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MASTER

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ABSTRACT

Wave heating of ELMO BUMPY TORUS-SCALE (EBT-S) in the ion-cyclotron range of frequencies will be strongly influenced by the geometry of the plasma. In particular, the short finite length of the mirror sections means that the electron bounce frequency is of comparable magnitude to the ion-cyclotron frequency. Consequently, the bouncing motion of trapped particles impacts the electron absorption of wave energy. Furthermore, the varying magnetic field strength along magnetic field lines influences the ion-cyclotron absorption of waves because the ion-cyclotron resonance conditions are satisfied only at discrete points along the field lines. Expressions are given for trapped and passing electron absorption as well as ion-cyclotron absorption. A numerical example is also discussed.

I. INTRODUCTION

Waves with frequencies comparable to the ion-gyrofrequency may heat either electrons or ions or both electrons and ions. The precise magnitude and spatial location of the heating generally depends on the plasma geometry and parameters as well as wave propagation characteristics within the plasma.

In particular for Elmo Bumpy Torus-Scale (EBT-S), a plasma confinement device which consists of a set of 24 toroidally linked magnetic mirrors, stabilized by a hot electron annulus,[1] the bounce frequency of trapped electrons is only slightly smaller than the ion-cyclotron frequency. Quantitatively, for a proton plasma with an electron temperature (T_e) of 600 eV, a mirror sector length (L_s) of 39 cm, and a magnetic field strength (B_0) of 7×10^3 gauss, the ratio of the electron bounce to ion-cyclotron frequencies is

$$\frac{\omega_{be}}{\Omega_i} \approx \frac{(v_e/L_s)}{\Omega_i} \approx 0.39 \qquad (1)$$

where ω_{be}, v_e and Ω_i are the electron bounce frequency, electron thermal speed, and the ion-cyclotron frequency, respectively. In addition, the electron collision frequency,[2]

$$\nu_e = \frac{25.3 - 1.15 \log(n) + 2.3 \log(T_e)}{3.5 \times 10^5} \frac{n}{T_e^{3/2}} \approx 1.0 \times 10^4 \text{ Hz} \qquad (2)$$

is very much smaller than the electron bounce frequency for the assumed density (n) of 3×10^{12} cm^{-3}. It follows that the bounce motion of trapped electrons in a general sense should impact the absorption of waves in the ion-cyclotron range of frequencies.

The geometry of EBT-S also affects the ion-cyclotron absorption of waves. Specifically, the wave frequency is equal to the fundamental or harmonic of the ion-cyclotron frequency only at discrete points along a field line (see Figure 1). In addition, the ion-cyclotron frequency varies along a typical ion's trajectory in real space while the ion bounce frequency (for trapped ions), ω_{bi}, and the ion transit frequency, ω_{ti}, are both much smaller than the ion-cyclotron frequency [i.e.,

$$\omega_{bi} \approx v_i/L_s \ , \qquad \omega_{ti} \approx v_i/R_T << \Omega_i \tag{3}$$

where v_i and R_T are the ion thermal speed and the major radius of the torus (150 cm), respectively]. It follows that the phase of the ion-cyclotron gyration is in general effectively randomized for successive ion passes through the resonance regions, and appropriate manipulations of the infinite homogeneous expression for ion-cyclotron absorption can be used to evaluate the ion heating rate in EBT-S.[3-7]

The remainder of this report is divided into four sections. In the second section, a model for the trapped electron absorption of waves is developed. In the third section, an analogous model for the absorption of waves by passing electrons is developed. The fourth section is concerned with the development of a model for ion absorption, while the last section is a summary and discussion of the absorption models.

II. TRAPPED ELECTRON ABSORPTION

As the basis for a calculation of the trapped electron absorption of waves, the following magnetic field model is assumed:

$$B(x,y,z) = B_o(x,y)\ (1 + z^2/L^2) \tag{4}$$

where z is the coordinate along the field line, and the coordinates, x and y, are orthogonal to the field line. In the absence of waves the electron motion along the field lines satisfies the equation

$$m_e \frac{d^2 z_o}{dt^2} = -\ 2\ \mu B_o(x,y)\ z_o/L^2 \tag{5}$$

where

$$\mu = mv_\perp^2/2B_o(x,y,z) \tag{6}$$

is the magnetic moment, assumed to remain constant. Also, t, m_e and $v_\perp$ are the time, electron mass and the perpendicular velocity, respectively. The general solution to Eq. (5) is

$$z_o(t) = \beta \sin\ (\omega_b t + \alpha) \tag{7a}$$

$$dz_o(t)/dt = \beta\ \omega_b \cos\ (\omega_b t + \alpha) \tag{7b}$$

where

$$\omega_b = [2\mu B_o(x,y)/m_e L^2]^{\frac{1}{2}} = (v_\perp/L)_{z=0}\ . \tag{8}$$

In writing Eq. (8) the dependence of ω_b on the coordinates x and y has been suppressed. The constants β and α are determined from the initial condition criteria

$$\left.\frac{dz_o(t)/dt}{z_o(t)}\right|_{t=0} = \omega_b \cot(\alpha) \quad , \tag{9a}$$

$$\left.\frac{1}{2} m v_{\|}^2\right|_{z=0} + \mu B_o(x,y) = \mu B_o(x,y)\ (1 + \beta^2/L^2) \quad . \tag{9b}$$

In writing Eq. (9b) both the constancy of the magnetic moment and the particle energy in a static magnetic field have been invoked. It follows from Eq. (9b) that

$$\beta = L(v_{\|}/v_{\perp})_{z=0} \tag{10}$$

where $v_{\|} = dz_o(t)/dt$.

