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RECONNECTION OF MAGNETIC LINES IN AN IDEAL FLUID

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Abstract

The rate of reconnection of magnetic lines at an X-point, also growth of a "tearing" configuration have always been related to the presence of resistivity or other dissipative mechanisms. These phenomena, exhibiting nonconservation of magnetic line topology, are shown to occur in an ideal, nondissipative fluid, thereby violating beliefs, theorems, and calculations of over a century (including the mathematically equivalent questions involving vortex lines in an ideal fluid).

I. INTRODUCTION

There are two distinct ideal models which we consider in describing magnetic line reconnection or tearing. Type one, the more important one practically, contains singular surfaces (usually a separatrix) across which the fluid state is connected by jump conditions (analogous to discontinuities at shock surfaces connected by the Hugoniot conditions, or the more appropriate though less familiar analogy of a throttling process across a porous plug). Type two is a strict solution of some form of the ideal equations of motion which exhibits mass and flux flow across a separatrix. Type one (previously described, both general theory[1] and numerical examples[2]) contradicts results of the kind which relate the rate of line breaking or the tearing growth rate to the magnitude of the plasma resistivity, but it does not contradict the basic theoretical concept since (as in a shock) there may be dissipation hidden in the discontinuity layer. The significance of an example of type one is that it reverses cause and effect; instead of the resistivity producing a certain rate of flow across a separatrix, the rate of flow is determined by external, global boundary conditions and other constraints, and the resistivity and thickness of the layer adjust to produce the required reconnection rate.

Type two examples, given in this paper, are at present more limited; but they have the additional interest of directly violating one of the oldest principles of fluid dynamics and magnetofluid dynamics. How could such a situation arise? The answer is very common: theorems which apply to physical situations are ordinarily proved under restrictive conditions which are almost inevitably ignored when the result is quoted; [for example, the thermodynamic model of two boxes connected with a capillary gives rise to the statement that "the entropy of the universe is increasing"]. Although our explicit counterexamples speak for themselves, we shall give a sequence of outlined proofs of the invariance of magnetic lines under more and more general conditions to demonstrate the origin of the limit beyond which the result is no longer valid; we shall also give more general counterexamples which, though plausible, are not yet mathematically proved.

In a theory of adiabatic processes in ideal MHD[1], it is natural to include flux conservation as a constraint since this is a property of the equations of motion. In a complex topology (Fig. 1), the flux inside each region must be preserved (in particular, no new islands can appear). This constraint in a complex topology is found to lead to the unacceptable conclusion that the external parameters (such as current in a coil, position of a wall, etc.) cannot be varied at will. This defect was overcome by introducing the (type one) concept of a generalized adiabatic process[1] in which the total flux (sum of three regions in Fig. 1) is conserved rather than each flux individually. The original justification provided was that, upon varying an external parameter, large current concentrations would appear near the separatrix, violating the nondissipative assumption locally. As in the theory of shock waves, jump conditions can be specified, allowing one to formulate and solve an inherently dissipative problem without overt reference to the magnitude or even presence of dissipation, no less its mechanism (resistivity, viscosity, etc.). This generalized adiabatic concept was used successfully to construct simulation codes[2] (including adiabatic creation, growth, and shrinking of islands), and was further confirmed by later resistive transport codes[3] which were sufficiently efficient and accurate to be run with small resistivity, confirming the localized current layer (Fig. 2), and confirming that the rate of island growth (line breaking) in this problem is nonzero in the limit of small resistivity.

From the beginning, there was one puzzling feature of the generalized adiabatic jump conditions as compared to the Hugoniot conditions in a shock. The purpose of the relaxation of strict adiabatic constraints was to take into account hidden dissipation; but there is a large class of problems in which there is no net entropy increase, even with mixing of plasma and magnetic flux from two regions into one (e.g. tearing), or splitting of one region into two (e.g. Doublet). This conceptual difficulty is resolved in the present paper by an example (type two) of a strict solution of the (linearized) equations of motion of ideal MHD allowing mass and magnetic flux transfer across a separatrix without intervention of jump conditions. In other words, in at least some cases, the generalized adiabatic solutions appear to be conventional solutions of the ideal equations of motion (in addition, as supported by numerical evidence, they are a valid nonuniform limit of resistive solutions).

The crucial point which allows transfer of plasma and flux across a separatrix is that the plasma have an adiabatic type singularity near the separatrix. Such a flow is not possible with a conventional separatrix equilibrium. One of the crucial features of the adiabatic formulation of plasma equlibrium problems is that instead of giving a pressure profile, $p(\psi)$, one specifies an adiabatic profile, $\mu(\psi)$, where

$$p = \mu(\psi)[\psi'(V)]^{\gamma} \qquad (1)$$

[$V(\psi)$ is the plasma volume within a surface ψ = const., $\psi(V)$ is the inverse function]. In a "standard" equilibrium (with smooth current profile) where $\psi \sim r^2$ near a hyperbolic critical point and $\nabla\psi \sim r$, the volume between a flux surface and the separatrix behaves like $V \sim \psi \log \psi$ which gives the adiabatically unacceptable conclusion that $\psi'(V) = 0$ ($p=0$) at the separatrix. In an adiabatically processed equilibrium, the flux contours will shift sufficiently to make $V \sim \psi$ and $\psi'(V)$ nonzero. This requires $\psi \sim rm$, $1 < m < 2$, near the X-point, which, in turn, requires a weak current singularity near the separatrix.[1] Without referring to adiabatic "processing" one can see that a standard separatrix, defined by finite pressure p and $dp/d\psi$ is incompatible with an adiabatic separatrix where μ and $d\mu/d\psi$ are finite.

A standard X-point does not allow transfer of plasma or flux; an adiabatic X-point does. This paper would be logically complete if we merely displayed the ideal flow pattern across such an adiabatic type separatrix; but to be psychologically satisfying we should also show that this X-point behavior is not ad hoc but arises from a self-consistent, successful, and experimentally confirmed adiabatic theory (and we will indicate how it might also arise, nonadiabatically, in a nonlinear, nonsteady flow).

II. CONSERVATION OF FLUX

The classical equation describing conservation of flux is

$$\frac{\partial B}{\partial t} + \text{curl}\ (B\times u) = 0\ ,\quad \text{div}\ B = 0 \tag{2}$$

which, for an open surface, S, moving with velocity u, implies

$$\frac{d}{dt}\int_{S(t)} B\cdot dS = 0 \tag{3}$$

Breaking field lines in a vacuum (where (2) is not normally considered to hold) is trivial. For example, consider the field between two parallel perfectly conducting plates through which a perfectly conducting (field excluding) object moves (Fig. 3). The complementary example, a perfectly conducting fluid between two plates through which a nonconducting object moves (also Fig. 3) is a little more subtle; but some thought shows that the magnetic line mapping from one plate to the other is altered by motion of the object. Proofs that (2) does imply conservation of flux go back to Helmholtz[4] and Kelvin[5] (in the vortex formulation); more recent, magnetically oriented discussions are in Refs. 6 and 7.

