A THERMODYNAMIC APPROACH TO THE INELASTIC STATE VARIABLE THEORIES

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ABSTRACT

A continuum model is proposed as a theoretical foundation for the inelastic state variable theory of Hart. The model is based on the existence of a free energy function and the assumption that a strained material element recalls two other local configurations which are, in some specified manner, descriptive of prior deformation. A precise formulation of these material hypotheses within the classical thermodynamical framework leads to the recovery of a generalized elastic law and the specification of evolutionary laws for the remembered configurations which are frame invariant and formally valid for finite strains. Moreover, the precise structure of Hart's theory is recovered when strains are assumed to be small.

INTRODUCTION

The recognized inadequacy of the classical theories in many sensitive applications, such as the design of turbine and nuclear reactor components, has spurred considerable interest in the development of new constitutive theories capable of describing the rate and temperature dependent response of metals. A recent contribution in this effort has been the proposal by materials scientists of a number of so called "state variable" theories. The distinguishing feature of such theories is that the accumulated effect of deformation history on future material response is completely characterized by the current values of a finite number of state variables. Although the choice of state variables differs from one theory to the next, the existence of rate laws which govern their evolution is common to all.

Of principal concern here is the theory proposed by Hart [1] in which the state of a material element is assumed to be fixed by the current values of elastic and anelastic strain tensors, absolute temperature and a scalar parameter denoted as "hardness". The qualitative features of this theory are illustrated by the simple schematic model shown in Fig. 1. Here, $\underset{\sim}{e}$ represents the elastic or instantaneously recoverable strain, $\underset{\sim}{a}$, the anelastic strain which is to be regarded as momentarily locked in, and $\underset{\sim}{D}, \dot{\underset{\sim}{\varepsilon}}$ and $\dot{\underset{\sim}{\alpha}}$, the total, non-elastic and plastic rates of deformation. Element three in this model is best understood as a non-linear viscous element while two is described as a plasticity element. $\underset{\sim}{\sigma}$, $\underset{\sim}{\sigma}_a$ and $\underset{\sim}{\sigma}_f$ represent, respectively, the total observable stress and the components of observable stress active in each branch. The various scalar coefficients in the suggested tensor relationships

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}_a + \underset{\sim}{\sigma}_f$$

$$\underset{\sim}{D} = \dot{\underset{\sim}{\varepsilon}} + \dot{\underset{\sim}{e}}$$

$$\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{a}} + \dot{\underset{\sim}{\alpha}}$$

$$\underset{\sim}{\sigma} = \lambda \underset{\sim}{e} + 2\mu \underset{\sim}{I}\ \mathrm{tr}(\underset{\sim}{e})$$

$$\underset{\sim}{\sigma}'_a \propto \underset{\sim}{a}$$

$$\dot{\underset{\sim}{\alpha}} \propto \underset{\sim}{\sigma}'_a$$

$$\dot{\underset{\sim}{\varepsilon}} \propto \underset{\sim}{\sigma}'_f ,$$

in which prime denotes deviatoric part, are determined from empirical laws which generally depend on temperature and hardness.

A distinctive feature of this theory is that plastic flow and hardening effects are simulated without making use of yield criteria in the classical sense. As a consequence, the complications which arise when attempting to determine and match boundary conditions across an elastic-plastic interface are eliminated. Another feature is its ability to predict induced anisotropic effects such as the Bauschinger effect. Directional characteristics of this type are introduced through the anelastic strain tensor which reflects local deformation history. Also characteristic of this model is its ability to exhibit time dependent phenomenon such as creep and stress relaxation through the Kelvin-Voigt combination of elements one and three.

Despite the apparent scope of this theory it is important to emphasize that its empirical foundation and validation are based almost entirely on uniaxial tests.* For this reason the present three dimensional version is, for the most part, unsubstantiated. Moreover, questions which arise when the linear or "small strain" limits are exceeded have not been addressed as no distinction is made between reference and current configurations nor is their a precise definition of the "dot" time derivative. In the present paper this theory is given precise mathematical structure within which

*Developments in this area are outlined and amply referenced in [2].

questions of this type may be meaningfully considered. In the development to follow, which proceeds from the assumed existence of a free energy function, all strains shall be considered finite and full use shall be made of the second law of thermodynamics and the invariance constraints of modern constitutive theory. The general theory which results not only allows for the recovery of the equations of Hart in the small strain limit but also provides a rational basis for proposing non-linear generalizations fully consistent with the laws of thermodynamics and mechanics. It is recognized, however, that the subsequent development of more sophisticated multiaxial experimental capability will provide the only true litmus for assessing the correctness of our model.