The interaction of trapped electrons with waves having frequencies comparable to the ion-cyclotron frequency is dependent on the wave structure parallel to the magnetic field. In general, the wave electric and magnetic fields [i.e., $\vec{E}(x,y,z,t)$ and $\vec{B}(x,y,z,t)$] can be expanded in a Fourier series based on the length of a field line, L_f,

$$\vec{E}(x,y,z,t) = \exp(-i\omega t)\sum_{r=-\infty}^{\infty} \vec{E}_r(x,y)\ \exp(ik_r z) \tag{11a}$$

$$\vec{B}(x,y,z,t) = \exp(-i\omega t)\sum_{r=-\infty}^{\infty} \vec{B}_r(x,y)\ \exp(ik_r z) \tag{11b}$$

where

$$k_r = 2\pi r/L_f \quad . \tag{12}$$

With the assumed form for the wave fields, the rate at which trapped electrons absorb wave energy can be calculated using a perturbation technique to solve the equation of motion parallel to the field line to second order in the wave amplitude. A similar technique was used to derive the expression for the Landau damping absorption of wave energy for the physical situation when the unperturbed motion is a straight line.[8] For the wave fields given by Eq. (11) the perturbation of the particle motion about the unperturbed particle trajectory given by Eq. (7) satisfies the following equation to first order in the wave amplitude

$$\frac{d^2 z_1}{dt^2} + \omega_b{}^2 z_1 = \frac{1}{m_e} \sum_{r=-\infty}^{\infty} \gamma_r \exp\{i[k_r z_o(t) - \omega t]\} \tag{13}$$

where

$$\gamma_r = q_e \hat{z} \cdot \vec{E}_r(x,y) = i \mu k_r \hat{z} \cdot \vec{B}_r(x,y) \quad , \tag{14}$$

and q_e and $\hat{z}$ are the electron charge and the unit vector in the $\hat{z}$-direction, respectively. The subscript, 1, in Eq. (13) denotes a quantity to first order in the wave amplitudes and the dependence of γ_r on x and y has been suppressed. Substitution of Eq. (7) into the right-hand side of Eq. (13) and using the Bessel function identity,[9]

$$\exp[ix \sin(y)] = \sum_{p=-\infty}^{\infty} J_p(x) \exp(ipy) \tag{15}$$

permits Eq. (13) to be rewritten as

$$\frac{d^2z_1}{dt^2} + \omega_b{}^2 z_1 = A_1 \tag{16}$$

where

$$A_1 = \frac{1}{m_e} \sum_{r=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \gamma_r J_p (k_r\beta) \exp[i(p\omega_b-\omega)t + ip\alpha] . \tag{17}$$

In Eqs. (15) and (17), J_p is the Bessel function of the first kind and order p. For the initial conditions,

$$0 = \left.\frac{dz_1}{dt}\right|_{t=0} = \left. z_1 \right|_{t=0} , \tag{18}$$

Eq. (17) is readily evaluated using standard techniques for linear differential equations. The appropriate solutions are:

$$\frac{dz_1}{dt} = \frac{i}{2\omega_b m_e} \sum_{r=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \gamma_r J_p(k_r\beta) \exp(ip\alpha)$$

$$\left(\frac{(p\omega_b-\omega) \exp[i(p\omega_b-\omega)t] + \omega_b \exp(-i\omega_b t)}{(p+1)\,\omega_b - \omega} - \frac{(p\omega_b-\omega) \exp[i(p\omega_b-\omega)t] - \omega_b \exp(i\omega_b t)}{(p-1)\,\omega_b - \omega}\right) \tag{19a}$$

$$z_1 = \frac{1}{2\omega_b m_e} \sum_{r=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \gamma_r J_p(k_r\beta) \exp(ip\alpha)$$

$$\left(\frac{\exp[i(p\omega_b-\omega)t] - \exp(-i\omega_b t)}{(p+1)\,\omega_b - \omega} - \frac{\exp[i(p\omega_b-\omega)t] - \exp(i\omega_b t)}{(p-1)\,\omega_b - \omega}\right) . \tag{19b}$$

The second order correction to the electron trajectory (subscript 2) satisfies the following equation of motion

$$\frac{d^2 z_2}{dt^2} + \omega_b{}^2 z_2 = A_2 \tag{21}$$

where

$$A_2 = \frac{1}{m_e} \sum_{r=-\infty}^{\infty} \gamma_r \exp[ik_r\beta \sin(\omega_b t + \alpha)]\ (ik_r z_1{}^*) \quad . \tag{22}$$

The rate at which a single electron absorbs wave energy, dW/dt, is determined from the relation

$$\frac{dW}{dt} = \frac{m_e}{2} \left(\frac{dz_1}{dt} A_1^* + \frac{dz_0}{dt} A_2^* \right) + \text{c.c.} \quad . \tag{23}$$

With the assumption of random electron phase, α, at $t = 0$,[8] Eq. (23) can be averaged in the following way:

$$\left\langle \frac{dW}{dt} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \, \frac{dW}{dt} \quad . \tag{24}$$

Using Eqs. (7), (17), (19) and (22) the result of the averaging process (see Appendix A for details) is

$$\left\langle \frac{dW}{dt} \right\rangle = \frac{1}{2m_e} \sum_{r=-\infty}^{\infty} \sum_{r'=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} p^2 (\gamma_r \gamma_{r'}^* + \gamma_{r'} \gamma_r^*) \frac{\sin[(p\omega_b - \omega)t]}{p\omega_b - \omega} \left[\frac{J_p(k_r\beta)}{k_r\beta} \frac{d[J_p(k_{r'}\beta)]}{d(k_{r'}\beta)} + \frac{J_p(k_{r'}\beta)}{k_{r'}\beta} \frac{d\, J_p(k_r\beta)}{d(k_r\beta)} \right] . \tag{25}$$

Noting that

$$\lim_{t\to\infty} \frac{\sin[(p\omega_b-\omega)t]}{p\omega_b-\omega} = \pi\delta(p\omega_b-\omega) = \pi|L/p|\ \delta(v_\perp-L\omega/p) \tag{26}$$

permits Eq. (25) to be rewritten as

$$\langle\frac{dW}{dt}\rangle = \frac{\pi}{2}\frac{1}{m_e} \sum_{r=-\infty}^{\infty} \sum_{r'=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} p^2(\gamma_r\gamma_{r'}^* + \gamma_{r'}\gamma_r^*)$$
$$|L/p|\delta(v_\perp-L\omega/p) \left[\frac{J_p(k_r\beta)}{k_r\beta} \frac{d[J_p(k_{r'}\beta)]}{d(k_{r'}\beta)}\right.$$
$$\left. + \frac{J_p(k_{r'}\beta)}{k_{r'}\beta} \frac{d[J_p(k_r\beta)]}{d(k_r\beta)}\right] \quad . \tag{27}$$

It follows from Eqs. (26) and (27) that the absorption of wave energy in the parallel electron motion is due to a resonance resulting from finite perpendicular electron energy.