The concept of conservation of flux and of field line topology is inherently dependent on the possibility of unambiguously labeling a moving magnetic line (a possibility which is almost imbedded in the language used to talk about magnetic fields). To exhibit the full generality of these concepts, independent of whether the equations of motion are dissipative or ideal, macroscopic or microscopic, fluid dynamic or magnetic, we consider solenoidal fields B(x) which are subject only to div B = 0. As a first example take the simple tubular domain, Fig. 4. We are given a smooth field B with the property that through each interior point of the tube there is a unique magnetic line which intersects each end surface of the tube ($B_n = 0$ on S_o, $B_n < 0$ on S_1, $B_n > 0$ on S_2).

It is an easy matter to introduce coordinates (α,β) on S such that $d\alpha\, d\beta = B_n\, dS$.[8] Extending α and β along each line as a constant, defines $\alpha(x)$, $\beta(x)$ such that

$$B\cdot\nabla\alpha = 0 \ , \ B\cdot\nabla\beta = 0 \tag{4}$$

also

$$B = \nabla\alpha \times \nabla\beta \tag{5}$$

and for any surface element,

$$B\cdot dS = d\alpha d\beta \tag{6}$$

First of all, α and β are coordinates (4); second they are flux coordinates (6). The coordinates (α,β) are not unique; any other pair satisfying (5) [which implies (4) and (6)], $\alpha' = \alpha'(\alpha,\beta)$, $\beta' = \beta'(\alpha,\beta)$, has Jacobian one, $\partial(\alpha',\beta')/\partial(\alpha,\beta) = 1$.

Next consider a one-parameter smooth family of incompressible vector fields B(x,t) (t need not be related to time). Suppose for each t one chooses $\alpha(x)$ and $\beta(x)$ satisfying (5) such that $\alpha(x,t)$, $\beta(x,t)$ are smooth in t. Introducing the two vector fields E and u,

$$E = \frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta \tag{7}$$

$$u = \frac{1}{B^2}\left[\nabla\alpha\ \nabla\beta - \nabla\beta\ \nabla\alpha\right]\cdot\left(\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right) \tag{8}$$

we verify that

$$E + u \times B = 0 \tag{9}$$

$$\frac{\partial B}{\partial t} + \text{curl } E = 0 \tag{10}$$

$$\frac{\partial B}{\partial t} + \text{curl } (B\times u) = 0 \tag{11}$$

$$\frac{\partial\alpha}{\partial t} + u\cdot\nabla\alpha = 0\ ,\ \frac{\partial\beta}{\partial t} + u\cdot\nabla\beta = 0 \tag{12}$$

There seems to be a large amount of physics hidden in the variation of an arbitrary incompressible field (which is not even necessarily a magnetic field)![7] The "Maxwell" equation (10) simply states that E is a vector potential of the rate of change of B; if the full system of equations contains Maxwell's equations, E differs from the electric field by a gradient. The magnetic field line velocity, u, carries α and β; we <u>define</u> $\alpha(x,t)$ = const., $\beta(x,t)$ = const. to be a specific line. Since α and β are highly nonunique, u is also nonunique. If there are fluid equations in the full system (i.e. in addition to div B = 0), the fluid velocity will, in general, bear no relation to u in (8) which carries the field lines unless "Ohm's law" (9) governs the true velocity and electric field.

Since (α,β) for a given B are not unique, α and β can depend on t even if B does not. The flow, u, follows the arbitrary labels (α,β) which are assigned to the lines (this is a classical interchange).

The theorem just demonstrated [viz. that in a topologically simple, moving, incompressible field one can find flux coordinates (α,β) carried with a velocity u which satisfies (11)] has the following converse: given two vector fields, u and B, which satisfy (11) (and div B = 0), then one can find coordinates (α,β) which satisfy (5) and (12).[7]

Next we take as our domain the inside of a torus and assume that a smooth vector field B is given with the property that a cross cut S can be found such that through each interior point there is a unique field line which when extended in either direction intersects the cut (Fig. 5). This reduces the problem to the simple domain, Fig. 4, on identifying S_1 and S_2 as the two side of S. There remains the question of the continuation of α and β across the cut. This is not a serious matter since we are used to multivalued potentials. The properties of field lines when extended <u>indefinitely</u> in a torus are usually examined in terms of iterations of the mapping, $\alpha' = \alpha'(\alpha,\beta)$, $\beta' = \beta'(\alpha,\beta)$ across the cut, Fig. 5. However, in a general time varying field (satisfying div B = 0 only), the mapping, $\alpha' = \alpha'(\alpha,\beta,t)$, $\beta' = \beta'(\alpha,\beta,t)$ will also be time dependent. This is a very unsatisfactory means of assigning a name to a line. We <u>define</u> that a magnetic line coordinate system is established when the mapping is independent of time. Introducing the velocity field u via (8) in the cut domain, we see that u' = u is necessary and sufficient for the (α,β) map to be time-independent (this could be taken as an independent definition that field lines are not being reconnected). It is also easily shown that, if u is continuous across the cut, then an initially closed magnetic line will remain so; a closed magnetic surface will persist; an ergodic line will persist; the rotation number of a magnetic line on a surface will be invariant in time.

It is not clear whether there are physical examples of a nonsingle-valued velocity u across the cut. If u is a physical velocity [more precisely, if the perpendicular component of the physical velocity satisfies (11)], then it is usually continuous. Across a shock, where the physical velocity u is discontinuous, the (α,β) mapping can be shown to be time-independent since flux conservation is one of the shock jump conditions; (note that the shock requires a generalization of the present formulation to allow discontinuous B).

Up to now we have taken smooth fields B with no sudden shifts in direction or singularities (except for the aside concerning shocks).* To continue, consider a two dimensional field. It can be represented in terms of a single flux function,

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* There has been found an exact solution with intersecting shocks (type one) which reconnects lines; E. Hameiri, private communication.

$$B = n \times \nabla\psi \tag{13}$$

where

$$\frac{\partial\psi}{\partial t} + u\cdot\nabla\psi = 0 \tag{14}$$

If u is required to be finite, then $\partial\psi/\partial t = 0$ at every critical point, $\nabla\psi = 0$. This means that the maxima, minima, and separatrix values of ψ are fixed in time. No islands can appear, disappear, grow or shrink. This is the traditional conclusion.

It is possible to allow u to be unbounded to some extent and still be physically acceptable. If ψ is a smooth function, then in the neighborhood of a critical point, $\nabla\psi = 0$, we have $\psi - \psi_o \sim r^2$, $\nabla\psi \sim r$. Taking $\partial\psi/\partial t \neq 0$ at such a critical point implies that $u \sim 1/r$ which is not square integrable. If we take finite kinetic energy to be a valid restriction on u, then, again, we conclude that flux and topology are strictly conserved, even when there are critical points, $\nabla\psi = 0$.

