is a 4 x 3 array of nonnegative integers (b_{ij}), $i = 1,2,3,4; j = 1,2,3,$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} \text{, such that each } 2 \times 2 \text{ minor } \begin{bmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{bmatrix} \tag{10}$$

obeys $b'_{11}+b'_{22} = b'_{12}+b'_{21}$ Two standard parametrizations of the array B are given by

$$\begin{pmatrix} a+b-c & a+f-e & b+f-d \\ a+e-f & a+c-b & c+e-d \\ b+d-f & d+c-e & b+c-a \\ d+e-c & d+f-b & e+f-a \end{pmatrix} \ ; \quad \begin{pmatrix} \beta_1-\alpha_1 & \beta_2-\alpha_1 & \beta_3-\alpha_1 \\ \beta_1-\alpha_2 & \beta_2-\alpha_2 & \beta_3-\alpha_2 \\ \beta_1-\alpha_3 & \beta_2-\alpha_3 & \beta_3-\alpha_3 \\ \beta_1-\alpha_4 & \beta_2-\alpha_4 & \beta_3-\alpha_4 \end{pmatrix} \ , \quad \Sigma\alpha_i = \Sigma\beta_j . \tag{11}$$

The parameters (a,b,c,d,e,f) are the usual geometric ones correspond- to the 6 angular momenta that constitute the four triangles (a,b,c), (a,e,f), (b,d,f), (c,d,e) of a tetrahedron. The parameters $(\alpha_1,\alpha_2,\alpha_3,\alpha_4;\beta_1,\beta_2,\beta_3)$ are quantities associated with the row sums $R_i = \sum_j b_{ij}$ and column sums $C_j = \sum_i b_{ij}$: $\alpha_i = (S - R_i)/3$, $\beta_j = (S + C_j)/4$, $S = \sum_i R_i$.

The Racah function R is a function defined on the set $\mathcal{B}$ of all Bargmann arrays with values in the closed interval $[-1,1]$; that is, $B \xrightarrow{R} [-1,1]$, each $B \in \mathcal{B}$. The explicit definition is given by

$$R(B) = \left[\frac{\Pi b_{ij}!}{\Pi(\alpha_i+1)!}\right]^{1/2} \sum_k \frac{(-1)^k (k+1)!}{\prod_i (k-\alpha_i)! \prod_j (\beta_j - k)!} \quad . \tag{12}$$

The standard 6j-coefficient is obtained from this expression by $\begin{Bmatrix} abc \\ def \end{Bmatrix} = R(B)$ with B in terms of the parameters (a,b,c,d,e,f).

Letting P_n denote the group of n x n permutation matrices, the Regge group $\mathcal{R}$ may be realized as the matrix group of ordered pairs defined by $\mathcal{R} = \{(p_4,p_3) \mid p_4 \in P_4,\ p_3 \in P_3\}$. The action of $(p_4,p_3) = r \in \mathcal{R}$ on the Bargmann array $B \in \mathcal{B}$ is defined by $B \xrightarrow{r} B' = r \bullet B = p_4 B p_3^t$, where t denotes transposition. Since row and column sums are invariant under permutations of rows and columns of B, the Racah function R(B) is invariant under the action of the 144 element group $\mathcal{R}$; that is,

$$R(r \bullet B) = R(B),\ r \in \mathcal{R}\ ,\ B \in \mathcal{B}. \tag{13}$$

3. EXTENSION OF THE RACAH FUNCTION

We define the function $\gamma(x)$, $x \in \mathbb{R}$, by*

$$\gamma(x) = \begin{cases} \Gamma(x+1)/\sqrt{\pi}\ , & x \geq -1/2 \\ \sqrt{\pi}\ /\Gamma(-x)\ , & x \leq -1/2\ . \end{cases} \tag{14}$$

This function is <u>continuous</u> <u>and</u> <u>positive</u> for all $x \in \mathbb{R}$. Using $\gamma(x)$, we now define the extended Racah function $W_n : \mathbb{R}^5 \rightarrow \mathbb{R}$ by

$$W_n(\underline{x}) = \left[\frac{\prod_{i<j} \gamma(x_i+x_j-1)}{n!\ \prod_i \gamma(-x_i+X+n-1)}\right]^{1/2} \cdot \mathcal{P}_n(\underline{x})\ ,\quad X = \sum_{j=1}^{5} x_j, \tag{15}$$

where $\mathcal{P}_n(\underline{x})$ is the polynomial defined by (8). The function $W_n(\underline{x})$ clearly satisfies

$$W_n(p\underline{x}) = W_n(\underline{x}),\quad p \in S_5,\ \underline{x} \in \mathbb{R}^5, \tag{16}$$

since the 10 numerator γ-factors and the 5 denominator γ-factors are permuted among themselves under interchanges of x_1,x_2,x_3,x_4,x_5.

*Originally we defined a discontinuous $\gamma(x)$ by omitting $\sqrt{\pi}$ and with $x \geq 0$ and $x < 0$. H. Bacry suggested this be modified to (14).