To determine the total power absorbed by electrons per unit volume, P_t, Eq. (27) must be integrated over the electron velocity distribution function. In particular, for the thermal distribution function (i.e.,

$$f_e(\vec{v}) = n(2\pi\ v_e^2)^{-3/2} \exp[-(v_\perp^2 + v_\parallel^2)/2v_e^2] \quad , \tag{28}$$

$$P_t = 2\pi\int_0^\infty dv_\perp v_\perp \int_{-v_\perp\delta}^{v_\perp\delta} dv_\parallel \langle\frac{dW}{dt}\rangle\ f_e(\vec{v})$$
$$= (\frac{\pi}{8})^{1/2} \frac{n}{m_e v_e^3} \sum_{r=-\infty}^{\infty} \sum_{r'=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \int_0^\infty dv_\perp v_\perp (\gamma_r\gamma_{r'}^* + \gamma_r^*\gamma_{r'})$$
$$|Lp|\delta(v_\perp-L\omega/p)\quad \exp(-v_\perp^2/2v_e^2) \int_{-v_\perp\delta}^{v_\perp\delta} dv_\parallel \exp(-v_\parallel^2/2v_e^2)$$
$$\left[\frac{J_p(k_r\beta)}{k_r\beta} \frac{d[J_p(k_{r'}\beta)]}{d(k_{r'}\beta)} + \frac{J_p(k_{r'}\beta)}{k_{r'}\beta} \frac{d[J_p(k_r\beta)]}{d(k_r\beta)}\right] \quad . \tag{29}$$

In writing Eq. (29) the parameter, δ, denotes that only particles satisfying

$$|v_{\parallel}| < v_{\perp}\delta \tag{30}$$

are trapped. The parameter, δ, is related to the fraction of electrons which are trapped, f, through the expression:

$$f = \frac{2\pi}{(2\pi v_e^2)^{3/2}} \int_0^{\infty} dv_{\perp} v_{\perp} \int_{-v_{\perp}\delta}^{v_{\perp}\delta} dv_{\parallel} \exp[-(v_{\parallel}^2 + v_{\perp}^2)/2v_e^2]$$

$$= \frac{\delta}{(1 + \delta^2)^{1/2}} \tag{31a}$$

or

$$\delta = \left(\frac{f^2}{1 - f^2}\right)^{\frac{1}{2}} \quad . \tag{31b}$$

Because of the delta-function, the perpendicular velocity integration in Eq. (29) is readily evaluated. Assuming that the parameters, L and ω, are positive, the result is:

$$P_t = (\tfrac{\pi}{2})^{\frac{1}{2}} \frac{n}{m_e v_e^3} \sum_{r=-\infty}^{\infty} \sum_{r'=-\infty}^{\infty} \sum_{p=1}^{\infty} \omega L^2(\gamma_r \gamma_{r'}^* + \gamma_r^* \gamma_{r'})$$

$$\exp(-L^2\omega^2/2p^2v_e^2) \times \int_0^{L\omega\delta/p} dv_{\parallel} \exp(-v_{\parallel}^2/2v_e^2)$$

$$\left[\frac{J_p(k_r\beta)}{k_r\beta} \frac{d[J_p(k_{r'}\beta)]}{d(k_{r'}\beta)} + \frac{J_p(k_{r'}\beta)}{k_{r'}\beta} \frac{d[J_p(k_r\beta)]}{d(k_r\beta)}\right] \tag{32}$$

where

$$\beta = pv_{\parallel}/\omega \quad , \tag{33a}$$

$$\gamma_r = q_e \hat{z} \cdot \vec{E}_r(x,y) - i \frac{m_e}{2B_o(x,y)} \left(\frac{L\omega}{p}\right)^2 \hat{z} \cdot \vec{B}_r(x,y) \quad . \tag{33b}$$

If only one term in the fourier series, $r = r_o$, contributed to the heating of trapped particles, Eq. (32) would be:

$$P_t = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{n}{m_e v_e^3} \sum_{p=1}^{\infty} \omega L^2 \, |\gamma_{r_o}|^2 \exp(-L^2\omega^2/2p^2 v_e^2)$$

$$\int_0^{L\omega\delta/p} dv_\parallel \exp(-v_\parallel^2/2v_e^2) \frac{1}{k_{r_o}\beta} \frac{d[J_p^2(k_r\beta)]}{d(k_r\beta)} \quad . \tag{34}$$

Although a numerical evaluation of Eq. (32) is required to determine a quantitative estimate of the total trapped electron absorption of wave energy, several qualitative comments can be made concerning the equation. First, if more than one term in the summation over, r, contributes to the description of the wave amplitude, cross terms in Eq. (32) (i.e., $r \neq r'$) contribute to the wave absorption by trapped electrons. Second, the quantity P_t is not obviously positive definite since the Bessel functions and the amplitude factors with $r \neq r'$ may be negative. Hence, trapped electrons may give energy to waves rather than absorb it. However, as shown in Appendix B, those contributions to P_t with $r \neq r'$ are positive definite. Hence, the expression in Eq. (34) is positive definite. Third, as the harmonic number, p, increases a greater number of electrons are in resonance with the wave [i.e., $\exp(-L^2\omega^2/2p^2v_e^2)$ approaches one]. However, the summation over the harmonic number, p, does converge (see Appendix C). Fourth, Landau, transit time and cross Landau-transit time damping terms [proportional to $E_r E^*_{r'}$, $B_r B^*_{r'}$ and $E_r B^*_{r'}$ (or $E^*_r B_{r'}$), respectively] contribute to electron absorption of wave energy.