For ideal MHD stability, the Lagrangian virtual displacement, ξ, satisfies the same equation as u, viz. (14). Since this is a self-adjoint formulation with $\int \rho_o\xi^2\, dV$ as norm, for ξ to be an admissible displacement implies that ψ not be varied at a critical point (this is a precaution that should be observed when constructing numerical stability codes).

However, if we allow nonstandard behavior near a critical point, e.g. $\psi - \psi_o \sim r^m$, $1 < m < 2$, $\nabla\psi \sim r^{m-1}$, then $\partial\psi/\partial t \neq 0$ is allowable while still maintaining finite kinetic energy. We shall return to this point.

III. EQUATIONS OF MOTION AND ADIABATIC FORMULATION

The complete nondissipative formulation is

$$\left\{\begin{array}{ll} \frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho u) = 0 & \\ \rho(\frac{\partial u}{\partial t} + u\cdot\nabla u) + \nabla p = J\times B\,, & J = \frac{1}{\mu_o}\,\mathrm{curl}\,B \\ \frac{\partial p}{\partial t} + u\cdot\nabla p + \gamma p\,\mathrm{div}\,u = 0 & \\ \frac{\partial B}{\partial t} + \mathrm{curl}\,(B\times u) = 0\,, & \mathrm{div}\,B = 0 \end{array}\right. \tag{15}$$

The <u>adiabatic</u> system has been defined[1] as the above with inertia, $\rho\,du/dt = \rho(\partial u/\partial t + u\cdot\nabla u)$ removed:

$$\left\{\begin{array}{ll} \frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho u) = 0 & \\ \nabla p = J\times B\,, & J = \frac{1}{\mu_o}\,\mathrm{curl}\,B \\ \frac{\partial p}{\partial t} + u\cdot\nabla p + \gamma p\,\mathrm{div}\,u = 0 & \\ \frac{\partial B}{\partial t} + \mathrm{curl}\,(B\times u) = 0\,, & \mathrm{div}\,B = 0 \end{array}\right. \tag{16}$$

This can be obtained by a formal, "slow" scaling in which $\partial/\partial t$ and u are multiplied by a small parameter, ϵ ; ϵ enters homogeneously in all but the momentum equation [this is the nondissipative form of the Grad-Hogan transport formulation[9]].

One might expect solutions of (15) to converge to those of (16) if, for example, the initial state is static and the constraints are varied slowly. This seems plausible in a simple topology but seems to offer an immediate contradiction when there is a separatrix - these statements will be made more precise.

The crucial analytical and numerical features of the adiabatic system (16) is that u does not "march". The velocity must so adjust that at every instant the pressure, magnetic field, and current (which do march) are in static pressure balance, $\nabla p = J\times B$.

The original theoretical attack on (16) (more precisely on a similar resistive model) was to obtain an equation for the velocity field by differentiating the static equilibrium with respect to time, then replacing $\partial p/\partial t$, $\partial B/\partial t$, $\partial J/\partial t$ by using the remaining equations (described in Refs. 9 and 10 with more general transport in Ref. 11). A greatly preferable procedure (both analytically and numerically) was found to be to eliminate the velocity field from the problem.[1,2] Our present task will be to reinsert the velocity field; i.e. after solving for the adiabatic evolution of thermodynamic and magnetic field quantities, to retrace the steps by which u was eliminated.

Assuming that there is a family of flux surfaces identified by a flux function, ψ, we introduce the volume, $V(\psi)$, within a surface ψ and the inverse function $\psi(V)$, also $p(V)$, $\rho(V)$ [it is assumed that $\rho = \rho(\psi)$ is constant on a flux surface].

We also introduce the microcanonical volume average, restricted to a flux surface: for a general function $\phi(x)$,

$$\langle\phi\rangle = \oint \phi \, dS/|\nabla V|, \quad \oint dS/|\nabla V| = 1 \tag{17}$$

If $\phi(x,t) = F(V,t)$ is constant on flux surfaces, we find that

$$\partial F/\partial t \equiv \phi_t = \langle\partial\phi/\partial t\rangle \quad . \tag{18}$$

Averaging the mass and energy equations yields

$$\rho_t + (\rho U)' = 0 \tag{19}$$

$$p_t + Up' + \gamma pU' = 0 \tag{20}$$

where

$$U = \oint u \cdot dS = \langle u \cdot \nabla V\rangle \ , \quad ' = \partial/\partial V \ . \tag{21}$$

The average pressure balance can be put in the form of a second order ordinary differential equation for $\psi(V)$ involving the two profiles $p(\psi)$ and $I(\psi)$ (poloidal current within ψ) and a 2×2 inductance matrix $L_{ij}(V)$ which depends on the geometrical shape of the flux surfaces only (Ref. 1, Appendix). A partial consequence of the averaged magnetic field equation is

$$\psi_t + U\psi' = c(t) \quad . \tag{22}$$

Specializing to two dimensions (V = area), with no third field component ($B_z = 0$), leaves (19) and (20) unchanged;

$$\frac{\partial\psi}{\partial t} + u\cdot\nabla\psi = c(t) \tag{23}$$

becomes the total content of the magnetic field equation, and the pressure balance

$$\Delta\psi = - p'/\psi' = - \dot{p}(\psi) \ , \quad \dot{} = \partial/\partial\psi \tag{24}$$

has as its average

$$(K\psi')' = - p'/\psi' \tag{25}$$

where

$$K(V) = \langle|\nabla V|^2\rangle = \oint \nabla V\cdot dS \tag{26}$$

is the only relevant inductance coefficient in this special case. Introducing the total mass,

$$M = \int \rho \, dV \ , \ M' = \rho \tag{27}$$

and integrating (19), we have

$$M_t + UM' = a(t) \ . \tag{28}$$

The integration constants a(t) and c(t) are usually set equal to zero. Their choice depends on where we fix the physically irrelevant reference points $\psi = 0$ and $M = 0$ [in the resistive problem, fixing $\psi = 0$ at a convenient location usually gives a nonzero c(t)]. The choice $a = c = 0$ has the important conceptual consequence that, in virtue of (22) and (28), $M(\psi)$ becomes a time-independent profile (adiabatic invariant). By manipulating with c(t), the flux at a separatrix can be made to vary at will. Of course, this does not imply line breaking or reconnection. The crucial point is to change the total flux within a bounded separatrix or to observe mass flow across the separatrix.

Similarly, from (19) and

$$\psi_t' + (U\psi')' = 0 \tag{29}$$

we observe that

$$\rho/\psi' \equiv \tau(\psi) = dM/d\psi \tag{30}$$

is an ivariant profile. Also

$$p/(\psi')^{\gamma} \equiv \mu(\psi) \tag{31}$$

$$p/\rho^{\gamma} \equiv \sigma(\psi) = \mu/\tau^{\gamma} \tag{32}$$

are invariants of the adiabatic equations of motion; [including the third field component, B_z, gives rise to the rotational transform as an additional invariant profile].