We next describe how the 6j-coefficients are obtained from the function $W_n(\underset{\sim}{x})$. First, we introduce the nonstandard parametrization (x,y,z,u,v) of the Bargmann array B defined by

$$B = \begin{pmatrix} n & x+y+z+u+n-1 & x+y+z+v+n-1 \\ -y-z & x+u-1 & x+v-1 \\ -z-x & y+u-1 & y+v-1 \\ -x-y & z+u-1 & z+v-1 \end{pmatrix} . \qquad (17)$$

Using these parameters to define the function R_n by $R_n(x,y,z,u,v) = R(B)$, where B is given by (17) and R(B) by (12), it is straightforward, but nontrivial, to prove that

$$R_n(x,y,z,u,v) = (-1)^{\Phi}\, \pi^{-\frac{1}{2}}\, W_n(x,y,z,u,v), \quad \Phi = u+v. \qquad (18)$$

Thus, the 6j-coefficients are recovered, up to $\pi^{\frac{1}{2}}$ and a phase factor, from $W_n(\underset{\sim}{x})$ by restricting the point $\underset{\sim}{x} = (x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5$ to the domain of (x,y,z,u,v) as determined by the Bargmann array.

4. GROUP OF THE EXTENDED 6j-COEFFICIENTS

Each element $r \in \mathscr{R}$ effects the transformation $B \xrightarrow{r} B'$, which, in turn, uniquely determines a 7 x 7 matrix transformation $\underset{\sim}{z} \xrightarrow{r} \underset{\sim}{z}' = m(r)\underset{\sim}{z}$ of the variables $\underset{\sim}{z} = (\underset{\sim}{\xi},n,1)$, $\underset{\sim}{z}' = (\underset{\sim}{\xi}',n',1)$, where $\underset{\sim}{\xi} = (x,y,z,u,v)$, $\underset{\sim}{\xi}' = (x',y',z',u',v')$. The variables $(\underset{\sim}{\xi},n)$ and $(\underset{\sim}{\xi}',n')$ are the parametrization (17) of B and B'. The 1 in z and z' takes into account that $(\underset{\sim}{\xi},n)$ and $(\underset{\sim}{\xi}',n')$ are related by a linear transformation plus a translation. Thus, we obtain the matrix group $M = \{m(r) | r \in \mathscr{R}\}$ isomorphic to $\mathscr{R}$. For economy of notation, it is convenient to define the action of M on $\underset{\sim}{\xi}$ by $m \circ \underset{\sim}{\xi} = \underset{\sim}{\xi}' = \Pi(m\underset{\sim}{z}) = \Pi(\underset{\sim}{z}')$, $m \in M$, where Π is the projection of the first five components of $\underset{\sim}{z}' = m\underset{\sim}{z}$ (column matrices).

The Regge symmetry $R(B) = R(r \bullet B) = R(B')$ given by (13) is now expressed first as $R_{n'}(m \circ \underset{\sim}{\xi}) = R_n(\underset{\sim}{\xi})$ [see the definition of R_n following (17)], and then by (18) as

$$W_{n'}(m \circ \underset{\sim}{\xi}) = W_n(\underset{\sim}{\xi}), \quad n \xmapsto{m} n'. \qquad (19)$$

The phase factor $(-1)^{\Phi}$ in (18) is invariant under Regge transformations, since $u+v = R_1+C_1-4n+2$. In all there are 12 possible functions $W_{n'}$ in (19) corresponding to the 12 entries in B, any one of which can be brought to the upper left corner. Relation (19) is just a re-expression of the Regge symmetries in terms of $(\underset{\sim}{\xi},n)$ and $(\underset{\sim}{\xi}',n')$. This form is, however, adapted to the S_5 symmetry of $W_n(\underline{x})$, $\underline{x} \in \mathbb{R}^5$.

Let $P_{5,1,1} = P_5 \oplus 1 \oplus 1$ (matrix direct sum). Replacing $\underline{\xi}$ in (19) by $p \circ \underline{\xi}$, $p \in P_{5,1,1}$, we obtain $W_{n'}(m \circ (p \circ \underline{\xi})) = W_n(p \circ \underline{\xi}) = W_n(\underline{\xi})$, and since $m \circ (p \circ \underline{\xi}) = (mp) \circ \underline{\xi}$, we find

$$W_{n'}(g \circ \underline{\xi}) = W_n(\underline{\xi}),\ g \in G,\ B \text{ parametrized by } (\underline{\xi}, n), \qquad (20)$$

where G is the group generated by the 144 element group M (isomorphic to $S_4 \times S_3$) and the 120 element group $P_{5,1,1}$ (isomorphic to S_5). In (20) the nonnegative integer n' is related to n by $n \overset{m}{\longmapsto} n'$, where m is uniquely given by $g = mp$, $m \in M$, $p \in P_{5,1,1}$ for each $g \in G$. Relation (20) is our main result. It extends the symmetries of the Racah coefficients to the group G by extending the domain of these coefficients.

The group G is conveniently given as a union of double cosets:

$$G = \bigcup_{i=1}^{16} Hc_iH\ ,\ H = P_{5,1,1},\ \ |G| = 2^9 \cdot 3^2 \cdot 5 = 23{,}040. \qquad (21)$$

The double coset representatives c_i are known explicitly, but space does not allow their presentation here. The order |G| of G is found from a well-known formula for the number of elements in a double coset. The elements in each matrix c_i, hence each $g \in G$, consist only of the numbers 0, ± 1, $\pm 1/2$. It is also useful to note that G contains transformations such as $d \longmapsto -d-1$ (interchange x and u) that preserve the angular momentum eigenvalue $d(d+1)$.

Other papers giving partial results on the significance of Bailey's identity for symmetries of 6j-coefficients are those of Rao *et al.*[6] and Venkatesh[7].

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