III. PASSING ELECTRON ABSORPTION

The plasma model and calculative technique used to derive the expression for passing electron absorption of wave energy is analogous to that used in the previous section with the exception that the z-directed magnetic field is assumed to be of constant strength B_o. In the absence of wave fields the trajectory of the electrons along the magnetic field is:

$$z_o'(t) = v_c t + z_c \qquad (35a)$$

$$\frac{dz_o'(t)}{dt} = v_c \qquad (35b)$$

where v_c and z_c are constants such that the initial particle position and velocity at t = 0 are z_c and v_c, respectively.

In the presence of the wave fields of the form specified by Eqs. (11a) and (11b) the perturbed particle trajectories $z_1'(t)$ and $z_2'(t)$ satisfy

$$m_e \frac{d^2 z_1'}{dt^2} = \sum_{r=-\infty}^{\infty} \gamma_r \exp[i(k_r z_o' - \omega t)] \qquad (36a)$$

$$m_e \frac{d^2 z_2'}{dt^2} = \sum_{r=-\infty}^{\infty} \gamma_r \exp[i(k_r z_o' - \omega t)](ik_r z_1'^*) \quad . \qquad (36b)$$

As in the previous section the subscripts, "1" and "2," denote quantities which are first and second order in wave amplitudes, respectively. Also, for the sake of brevity, the dependence of $z_o'(t)$, $z_1'(t)$ and $z_2'(t)$ on time has been suppressed in writing Eqs. (36a) and (36b). With the initial conditions,

$$0 = \left.\frac{dz_1'}{dt}\right|_{t=0} = z_1'\Big|_{t=0} \tag{37}$$

the solution to Eq. (36a) is:

$$\frac{dz_1'}{dt} = -i \sum_{r=-\infty}^{\infty} \frac{\gamma_r}{m_e} \frac{1}{k_r v_c - \omega} \left\{ \exp[i(k_r v_c - \omega)t] - 1 \right\} \exp(ik_r z_c) \tag{38a}$$

$$z_1' = i \sum_{r=-\infty}^{\infty} \frac{\gamma_r}{m_e} \frac{1}{k_r v_c - \omega} \left(i \frac{\exp[i(k_r v_c - \omega)t] - 1}{k_r v_c - \omega} + t \right) \exp(ik_r z_c) \quad , \tag{38b}$$

$$\frac{d^2 z_2'}{dt^2} = \frac{-1}{m_e^2} \sum_{r=-\infty}^{\infty} \sum_{r'=-\infty}^{\infty} k_{r'} \gamma_{r'} \gamma_r^* \frac{\exp[i(k_r' v_c - \omega)t]}{k_r v_c - \omega} \times \left[i \frac{\exp[-i(k_r v_c - \omega)t] - 1}{k_r v_c - \omega} - t \right] \exp[i(k_{r'} - k_r) z_c] \ . \tag{38c}$$

The rate at which a single electron absorbs energy from the wave is given by the expression:

$$\frac{dW'}{dt} = \frac{m_e}{2} \left(\frac{dz_1'}{dt} \frac{d^2 z_1'^*}{dt^2} + \frac{dz_o^*}{dt} \frac{d^2 z_2'}{dt^2} \right) + \text{c.c.} \tag{39}$$

Upon averaging over the initial electron position and substituting Eqs. (38a), (38b) and (38c) into Eq. (39), there results the following expression for the average rate at which wave energy is absorbed per unit volume by electrons of speed v_c:

$$\langle \frac{dW'}{dt} \rangle \equiv \frac{1}{L_f} \int_0^{L_f} dz_c \frac{dW'}{dt}$$

$$= \sum_{r=-\infty}^{\infty} \frac{|\gamma_r|^2}{m_e} \frac{d}{dv_c} \left(\frac{v_c \sin[(k_r v_c - \omega)t]}{k_r v_c - \omega} \right) \quad . \tag{40}$$

For values of time going to infinity Eq. (40) can be rewritten as:

$$\langle \frac{dW'}{dt} \rangle = \pi \sum_{r=-\infty}^{\infty} \frac{|\gamma_r|^2}{m_e} \frac{d}{dv_c} [v_c \delta(k_r v_c - \omega)] \quad . \tag{41}$$

Note that unlike the case of trapped electrons cross-terms (i.e., $\gamma_r \gamma^*_{r'}$ with $r \neq r'$) do not contribute to the wave absorption by passing electrons. However, as in the calculation of trapped electrons Landau, transit time and cross Landau-transit time damping contribute to electron absorption of wave energy.

To calculate the total rate at which wave energy is absorbed by passing electrons, Eq. (41) must be integrated over the velocity distribution function for passing electrons. Assuming the thermal distribution function given by Eq. (28), the following expression for the absorption rate results:

$$P_p = 2\pi \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{|v_{\parallel}|/\delta} dv_{\perp} v_{\perp} \langle dW'/dt \rangle f_e(\vec{v})$$

$$= \frac{1}{2(\pi)^{\frac{1}{2}}} \frac{\omega_{pe}^2}{\omega} \sum_{r=-\infty}^{\infty} |x_r|^3 \exp(-x_r^2) \left\{ |E_r|^2 [1 - (1 + \delta^{-2}) \exp(-b_r)] \right.$$

$$+ \left| \frac{m_e v_e^2 B_r k_r}{q_e B_o} \right|^2 \left[2 - \left(b_r^2 (1 + \delta^{-2}) + 2b_r + 2 \right) \exp(-b_r) \right]$$

$$\left. + i \frac{k_r v_e^2 m_e}{q_e B_o} (B^*_r E_r - B_r E^*_r) \left[1 - \left(b_r (1 + \delta^{-2}) + 1 \right) \exp(-b_r) \right] \right\} \quad . \tag{42}$$

In Eq. (42),

$$x_r = \omega/[(2)^{\frac{1}{2}} k_r v_e] \quad ,$$

$$b_r = x_r^2/\delta^2 \quad ,$$

$$\omega_{pe}^{\ 2} = 4\pi\ n\ q_e^2/m_e \quad . \tag{43}$$

In general, P_p may be positive, negative or zero depending on wave and plasma parameters.