Eliminating p in favor of μ in (24) gives a <u>single equation</u> which governs the adiabatic evolution of ψ(x,y,t)

$$\Delta\psi = -\dot{\mu}(\psi')^{\gamma} - \gamma\mu(\psi')^{\gamma-1}\,\psi'' \;, \quad \dot{} = d/d\psi \tag{33}$$

This is an unusual equation which has been studied intensively.[1,12,13] Just as the nonlinear elliptic PDE (24) allows ψ(x,y) to be determined, given p(ψ) and suitable (elliptic) boundary data and constraints (e.g. coils), the <u>GDE</u> (33) allows ψ(x,y) to be determined given μ(ψ) and appropriate (nonelliptic) data.[1,12] Time variation in an adiabatic formulation consists of successive snapshots taken with the varied boundary data or constraints; $\mu(\psi)$ is fixed.

Given a family of adiabatic solutions $\psi(x,y,t)$ and $\psi(V,t)$ depending on the parameter t, U is then obtained from

$$U = \psi'/\psi_t \ , \tag{34}$$

the normal component of u from

$$u_n = (c - \partial\psi/\partial t)/|\nabla\psi| \tag{35}$$

and the parallel component (within an added constant on each line) using (31) and

$$\mathrm{div}\ u = -\frac{1}{\gamma p}\left(\frac{\partial p}{\partial t} + u\cdot\nabla p\right) \tag{36}$$

It is interesting to note that in the Grad-Hogan resistive theory, from which velocity is eliminated as in the adiabatic theory, the reinserted velocity field is square integrable near a separatrix, whereas in the Pfirsch-Schlüter resistive theory it is not.

IV. CORNER EQUILIBRIA

Consider the two-dimensional equation

$$\Delta\psi = \lambda|\psi|^n , \qquad -1 < n < 0 . \tag{37}$$

The parameter λ can be scaled out; it is significant primarily through its sign. We look for a solution in the sector $0 < \theta < \theta_o$ with $\psi = 0$ at $\theta = 0$ and at $\theta = \theta_o$ and adopt the form

$$\psi = r^m g(\theta) . \tag{38}$$

For the present, take $g > 0$, $\psi > 0$ in the sector. Substitution of (38) into (37) yields

$$m = \frac{2}{1-n} , \qquad 1 < m < 2 \tag{39}$$

$$g'' + m^2 g = \lambda g^n . \tag{40}$$

This equation can be integrated,

$$\theta = \int dg/v \tag{41}$$

where

$$v^2 = v_o^2 - m^2 g^2 + \alpha g^{1+n} , \ \alpha = 2\lambda/(1+n) . \tag{42}$$

The sector angle is given by

$$\theta_o = 2\int_o^{g_1} dg/v \tag{43}$$

where

$$v_o^2 - m^2 g_1^2 + \alpha g_1^{1+n} = 0 \tag{44}$$

In terms of $x = g/g_1$,

$$v^2 = m^2(g_1^2 - g^2) - \alpha(g_1^{1+n} - g^{1+n})$$

$$= m^2 g_1^2 [(1-x^2) - \frac{\alpha}{m^2 g_1^{1-n}} (1 - x^{1+n})]$$

$$\theta_o = \frac{2}{m} \int_0^1 (1-x^2)^{-1/2} [1-\beta \frac{1-x^{1+n}}{1-x^2}]^{-1/2} dx, \quad \beta = \frac{2\lambda}{(1+n) m^2 g_1^{1-n}} \tag{45}$$

The second factor in the integrand is bounded in $0 \leq x \leq 1$ if $\beta < 1$, and it is evident that θ_0 is monotone in β. For $\beta = 0$,

$$\theta_0 = \theta_{vac} = \pi/m \tag{46}$$

where θ_{vac} is the sector angle for $\Delta\psi = 0$, $\psi = r^m \sin m\theta$. We have

$$\begin{cases} \theta_o < \theta_{vac}, & \beta < 0 \\ \theta_o > \theta_{vac}, & \beta > 0 \end{cases} \tag{47}$$

Also, $\theta_o \to 0$ for large negative β and θ_o can be made arbitrarily large with β positive ($\beta < 1$).

This result is surprisingly general. It can be shown that in an arbitrary simple bounded domain (with or withour corners, Fig. 6) the nonlinear equation (37) subject to $\psi = 0$ on the boundary has a unique solution, $\psi(x,y)$, with sign opposite to λ everywhere in the interior. This seems to contradict the second corner solution above with $\theta_o > \theta_{vac}$ (β has the same sign as λ). There is no strict contradiction since the standard elliptic existence theorem referred to does not apply to an unbounded sector. To consider just the relation of the sign of ψ to the sign of λ, we employ the Green's function of the sector (always negative),

$$\psi = \lambda \int G|\psi|^n \, dx \, dy$$

This implies that the sign of ψ is opposite to that of λ <u>except</u> that a simple estimate shows that the integral ceases to converge for $r \to \infty$ when $\theta_o > \theta_{vac}$; thus there is no contradiction in the unbounded sector.

We see that the large angle solution, $\theta_o > \theta_{vac}$, is not valid for the neighborhood of a corner <u>inside</u> an island (cf. Fig. 1). However, there is no reason to discard it in a nonsimple region (e.g. outside the figure-eight in Fig. 1). The reader can verify that the sign of ψ can change in an annulus.

A similar argument gives a somewhat different conclusion for bounded current, say $\Delta\psi = 1$. It is easy to construct separatrix solutions of $\Delta\psi = 1$ as in Fig. 1. Suppose $\psi = 0$ at the separatrix. The same arguments as above shows that $\psi < 0$ inside the figure-eight and clearly $\psi > 0$ outside. Near the separatrix, the representation

$$\psi \sim \frac{1}{2} ax^2 + \frac{1}{2}(1 - a)y^2 \qquad (a < 0 \text{ or } a > 1)$$

shows that $\psi > 0$ on the obtuse side. We conclude that the inner angle of the figure-eight is always acute (for finite current).

The similar statement for the singular current, (37), is that an interior corner of an island is smaller than θ_{vac} (for the given m or n), but the exterior angle can be either larger or smaller. To make this statement meaningful, we must match the corner solution, $0 < \theta < \theta_o$, to one in the supplementary corner, $\theta_o < \theta < \pi$ (using symmetry to obtain a full neighborhood of the origin, Fig. 7).