The theory set forth in this paper is based on an ideal material with "selective memory". To make this precise it is assumed that the deformed element remembers two other configurations: one which would be recovered instantaneously upon step removal of the supporting load, and a second which represents the rest or zero strain energy configuration. By distinguishing between the rest and unstressed configurations we introduce anelasticity into our model whitch reflects the tendency of a typical metal to continue to recover, thereby releasing energy, for a measurable period following the removal of load. In addition, the rest and initial or reference configurations need not coincide. This introduces non-recoverable deformation into our model and reflects the fact that two subsequent zero strain energy configurations may generally differ. The relative distortions between the current and remembered configurations, together with absolute temperature, temperature gradient and a collection of scalars make up the complete list of state variables. Besides fixing the current values of stress, free energy, entropy and heat flux through constitutive equations consistent with the second law of thermodynamics, these state variables are also assumed to fix the evolution of the unstressed and rest configurations through constitutive equations for their rates of change. As previously mentioned, this theory will be formally valid for large deformations and will also be capable of accounting for material inhomogeneity and certain types of anisotropy. Although the development is carried to completion only for isotropic materials it may be extended to handle materials which recall a rest configuration with invariant directional characteristics. This sort of anisotropy, of course, depends on the characterization of plastic flow as a simple rearrangement of material along slip lines, inducing no further local anisotropy or texture.

In the fully isotropic theory it is also assumed that there exists no mechanism for the maintainance of dilatation in the absence of stress. It is therefore necessary to impose an incompressibility requirement on the flow rules for the unstressed and rest configurations. Most important, however, is the complete characterization of plastic flow as a simple non-dilatational rearrangement of material, carrying an isotropic rest configuration into a new isotropic rest configuration, without altering the viscoelastic properties of the material. Although this could be relaxed to some degree it does allow us to characterize material response as being purely viscoelastic with the added complication of a continuously deforming viscoelastic reference.

The theory proposed here is cast within the general state variable structure provided by Onat [3,4] which, although based on a rather precise and distinctly different notion of state, has apparently not been widely noticed or accepted. Specifically, two elements are said to have the same state and orientation at a given instant if their measurable future response to identical but otherwise arbitrary temperature and deformation processes (stimuli) is indistinguishable. Consideration is then restricted to materials for which the instantaneous state and orientation, i.e., future response to ongoing stimuli, is determined by a finite number of tensor state variable fields defined over the current element configuration. It is noteworthy that this characterization of state variables reduces to the classical one, namely that the state variables determine the instantaneous values of stress, free energy, entropy and heat flux, only for materials which respond continuously to continuous stimuli. We note, after recalling the specifics of the two constant Kelvin-Voigt model for one-dimensional solid viscoelasticity, that this is not always the case. For this simple model smooth deformation is required in order to elicit continuous stress response even though the specification of total strain suffices to determine the state in the sense intended by Onat. Only a part of the stress, i.e., the stress carried in the elastic branch, would reflect the state while the remaining part would reflect state transition.

The important features of this general state variable structure are reviewed in the initial section. The kinematic measures necessary to describe the material deformation and account for the remembered element configurations are then introduced in Section 2. A general theory capable of including material anisotropy is set forth in Section 3 and then quickly particularized in accordance with the specific qualitative features discussed previously. In the closing sections we demonstrate that, to first order in strains, the equations proposed by Hart result from considering the simplest theory incorporating energy dissipation in the non-elastic (viscous) and plastic components of the flow, the vanishing of stress with elastic strain, and a constant temperature minimum for free energy when the current and remembered configurations coincide.

1. STATE VARIABLE STRUCTURE

In this section we present a review and slight extension of the state variable structure proposed by Onat [3,4]. We consider a class of simple materials whose "state and orientation" at any instant of time t are locally determined by the current values of the mass density ρ, the absolute temperature θ, the spacial temperature gradient $\vec{g} = \vec{\nabla}\theta$, and a finite collection, $\{\tilde{q}_\alpha\}_{\alpha=1}^{N}$, of tensor state variables defined pointwise over the current configuration. The fixing of the state and orientation at time t, as defined by Onat, determines the response of the material to future stimulus in the sense that Cauchy stress $\underset{\sim}{\sigma}$, specific Helmholtz free energy per unit mass ψ, specific entropy η, and outward heat flux $\vec{h}$ at a future instant $(t+\varepsilon)$ are determined by the local deformation and temperature history in the intervening period $t \rightarrow (t+\varepsilon)$. This may be expressed in frame indifferent form through the operational equation