IV. ION ABSORPTION

Because of the strong variation of magnetic field strength in EBT-S (i.e., magnetic field strength of 0.7 tesla on axis at the midplane and a mirror ratio of two), ions are heated in a thin layer at surfaces of constant magnetic field strength (see Figure 1) where the wave frequency is equal to the fundamental or harmonic of the ion gyrofrequency, i.e.,

$$\omega = p_i \Omega_i \tag{44a}$$

$$p_i = 1,\ 2,\ \ldots\ldots \quad . \tag{44b}$$

The formalism which will be used to derive the ion heating rate assumes that within the thin layer the infinite and homogeneous expression for cyclotron absorption is valid and that to lowest order wave absorption does not alter the wave structure within the absorption layer.[3-7] Hence, the expressions derived for ion absorption are generally expected to be most generally applicable for the harmonic cyclotron damping processes, but may also be valid for fundamental heating by fast waves in a dense single ion species plasma.[3,8] The assumption of cyclotron heating within a thin layer is reasonable in EBT-S since the gyrofrequency varies along the ion trajectory while the ion bounce frequency is very much smaller than the gyrofrequency (i.e., $v_i/L_s < \Omega_i$) and so the phase of the ion gyration can in general be expected to be uncorrelated during successive passes through the resonance layers.

Now in an infinite and homogeneous plasma the rate at which wave energy is absorbed per unit volume by ions with a thermal velocity distribution function,

$$f_i(\vec{v}) = n(2\pi v_i^2)^{-3/2} \exp(-v^2/2v_i^2) \quad , \tag{45}$$

is

$$P_i = \frac{1}{4\pi}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{r=-\infty}^{\infty} |E_{xr} + iE_{yr}|^2 \sum_i \frac{p_i}{(p_i - 1)!}\left(\frac{1}{2}\right)^{p_i}$$

$$(k_{\perp r}v_i/\Omega_i)^{2p_i-2} \times \frac{\omega_{p_i}^2}{|k_r|v_i} \exp\left[-\frac{(\omega - p_i\Omega_i)^2}{2k_r^2v_i^2}\right] \qquad (46)$$

where $k_{\perp r}$ is the perpendicular wavenumber associated with the Fourier component of the wave amplitude with parallel wavenumber k_r. The quantity, $|E_{xr} + iE_{yr}|$, is the magnitude of the left-hand component of the wave amplitude associated with parallel wavenumber k_r. The summation over, i, indicates ion species satisfying the gyroresonance condition Eq. (44a). Also $k_{\perp r}v_i/\Omega_i$ has been assumed to be much smaller than one and the quantity, ω_{pi}, is the ion plasma frequency for species i.

The assumption that ion heating is taking place within a very thin layer along a field line is valid in the limit of $k_r v_i/\Omega_i$ going to zero. For an asymptotically thin resonance layer, the magnetic field strength can be assumed to be linearly expandable (except for the intersection of the layer with the plane defining the fattest cross section of a mirror section (see Figure 1) along a field line about the point where the wave frequency satisfies the resonance criterion, Eq. (43a), i.e.,

$$\Omega_i(z) = \Omega_{io}(1 + z/L_1) \qquad (47a)$$

$$\omega = p_i\Omega_{io} \quad . \qquad (47b)$$

Substitution of Eqs. (46a) and (46b) into Eq. (45) and assuming that the only strong variation of the resulting expression is in the exponent, there results the following average expression for the ion heating rate per unit volume along a field line per mirror section (i.e., two resonance regions):

$$\langle P_i \rangle_1 = \frac{2}{L_f} \int_{-\infty}^{\infty} dz\, W_i$$

$$= \frac{L_1}{2L_f} \sum_{r=-\infty}^{\infty} |E_{xr} + iE_{yr}|^2 \sum_i \frac{\omega_{p_i}^2}{\Omega_{io}} \frac{(0.5)^{p_i}}{(p_i - 1)!}$$

$$(k_{\perp r} v_i / \Omega_{io})^{2p_i - 2} \quad . \tag{48}$$

Another informative calculation is to consider the case where the resonance region intersects the plane which defines the fattest cross section of a mirror section (see Fig. 1). For this case the magnetic field strength along the central field line can be taken to have the form,

$$\Omega_i(z) = \Omega_{io}(1 + z^2/L_2^2) \tag{49a}$$

$$\omega = p_i \Omega_{io} \quad . \tag{49b}$$

Following the procedure used to derive Eq. (48), the average expression for the ion heating rate per unit volume along a field line per mirror section is:

$$\langle P_i \rangle_2 = \frac{1}{L_f} \int_{-\infty}^{\infty} dz\, W_i$$

$$= 0.22 \frac{L_2}{L_f} \sum_{r=-\infty}^{\infty} |E_{xr} + iE_{yr}|^2$$

$$\times \sum_i \frac{\omega_{p_i}^2/\Omega_{io}}{(|k_r| v_i/\Omega_{io})^{1/2}} \frac{(0.5)^{p_i}}{(p_i - 1)!} (p_i)^{1/2} (k_{\perp r} v_i/\Omega_{io})^{2p_i - 2} \quad . \tag{50}$$

Several features of Eqs. (48) and (50) are noteworthy. First, the ion heating rate expressions, particularly Eq. (48), are very similar in form to those previously derived for the cyclotron damping process within a thin layer.[3-7] However, unlike the earlier work the contribution to the heating rates of each fourier amplitude has been taken into account. Second, Eq. (48) is independent of the parallel wavenumber, k_r, while Eq. (50) is inversely dependent on the square root of k_r. Hence, ion heating at the center of a mirror section is enhanced by long parallel wavelengths. Third, for fundamental ion heating (i.e., $p_i = 1$) both Eqs. (48) and (50) are independent of the perpendicular wavenumber. Eq. (48) is also independent of temperature while Eq. (50) is inversely proportional to the square root of temperature. It follows that for the heating model Eq. (50) ion heating efficiency is enhanced by low ion temperatures. Fourth, the average ion heating rate is increased if the geometry factor, L_1/L_f or L_2/L_f, is maximized. Consequently, resonance regions located near the center of a mirror section are likely to result in increased ion heating efficiency. These resonance regions are also beneficial from the standpoint of ion heating since they have the largest surface area of any constant magnetic field strength surfaces in EBT.