We demand that the vector $\nabla\psi$ be continuous, i.e. that ψ (automatic) and $\partial\psi/\partial n$ be continuous across a ray on which $\psi = 0$. Suppose the solution in $0 < \theta < \theta_o$ is fixed. Since $\partial\psi/\partial n = r^{m-1}g' = r^{m-1}v_o$ we must take the same exponent m (and n) in the two sectors; $g' = v_o$ must have the same magnitude but change sign (ψ changes sign) and $\bar{\lambda}$ in the second region must be adjusted to obtain a sector angle $\bar{\theta}_o = \pi - \theta_o$. To verify this, we first remark that carrying out the previous calculation with $g < 0$ is equivalent to changing the sign of λ, i.e. it gives the same formulas (44) and (45), but with g_1 replaced by $\bar{g}_1$ and

$$\begin{cases} \bar{\alpha} = -2\bar{\lambda}/(1+n) \\ \bar{\beta} = \bar{\alpha}/m^2|\bar{g}_1|^{1-n} = -2\bar{\lambda}/(1+n)m^2|\bar{g}_1|^{1-n} \end{cases} \quad (48)$$

Equation $(\overline{45})$ determines a unique value $\bar{\beta}$ to produce $\bar{\theta}_o = \pi-\theta_o$. Equation $(\overline{44})$ can be rewritten

$$v_o^2 = m^2\bar{g}_1^2 - \bar{\alpha}|\bar{g}_1|^{1+n} = m^2\bar{g}_1^2\ (1 - \bar{\beta}) \tag{49}$$

which determines $\bar{g}_1$, since v_o^2 is unchanged. Thus we have extended the corner solution to include an entire neighborhood of the origin, Fig. 7, with $\psi = 0$ at two intersecting straight lines and $\nabla\psi$ continuous.

We have already established that an interior corner has $\theta_o < \theta_{vac}$. Since $1 < m < 2$, θ_{vac} is obtuse. But, depending on whether θ_o is smaller or larger than the supplement of θ_{vac}, $\bar{\theta}_o = \pi - \theta_o$ can be either larger or smaller than θ_{vac}. Also, θ_o can be acute or obtuse. If $\bar{\theta}_o > \theta_{vac}$, the current $(\Delta\psi)$ reverses across the separatrix; if $\bar{\theta}_o < \theta_{vac}$, the current has the same sign on both sides. If $\theta_o = \pi - \theta_{vac}$, then the current in the exterior corner is zero (or it can be finite and nonzero by fitting to $\Delta\psi = 1$ instead of $\Delta\psi = 0$; this requires adding a higher order correction since $\psi \sim r^2$ is small compared to $\psi \sim r^m$).

To summarize, from the corner analysis, two parameters, θ_o and n can be given arbitrarily [subject to $-1 < n < 0$ and $\theta_o < (1-n)\pi/2$], and an appropriate solution can be found in an entire neighborhood of the origin. The current reverses or not depending on whether $\theta_o < (1+n)\pi/2$ or not.

Next we turn to a boundary layer analysis of the entire separatrix and of a smooth closed curve (Fig. 6), each with the singularity given in (37). We have already seen that a global analysis restricts the class of valid corner solutions. Near the separatrix (or smooth boundary curve), away from the corner, the solution is approximately one-dimensional. Consider first the smooth curve. Taking y as the (inward pointing) normal coordinate, and assuming that $\psi > 0$, we have

$$\psi_{yy} = \lambda\psi^n \tag{50}$$

which integrates to

$$y = \int_0^{\psi} \frac{d\psi}{u} \tag{51}$$

where

$$u^2 = u_o^2 + \alpha\psi^{1+n} , \quad \alpha = 2\lambda/(1+n) . \tag{52}$$

The volume (oriented inward so that $\psi' > 0$) is

$$V = \oint y\, ds = \int_0^{\psi} d\psi \oint \frac{ds}{u} \tag{53}$$

Therefore,

$$\frac{1}{\psi'} = \oint \frac{ds}{u} \tag{54}$$

In this integral, $u_o(s)$ ($\partial\psi/\partial y$ at the boundary) is assumed to be given.

For a smooth curve

$$\left(\frac{1}{\psi'}\right)' = \psi' \frac{d}{d\psi} \left(\frac{1}{\psi'}\right) = - \lambda\, \psi'\, \psi^n \oint \frac{ds}{u^3} \tag{55}$$

(this formula cannot be used near a corner where u becomes zero). The most important conclusion that we draw is that ψ'' has the same singularity (viz. ψ^n) as $\Delta\psi$, and even has the same sign, $\psi''/\lambda\psi^n = (\psi')^3 \oint ds/u^3$. This is not surprising since intuitively $\Delta\psi \sim \psi_{YY} \sim \psi''$.

This boundary layer analysis applies to the solution of (37) which we know exists and is unique. However, if this solution is to represent an adiabatic equilibrium we would want $\dot{\mu} = d\mu/d\psi$ to be finite. The averaged pressure balance

$$(K\psi')' = -\dot{\mu}(\psi')^{\gamma} - \dot{\gamma}\mu(\psi')^{\gamma-1}\,\psi'' = \lambda\psi^{n} \tag{56}$$

implies that ψ'' has a sign opposite to $\lambda\psi^{n}$ (since the singular term in ψ'' is assumed to dominate the finite term in $\dot{\mu}$). Thus, although there is a perfectly legitimate standard (elliptic) solution with the indicated current singularity, it must have unbounded $\dot{\mu}$, and there is no adiabatic singular solution in the case of a smooth boundary, (nor should we expect one).

For a separatrix with a corner, the contribution to ψ'' from the neighborhood of the corner must be the same order as the distant contribution, and must be the reverse in sign. The boundary layer analysis for a global adiabatic separatrix will be presented elsewhere. For reasons that will presently become clear, we are interested in corner solutions whether or not they are compatible with a global adiabatic solution.

V. LINE BREAKING FLOWS

We start with the explicit equilibrium solution of a corner equilibrium as given in the last section. Ignoring the fact that this type of solution was suggested by adiabatic considerations, we construct from it a solution of the full equations of motion (15). To be more precise, we construct a solution of the full equations of motion linearized about the specified static equilibrium. It is well known that there are enormous subtleties in relating a linearized solution to the solution of the original nonlinear system. However, exactly this linearization is very widely used; for example, more than 95% of the literature of MHD stability is based on these equations. To be still more precise, we do somewhat better than the usual linearization. Assuming that the flow velocity, u, is small, we drop $\rho u \cdot \nabla u$ from the momentum equation (this yields the adiabatic system (16) with $\partial u/\partial t$ reinserted). But no other approximation is made; e.g. we do not replace ρ by $\rho_o + \rho'$ where ρ_o is the original equilibrium quantity and ρ' its perturbation. This allows us to distinguish between the Eulerian $\partial/\partial t$ and Lagrangian $\partial/\partial t + u \cdot \nabla$ which are normally identified in a linearization. For example, the explicit solution will have $p(x)$ unchanging in time, $\partial p/\partial t = 0$, but the entropy and density are carried with the fluid, e.g. $\partial \eta/\partial t + u \cdot \nabla \eta = 0$. This makes the solution believable for longer periods of time than a conventional linearization.

To start take $\partial \psi/\partial t = 1$. This is not a trivial relabeling of magnetic lines because it will be coupled with a mass flow across the separatrix, the mass flow following the magnetic flux flow implied by $\partial\psi/\partial t \neq 0$. In particular, a specific magnetic line carrying with it mass at a certain density will split in two as it crosses the separatrix; another pair of symmetric lines will touch as it reaches the separatrix, then leave in two different quadrants with the halves of the individual lines reconnected differently. To be a solution of the equations of motion it is clear that the configuration must be symmetric; more precisely, lines which merge must carry the same thermodynamic state; a line which splits leaves two lines carrying the same state. We shall return to the question of generalized adiabatic solutions which do not have this restriction.