$$[\underset{\sim}{T}(t+\varepsilon),\ \psi(t+\varepsilon),\ \eta(t+\varepsilon),\ \vec{H}(t+\varepsilon)] = \Sigma[\rho(t),\ \theta(t),\ \vec{g}(t),\ \tilde{q}_\alpha(t);\ \underset{\sim}{C}_\varepsilon^t,\ \theta_\varepsilon^t,\ \vec{G}_\varepsilon^t];$$

$$\varepsilon > 0\ , \qquad (1.1)$$

where

$$\begin{aligned} \underset{\sim}{T} &= J\underset{\sim}{F}^{-1}\underset{\sim}{\sigma}(\underset{\sim}{F}^{-1})^T \quad ; \qquad J \equiv \det(\underset{\sim}{F}) \\ \underset{\sim}{C} &= \underset{\sim}{F}^T\underset{\sim}{F} \\ \vec{G} &= \underset{\sim}{F}^T\vec{g} \\ \vec{H} &= J\underset{\sim}{F}^{-1}\vec{h}\ , \end{aligned} \qquad (1.2)^*$$

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* These quantities represent, respectively, the symmetric Piola-Kirchhoff stress tensor, the right Cauchy Green tensor and the referential temperature gradient and heat flux vector referred to the element configuration at time t. In terms of these quantities the Clausius Duhem inequality takes the form

$$-\rho(t)\dot{\eta}\dot{\theta} - \rho(t)\dot{\psi} + \frac{1}{2}\,\underset{\sim}{T}\cdot\dot{\underset{\sim}{C}} + \frac{1}{\theta}\,\bar{H}\cdot\bar{G} \geq 0 \quad .$$

and $\underset{\sim}{F}$ represents the non-singular deformation gradient measured from the element configuration at time t (i.e. $\underset{\sim}{F}(t) = \underset{\sim}{I}$). The notation

$$f_\varepsilon^t \equiv \{f(\tau): \quad t < \tau \leq t+\varepsilon\} \tag{1.3}$$

is used to indicate dependence of the operator Σ on the strain and temperature history over the intervening period $t \rightarrow (t+\varepsilon)$. It should be observed that if the material under consideration responds continuously to continuous stimuli then the current values of the response variables are determined by the current values of the state and orientation variables, i.e.,

$$(\underset{\sim}{\sigma},\psi,\eta,\vec{h}) = R(\rho,\theta,\vec{g},\tilde{q}_\alpha) \quad . \tag{1.4}$$

In order to see this consider the future stimulus

$$\begin{aligned} \underset{\sim}{C}(\tau) &= \underset{\sim}{I} \\ \theta(\tau) &= \theta(t) \qquad\qquad t < \tau \leq t+\varepsilon \quad , \\ \vec{G}(\tau) &= \vec{G}(t) = \vec{g}(t) \end{aligned} \tag{1.5}$$

which extends the overall history continuously through t. For this particular stimulus (1.1) reduces to

$$[\underset{\sim}{\sigma},\psi,\eta,\vec{h}](t+\varepsilon) = \Sigma[\rho,\theta,\vec{g},\tilde{q}_\alpha;\ \underset{\sim}{C}_\varepsilon^t,\theta_\varepsilon^t,\vec{G}_\varepsilon^t] \equiv \mathcal{R}(\rho,\theta,\vec{g},\tilde{q}_\alpha,\varepsilon) \quad . \tag{1.6}$$

From continuity it then follows that

$$[\underset{\sim}{\sigma},\psi,\eta,\vec{h}](t) = \lim_{\varepsilon\rightarrow 0} \mathcal{R}(\rho,\theta,\vec{g},\tilde{q}_\alpha,\varepsilon) \equiv R(\rho,\theta,\vec{g},\tilde{q}_\alpha) \quad . \tag{1.7}$$