E. SUMMARY AND DISCUSSION

As a quantitative example the electron and ion heating expressions were evaluated with the following parameters appropriate to an EBT-S hydrogen plasma being heated at the first harmonic of the ion-cyclotron frequency: B_o = 0.7 tesla, n = 3 x 10^{12} cm^{-3}, T_e = 600 eV, T_i = 500 eV, L_f = 942.5 cm, L = L_1 = L_2 = 39.3 cm, p = 2, $\omega = 2\Omega_i$, and $k_\perp$ = 0.3 cm^{-1}. Only one term in the summation over Fourier amplitudes, $r = r_o$, was retained in Eqs. (42), (48) and (50). These equations, along with Eq. (34) for trapped electrons, were the ones solved for the electron and ion heating rates. Wave amplitudes, $|\hat{z} \cdot \vec{E}| = |E_x + iE_y|$ = 1 statvolt/cm and $|\hat{z} \cdot \vec{B}|$ = 1 gauss, were assumed. Extrapolation of the heating rates to other values of the wave amplitudes is straightforward since the heating rate expressions are proportional to the square of the wave amplitudes. (It should also be pointed out that in actual experiments the parallel wave electric field will probably be much smaller than the perpendicular component.) To assess the relative importance of trapped electrons on the total electron dissipation of the wave, two values for the trapped electron fraction (i.e., 0.8 and 0.0) were assumed. Also, the expressions were calculated for different values of the parallel wavenumber. The results of the calculations are summarized in Table I.

The entries in Table I corresponding to Eqs. (34) and (42) indicate that with the assumption of eighty percent trapped electrons the contributions of trapped electrons to total wave dissipation is positive while that of passing electrons may be positive or negative. Hence, total wave dissipation,

$$P_e = P_t + P_p \tag{51}$$

can be positive or negative. The physical significance of negative dissipation is that electron thermal energy is being transformed into wave energy. Furthermore, as long as the ion wave absorption rate is of larger magnitude than electron deabsorption, the wave is stable and ions are being heated at the expense of wave and electron energy.

Table I demonstrates that the contribution of trapped electrons to total electron dissipation can be comparable to or greater than the contribution from passing particles. The significance of trapped electron absorption relative to passing electron absorption (or deabsorption) is most apparent for parallel phase velocities much greater than (i.e., with k_{r_0} = 0.00667 cm^{-1}) the electron thermal speed. In addition, a comparison of the passing electron absorption rates for 0.8 and 0.0 trapped electron fractions indicates that the number of passing particles has an important effect on the passing electron absorption rate as long as the parallel phase velocity is comparable to or smaller than the electron thermal speed.

The calculated values for the two ion heating expressions are noted in Table I. The average ion absorption rate for resonance locations lying away from the midplane of a mirror section ($<P_i>_1$) is given explicitly in Eq. (48) while for resonance locations lying on the midplane of a mirror the average ion absorption rate ($<P_i>_2$) is given explicitly by Eq. (50). There is no parametric dependence of the ion heating expressions on trapped electron fraction. Furthermore, the expression for $<P_i>_1$ does not depend on the parallel wavenumber. The values for $<P_i>_2$ in Table I reflect the inverse square root dependence on the parallel wavenumber. A significant result is that for comparable plasma and wave parameters ion heating is stronger at the midplane than away from the midplane particularly for waves with parallel wavenumber going to zero.

Ion heating in EBT-S will probably benefit from a choice of wave frequencies such that the surfaces of constant magnetic field strength, which satisfy the ion-cyclotron resonance criterion, lie close to the center of a mirror section. The basis for this assertion is that such surfaces have larger area and magnetic field strength gradient scale lengths than other constant magnetic field surfaces in EBT. Also, the relative magnitude of direct ion heating relative to direct electron heating will tend to be increased for long parallel wavelength modes.

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APPENDIX A

In this appendix the details in the derivation of Eq. (25) are given. Using Eqs. (17) and (19a) permits the first term on the right-hand side of Eq. (23) to be written in the following way:

$$\frac{m_e}{2}\left(\frac{dz_1}{dt} A_1^* + c.c.\right) = \frac{i}{4\omega_b m_e} \sum_{r=-\infty}^{\infty} \sum_{r'=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \sum_{p'=-\infty}^{\infty}$$

$$\gamma_r \gamma_{r'}^* \, J_p(k_r\beta) \, J_{p'}(k_{r'}\beta) \, \exp[-i(p'\omega_b-\omega)t + i(p-p')\alpha]$$

$$\left(\frac{(p\omega_b-\omega)\exp[i(p\omega_b-\omega)t] + \omega_b \exp(-i\omega_b t)}{(p+1)\omega_b-\omega}\right.$$

$$\left. - \frac{(p\omega_b-\omega)\exp[i(p\omega_b-\omega)t] - \omega_b \exp(i\omega_b t)}{(p-1)\omega_b-\omega}\right) + c.c. \qquad \text{(A1)}$$