Specifically, take

$$\psi = \psi_o + t \tag{57}$$

where ψ_o is the corner equilibrium solution. The magnetic field

$$B = n \times \nabla\psi \tag{58}$$

is unchanged in time as are J = curl B and p. The velocity component normal to B is

$$u \cdot \nabla\psi = -\, \partial\psi/\partial t = -\, 1 \tag{59}$$

The vector normal component is

$$u_\perp = -\, \nabla\psi/|\nabla\psi|^2 \tag{60}$$

This component is singular only at the X-point and it is easily seen to be square integrable for $-1 < n < 0$.

Next we evaluate div u from the combined entropy/mass equation for p. Since $\partial p/\partial t = 0$

$$u\cdot\nabla p + \gamma\, p \text{ div } u = 0$$

or

$$\text{div } u = -\dot{p}/\gamma p \quad , \quad \dot{} = d/d\psi \tag{61}$$

Also we recall that

$$\dot{p} = -\Delta\psi = -\lambda|\psi|^n \, , \; p = p_o - \frac{\lambda|\psi|^n}{(n+1)}\,\psi \tag{62}$$

Writing

$$u = u_\perp + u_\parallel \quad , \quad u_\parallel = \sigma B \quad , \tag{63}$$

we obtain

$$\text{div } u_\parallel = B\cdot\nabla\sigma = \text{div}(\nabla\psi/|\nabla\psi|^2) - \lambda|\psi|^n/\gamma p \tag{64}$$

which determines σ within an added constant on each flux contour.

The velocity component $u_\parallel$ is singular not only at the X-point, $r = 0$, but along the entire boundary of the sector, $\psi = 0$. Near the corner, $u_\parallel$ is square integrable for $m < 2$, in other words, always. Near the edges, $u_\parallel$ is square integrable for $m > \frac{3}{2}$, $n > -\frac{1}{3}$, which restrictions we adopt.

The momentum equation is satisfied since u is steady and $\partial u/\partial t = 0$. The only equations left to examine are mass and entropy (the combination of the two, which involves $\partial p/\partial t$, has already been used):

$$\frac{\partial\rho}{\partial t} = - u\cdot\nabla\rho - \rho \text{ div } u = \dot{\rho} - \rho\dot{p}/\gamma p = \rho[\log(\rho/p^{1/\gamma})]^{\cdot} \qquad (65)$$

$$\frac{\partial\eta}{\partial t} = - u\cdot\nabla\eta = \dot{\eta}$$

The initial density or entropy profile, $\rho(\psi)$ or $\eta(\psi)$ is arbitrary [the equilibrium equation relates only to $p(\psi)$]. If η = const. initially, then this persists, $\partial\eta/\partial t = \partial\rho/\partial t = 0$. More generally, the convection of η and ρ gives rise to the indicated time variation of ρ and η. The reason for the nontrivial convection is that p is invariant in Eulerian and η in Lagrangian (moving) coordinates.

A constant field component, B_z, can be superposed on the solution just given provided that an appropriate flow velocity u_z is also supplied. The condition curl E = 0 determines u_z within a constant on each ψ-contour. Although u_z is singular, it is square integrable if $n > -\frac{1}{2}$ as before.

Consider next the more complex problem of a time-varying adiabatic Doublet (Fig. 1). Assuming that the numerical evidence implies the mathematical existence of these solutions, we can reintroduce $u_\perp$ and then div u and $u_\parallel$ in the <u>adiabatic</u> formulation (16). In the case of two symmetric islands, mixing (shrinking islands) makes contact between two identical states and there is no obvious entropy increase - this is what is confirmed by use of the appropriate jump conditions. The velocities that go with this equilibrium have the same form as found in the exact solution of the corner, indicating that the adiabatic solution is a true solution of the full (linearized) equations of motion - with line-reconnection.

With asymmetric islands, the jump conditions give an entropy increase (ρ_1 and ρ_2 mix to form ρ_o). The jump in entropy is exactly the entropy of mixing and no more. Consistent with this nontrivial mixing, the adiabatic velocity component $u_\parallel$ turns out to have a δ-function at the separatrix. This is truly an adiabatic limiting solution only, not an exact solution of the equations of motion.

In the case of asymmetric splitting (growing islands in Fig. 1), the jump conditions show that $\rho_1 = \rho_2$ in the newly created regions, so that this case can be expected to be an actual solution of the equations of motion.

A maximum or minimum of ψ (e.g. the center of an island) is also a critical point, $\nabla\psi = 0$. There is no adiabatic evidence (theoretical or numerical) that the current is not smooth, but we may investigate the consequences of postulating $\psi \sim r^m$, $1 < m < 2$, at an extremum of ψ. Allowing $\partial\psi/\partial t \neq 0$ at this critical point yields a velocity field with finite kinetic energy. It must, nevertheless, be rejected on physical grounds, because $\partial\psi/\partial t \neq 0$ implies the presence of a <u>mass</u> source or sink; this is not allowable. We conclude that only at a hyperbolic critical point is this type of singularity relevant.

We offer now a conjecture as to the possible behavior at an X-point of the full <u>nonlinear</u> time-dependent ideal equations (15). At a two-dimensional X-point the velocity of an Alfvén wave is zero; (the projected signal speed in the plane remains zero with a nonzero third component, B_z). In the "normal" case with bounded current and $\psi \sim r^2$, the time required for a signal to reach the critical point is infinite. There will therefore be a buildup of irregularities (large derivatives) near this point. A very similar argument is used to show the instability of a compressive transonic nozzle flow in fluid dynamics (with subsequent development of a shock); a similar argument can also be given as the qualitative reason for dissipationless Alfvén wave heating at a resonance, $k \cdot B = 0$ (the propagation speed normal to the resonant surface approaches zero). We conjecture that "in general", a singularity of the type $\psi \sim r^m$, $1 < m < 2$, will develop after a finite period of time. This seems to be a difficult analytical question to resolve. A good two-dimensional numerical code could probably detect such a phenomenon. For a certain period of time the value of ψ at a separatrix (in the presence of waves) would be observed to remain approximately constant, but after a precise time, ψ should be seen to vary in a manner insensitive to choice of mesh size, etc. There will be an automatic problem with numerical accuracy near the X-point since the solution will become irregular whether or not the conjectured singularity arises. However, it should not be difficult to find that the global behavior (e.g. changing ψ at the X-point or a buildup of current) is reproducible.

The significance of the square-integrability condition for velocity is not simple. It is relatively clear that one should discard solutions or, at least, examine them very carefully if the kinetic energy is infinite. On the other hand, opening the doors to arbitrary square-integrable perturbations leads to a Pandora's box of disturbing possibilities. Consider, for example, the linearized equations of motion and a flux function

$$\psi = \psi_o + t\psi_1 \tag{67}$$

where ψ_o is smooth, $\nabla\psi_o \neq 0$, and ψ_1 has a cusp at which $\nabla\psi_1$ is unbounded. Since $u \cdot \nabla\psi_o = -\psi_1$, $u_\parallel$ is not only square integrable, it is bounded ($u_\parallel$ is square-integrable if $\nabla\psi_1$ is). But for arbitrarily small t, the perturbation ψ_1 changes the topology of the flux surface.

This is not particularly disturbing with regard to solutions of the equations of motion, but it is quite disturbing for variational (δW) stability where one is presumably working in a Hilbert space in which the displacement ξ ($\xi \cdot \nabla\psi_o = -\psi_1$) can easily alter the topology. Presumably, such nontopology-preserving displacements will usually increase the potential energy, in which case they are innocuous. First of all, no such theorem has been proved (or mentioned). Secondly, it is easy to construct counterexamples of ideal (nonresistive) tearing.[1] Thirdly, stability of a "standard" separatrix is entirely different from that of an adiabatic separatrix.[1] These questions will be broached elsewhere.

The explicit corner, line reconnection solution presented here and its relation to earlier adiabatic examples were presented and fully discussed at a Gordon Conference in Wolfeboro, New Hampshire, June, 1977, also at the Sherwood Theoretical Meeting in San Diego, May, 1977, at the Plasma Physics Division meeting of the APS in Atlanta, November, 1977, and at a number of intervening seminars and colloquia at various universities.

VI. CONCLUSION

We have shown by plausible arguments that a previously described class of adiabatic time-dependent solutions (type one) with complex topology has a subset (no mixing of regions with different density) in which reinsertion of the (small) velocity field probably gives a legitimate solution (type two) of the equations of motion in the limit of slow motion. We have also exhibited an explicit exact solution (type two) of the linearized ideal equations in a special separatrix configuration with flow across the separatrix. The role played by dissipation in magnetic line reconnection is, to a large extent, unknown.

Acknowledgments

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Figure Captions

Fig. 1: Complex Topology

Fig. 2: Resistive evolution from Belt Pinch to Doublet as external coil pinches the "waist". Current layer develops and moves with separatrix.

Fig. 3: Moving Object Breaking Magnetic Lines.

Fig. 4: Tubular Domain.

Fig. 5: Toroidal Domain, Magnetic Line Map.

Fig. 6: Interior Domain, With and Without Corner.

Fig. 7: Corner Solution.

E R R A T U M

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Report
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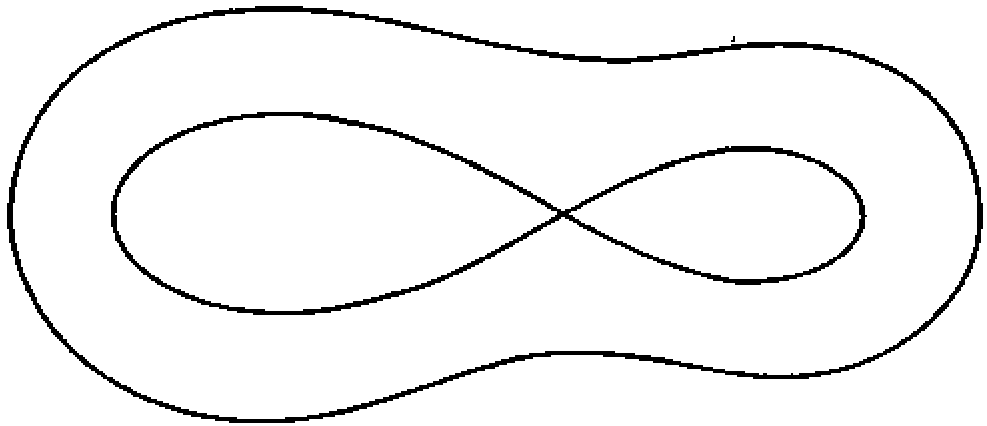

Figure 1

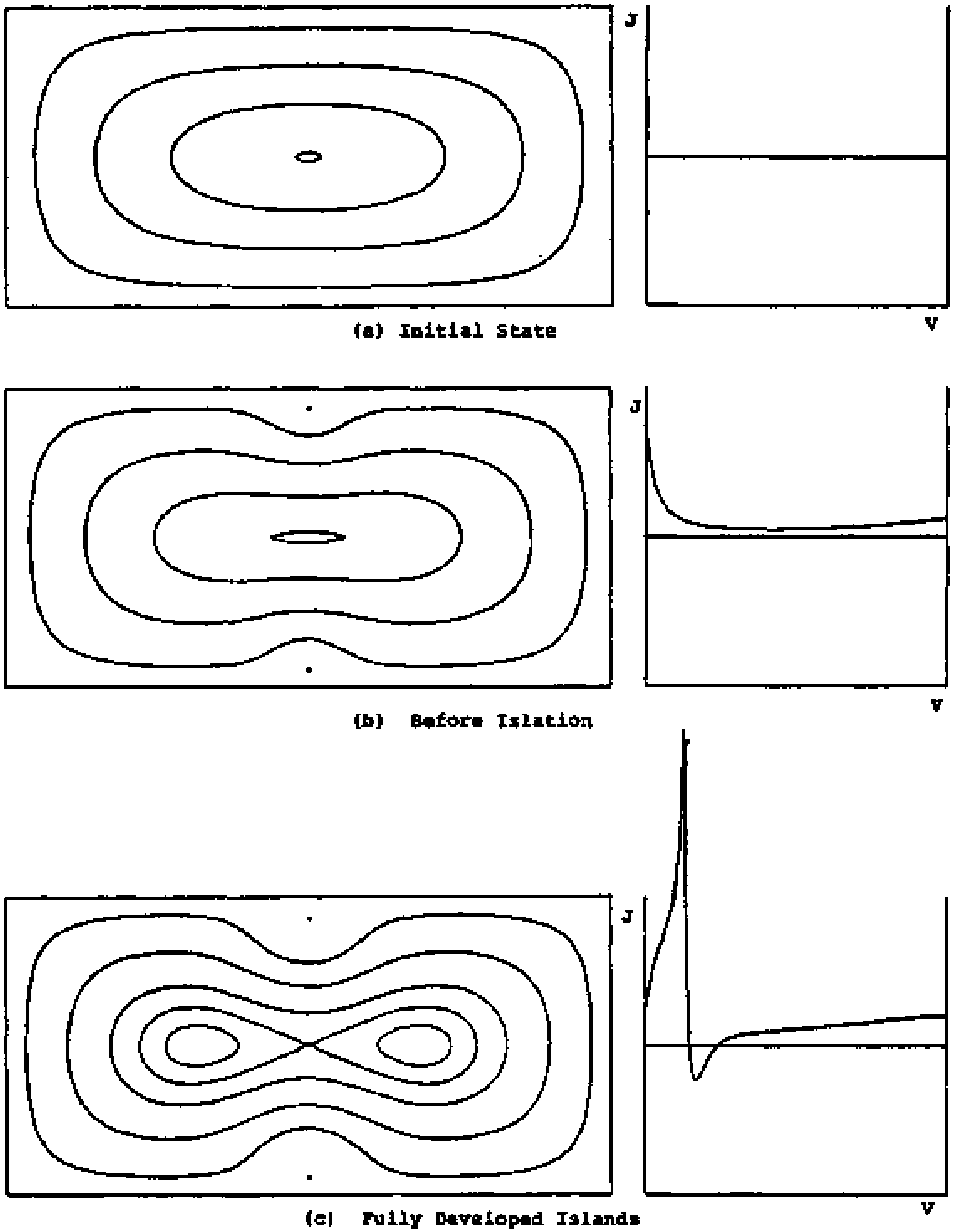
J
V
(a) Initial State
J
V
(b) Before Islation
J
V
(c) Fully Developed Islands