With the averaging specified in Eq. (24) only terms with p = p' are retained in Eq. (A1) and

$$\frac{m_e}{2} < \left(\frac{dz_1}{dt} A_1^* + c.c.\right) > = \frac{1}{2m_e} \sum_{r=-\infty}^{\infty} \sum_{r'=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \gamma_r \gamma_{r'}^*$$

$$J_p(k_r\beta) \, J_p(k_{r'}\beta) \left[\frac{\sin\,[(p+1)\omega_b t - \omega t]}{(p+1)\omega_b - \omega}\right.$$

$$\left. + \frac{\sin\,[(p-1)\omega_b t - \omega t]}{(p-1)\omega_b - \omega}\right]$$

$$= \frac{1}{2m_e} \sum_{r=-\infty}^{\infty} \sum_{r'=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \gamma_r \gamma_{r'}^* \Big[J_{p-1}(k_r\beta) \, J_{p-1}(k_{r'}\beta)$$

$$+ J_{p+1}(k_r\beta) \, J_{p+1}(k_{r'}\beta)\Big] \, \frac{\sin\,[(p\omega_b-\omega)t]}{p\omega_b-\omega} \quad . \qquad \text{(A2)}$$

The second term on the right-hand side of Eq. (23) can be written explicitly using Eqs. (7b), (19b) and (22) in the following way

$$\frac{m_e}{2}\left(\frac{dz_o}{dt}A_2^* + \text{c.c.}\right) = \frac{i}{2}\omega_b\beta\cos(\omega_b t + \alpha)\sum_{r=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}$$

$$k_r\gamma_r z_1^* \quad J_p(k_r\beta)\exp[i(p\omega_b-\omega)t + ip\alpha] \quad . \qquad \text{(A3)}$$

Noting that

$$\cos(\theta)\sum_{p=-\infty}^{\infty} J_p(z)\exp(ip\theta)$$

$$= \sum_{p=-\infty}^{\infty}(p/z)\,J_p(z)\exp(ip\theta) \qquad \text{(A4)}$$

permits Eq. (A3) to be rewritten as

$$\frac{m_e}{2}\left(\frac{dz_o}{dt}A_2^* + \text{c.c.}\right) = \frac{i}{4m_e}\sum_{r=-\infty}^{\infty}\sum_{r'=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{p'=-\infty}^{\infty}$$

$$\gamma_r\gamma_{r'}^*\; p\; J_p(k_r\beta)\; J_{p'}(k_{r'}\beta)\exp[i(p\omega_b-\omega)t$$

$$+ i(p-p')\alpha\,]\quad\left(\frac{\exp[-i(p'\omega_b-\omega)t] - \exp(i\omega_b t)}{(p'+1)\omega_b-\omega}\right.$$

$$\left. - \frac{\exp[-i(p'\omega_b-\omega)t] - \exp(-i\omega_b t)}{(p'-1)\omega_b-\omega}\right) + \text{c.c.} \quad . \qquad \text{(A5)}$$

With the averaging specified in Eq. (24), Eq. (A5) becomes

$$\frac{m_e}{2} < \left(\frac{dz_o}{dt} A_2^* + c.c. \right) > = \frac{1}{2m_e} \sum_{r=-\infty}^{\infty} \sum_{r'=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \gamma_r \gamma_{r'}^*$$

$$\left[(p-1) J_{p-1}(k_r \beta) J_{p-1}(k_{r'} \beta) - (p+1) J_{p+1}(k_r \beta) J_{p+1}(k_{r'} \beta) \right]$$

$$\frac{\sin[(p\omega_b - \omega)t]}{p\omega_b - \omega} \quad . \qquad \text{(A6)}$$

If Eqs. (A2) and (A6) are combined, then

$$< dW/dt > = \frac{1}{2} \sum_{r=-\infty}^{\infty} \sum_{r'=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \gamma_r \gamma_{r'}^* \, p$$

$$\left[J_{p-1}(k_r \beta) J_{p-1}(k_{r'} \beta) - J_{p+1}(k_r \beta) J_{p+1}(k_{r'} \beta) \right]$$

$$\frac{\sin[(p\omega_b - \omega)t]}{p\omega_b - \omega} \quad . \qquad \text{(A7)}$$

Eq. (25) follows from the Bessel function identities

$$J_{p-1}(z) = (p/z)J_p(z) + d[J_p(z)]/dz \qquad \text{(A8a)}$$

$$J_{p+1}(z) = (p/z)J_p(z) - d[J_p(z)]/dz \qquad \text{(A8b)}$$

and the symmetry of the Bessel functions in r and r'.

APPENDIX B

To prove that the contributions to P_t for $r = r'$ are positive definite it suffices to show that the integral in Eq. (32) is positive definite. Now for $r = r'$ the integral can be written as:

$$I = \int_0^b dv_\parallel \, \exp(-v_\parallel^2/2v_e^2) \, \frac{1}{av_\parallel} \, \frac{d[J_p^2(av_\parallel)]}{d(av_\parallel)} \quad , \tag{B1}$$

where

$$a = pk_r/\omega \tag{B2a}$$

$$b = L\omega\delta/p \quad . \tag{B2b}$$

Integrating by parts permits Eq. (B1) to be rewritten as

$$I = \frac{1}{a^2 v_\parallel} \, J_p^2\,(av_\parallel)\,\exp(-v_\parallel^2/2v_e^2)\Bigg|_{v_\parallel = b} + \frac{1}{a^2}\int_0^b dv_\parallel \left(\frac{1}{v_\parallel^2} + \frac{1}{v_e^2}\right)\exp\left(-v_\parallel^2/2v_e^2\right)J_p^2(av_\parallel) \tag{B3}$$

which is positive definite.