Figure 2

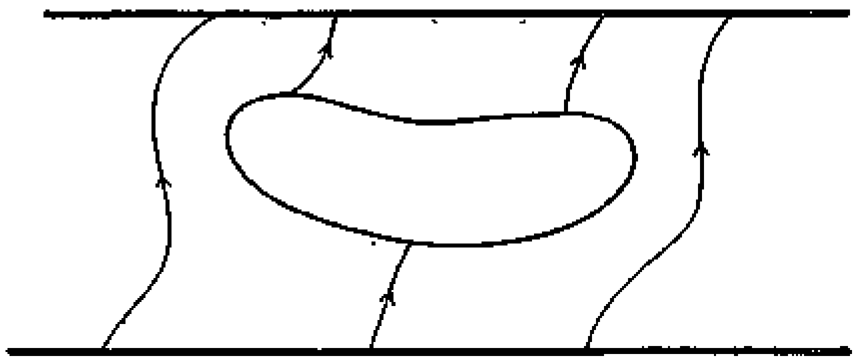

Figure 3

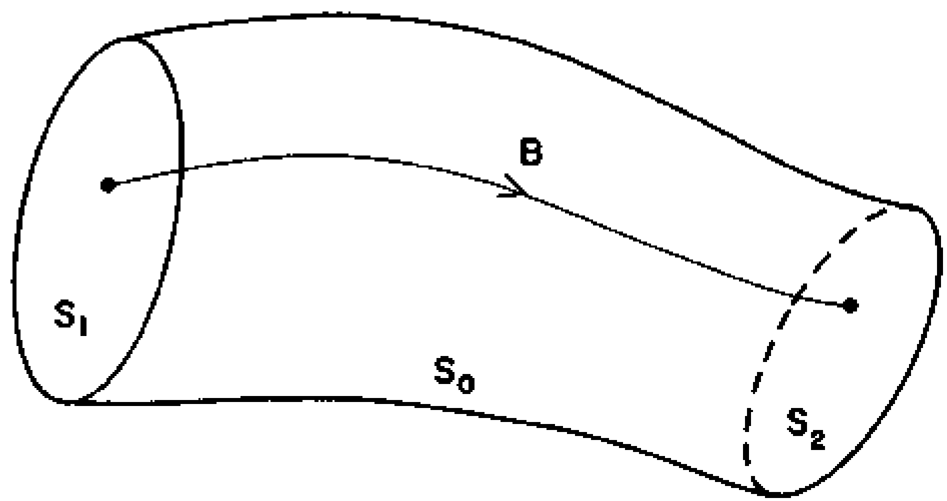

Figure 4

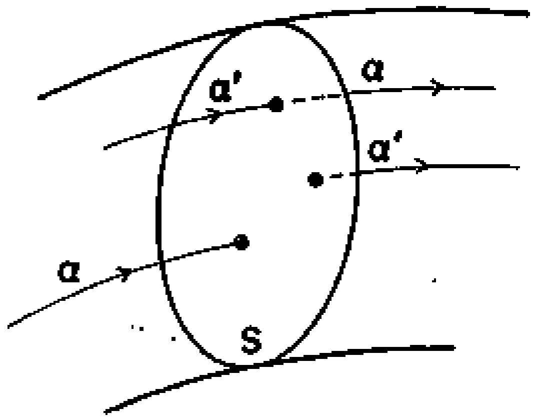

Figure 5

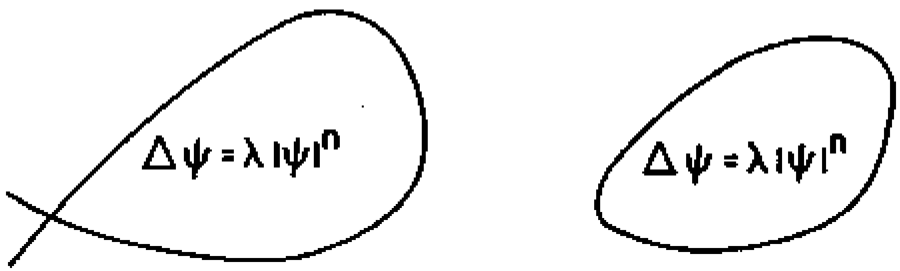

Figure 6

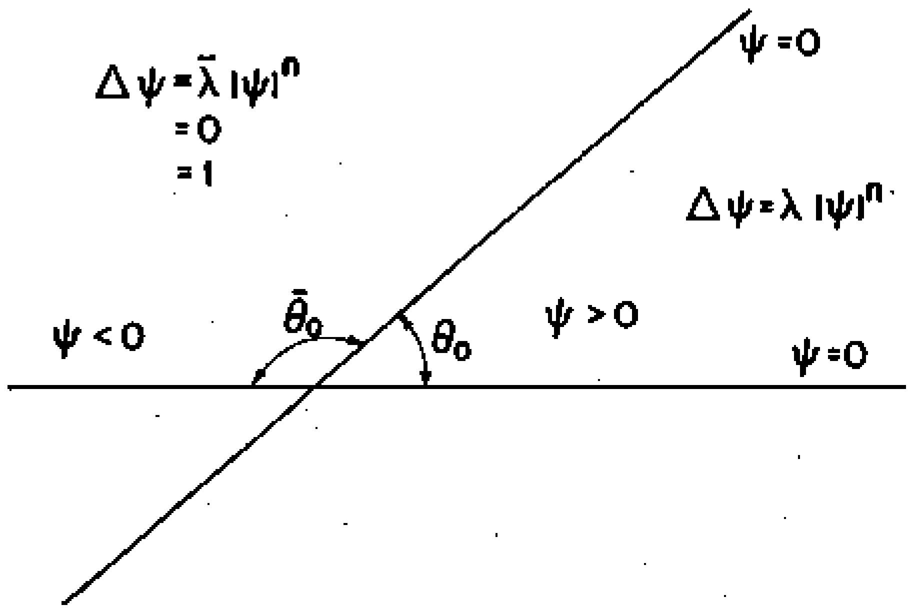

Figure 7