APPENDIX C

The proof that the summation over the harmonic number, p, converges in Eq. (32) is most easily shown using the ratio test. For values of the harmonic number, p, approaching infinity

$$J_p(z) \rightarrow \frac{(z/2)^p}{p!} \tag{C1}$$

$$d[J_p(z)]/dt \rightarrow \frac{(z/2)^{p-1}}{2(p-1)!} \tag{C2}$$

and the integral in Eq. (32) becomes

$$I_p \equiv \frac{1}{2} \int_0^{L\omega\delta/p} dv_{\parallel} \left[\frac{(k_r\beta/2)^p \, (k_{r'}\beta/2)^{p-1}}{p! \, (p-1)!} + \frac{(k_{r'}\beta/2)^p \, (k_r\beta/2)^{p-1}}{p! \, (p-1)!} \right]$$

$$= \frac{1}{2p} \, \frac{\omega}{(p!)^2} \left(\frac{k_r + K_{r'}}{k_{r'} k_r} \right) \left(\frac{k_r k_{r'} L^2 \delta^2}{4} \right)^p . \tag{C3}$$

Using Eq. (C3) it is clear that

$$I_{p+1}/I_p \xrightarrow[p\rightarrow\infty]{} 0 \tag{C4}$$

and so the series in Eq. (32) must converge.

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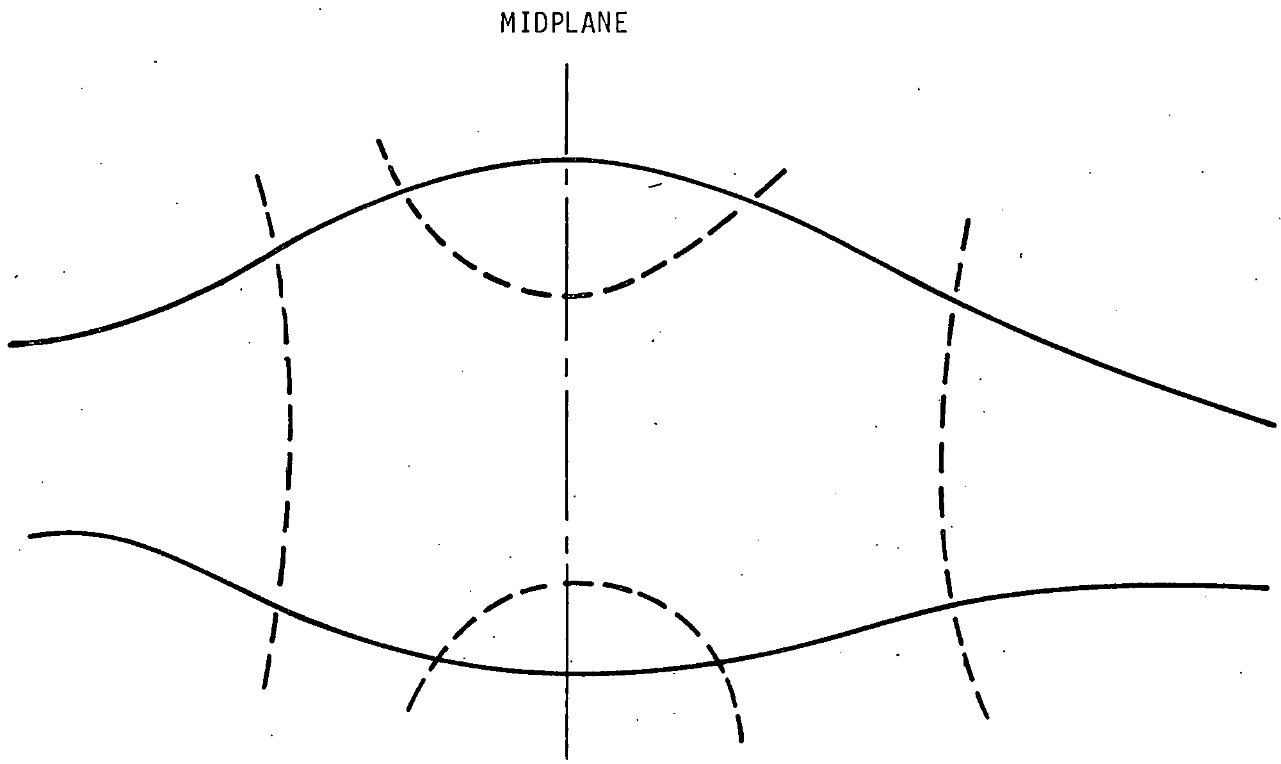
Table I. Calculated values for P_t, P_p, P_e, $\langle P_i \rangle_1$ and $\langle P_i \rangle_2$ for various values of k_{r_0} and for f = 0.8 and 0.0. All dimensioned quantities are in cgs units.

k_{r_0}	0.00667		0.04		0.08		0.12		0.16		0.20	
f	0.8	0.0	0.8	0.0	0.8	0.0	0.8	0.0	0.8	0.0	0.8	0.0
P_t	1.2×10^{10}	0.0	6.0×10^{11}	0.0	2.8×10^{12}	0.0	4.1×10^{12}	0.0	3.8×10^{12}	0.0	3.3×10^{12}	0.0
P_p	3.5×10^{-67}	3.5×10^{-67}	1.1×10^{12}	1.2×10^{12}	2.1×10^{12}	8.1×10^{12}	-6.1×10^{11}	5.1×10^{12}	-8.2×10^{11}	2.8×10^{12}	-6.2×10^{11}	1.6×10^{12}
P_e	1.2×10^{10}	3.5×10^{-67}	1.7×10^{12}	1.2×10^{12}	4.9×10^{12}	8.1×10^{12}	3.5×10^{12}	5.1×10^{12}	3.0×10^{12}	2.7×10^{12}	2.7×10^{12}	1.6×10^{12}
$\langle P_i \rangle_1$	3.9×10^{6}		3.9×10^{6}		3.9×10^{6}		3.9×10^{6}		3.9×10^{6}		3.9×10^{6}	
$\langle P_i \rangle_2$	5.2×10^{7}		2.1×10^{7}		1.5×10^{7}		1.2×10^{7}		1.1×10^{7}		9.4×10^{6}	

Figure 1. Figure illustrates general geometry of field lines (solid), surfaces of constant magnetic field strength (----), and the plane defining the fattest cross section of a mirror section (—— - ——) in EBT.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