If, however, smooth stimulus is required to elicit continuous response, as is the case with the Kelvin-Voigt model for viscoelasticity, we might then consider the smooth history extension

$$\begin{aligned} \underset{\sim}{C}(\tau) &= \underset{\sim}{I} + \dot{\underset{\sim}{C}}(t)\tau = \underset{\sim}{I} + 2\underset{\sim}{D}(t)\tau \\ \theta(\tau) &= \theta(t) + \dot{\theta}(t)\tau \qquad\qquad t < \tau \leq t+\varepsilon \, , \qquad (1.8) \\ \vec{G}(\tau) &= \vec{G}(t) + \dot{\vec{G}}(t)\tau = \vec{g}(t) + (\frac{\mathcal{D}}{\mathcal{D}t}\vec{g} - \underset{\sim}{D}\vec{g})\tau \, , \end{aligned}$$

where $\underset{\sim}{D}$ represents the deformation rate tensor and $\frac{\mathcal{D}}{\mathcal{D}t}$ represents the co-rotational or Jaumann derivative. By once again exploiting continuity it follows that

$$\begin{aligned} [\underset{\sim}{\sigma},\psi,\eta,\vec{h}](t) &= \lim_{\varepsilon\to 0} \Sigma[\rho,\theta,\vec{g},\tilde{q}_\alpha;\ \underset{\sim}{C}^t_\varepsilon,\theta^t_\varepsilon,\vec{G}^t_\varepsilon] \\ &= \lim_{\varepsilon\to 0} \mathcal{R}(\rho,\theta,\vec{g},\tilde{q}_\alpha,\underset{\sim}{D},\dot{\theta},\frac{\mathcal{D}}{\mathcal{D}t}\vec{g},\varepsilon) \qquad (1.9) \\ &\equiv R(\rho,\theta,\vec{g},\tilde{q}_\alpha,\underset{\sim}{D},\dot{\theta},\frac{\mathcal{D}}{\mathcal{D}t}\vec{g}) \end{aligned}$$

For materials of this type we thus observe that it will generally be necessary to specify, in addition, the instantaneous deformation rate and the temperature and temperature gradient rates in order to fix the response variables.

Having introduced the state and orientation variables $\tilde{q}_\alpha$ as tensor fields defined over the current element configuration we now investigate their sensitivity to simple element rotation. Such a rotation (Fig. 2) will, of course, alter only the spacial orientation of the element and not its state. The state and orientation variables $(\rho,\theta,\vec{g},\tilde{q}_\alpha)$, however, undergo a change as a result of this rotation which is expressed symbolically as

$$(\rho,\theta,\vec{g},\tilde{q}_\alpha) \overset{Q}{\underset{\sim}{\to}} (\rho,\theta,\underset{\sim}{Q}\vec{g},P_Q\tilde{q}_\alpha) \quad , \tag{1.10}$$

where Q represents the orthogonal rotation matrix. Now, since a simple prerotation of the rotated element by $\underset{\sim}{Q}^T$ will restore its original state and orientation it follows that the unrotated and rotated elements will respond identically to the respective stimuli $(\underset{\sim}{F}^t_\varepsilon,\theta^t_\varepsilon,\vec{g}^t_\varepsilon)$ and $(\underset{\sim}{F}^t_\varepsilon\underset{\sim}{Q}^T,\theta^t_\varepsilon,\vec{g}^t_\varepsilon)$ for arbitrary specifications of $\underset{\sim}{F}$, θ and $\vec{g}$. That is, we should observe the same Cauchy stress, free energy, entropy and spacial heat flux vector for each during the ensuing motion. In terms of the operator (1.1) this condition requires that

$$[\underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T,\psi,\eta,\underset{\sim}{Q}\vec{H}](t+\varepsilon) = \Sigma[\rho,\theta,\underset{\sim}{Q}\vec{g},P_Q\tilde{q}_\alpha;\ \underset{\sim}{Q}\underset{\sim}{C}^t_\varepsilon\underset{\sim}{Q}^T,\theta^t_\varepsilon,\underset{\sim}{Q}\vec{G}^t_\varepsilon] \tag{1.11}$$

for any future stimulus whenever and provided that

$$[\underset{\sim}{T},\psi,\eta,\vec{H}](t+\varepsilon) = \Sigma[\rho,\theta,\vec{g},\tilde{q}_\alpha;\ \underset{\sim}{C}^t_\varepsilon,\theta^t_\varepsilon,\vec{G}^t_\varepsilon] \quad . \tag{1.12}$$

Since all equations must be tensor equations, i.e., valid irrespective of coordinate system, (1.11) becomes

$$[\underset{\sim}{T},\psi,\eta,\vec{H}](t+\varepsilon) = \Sigma[\rho,\theta,\vec{g},\Gamma_Q(P_Q\tilde{q}_\alpha);\ \underset{\sim}{C}^t_\varepsilon,\theta^t_\varepsilon,\vec{G}^t_\varepsilon] \tag{1.13}$$

under the coordinate transformation

$$y_i = Q_{ij}x_j \quad , \tag{1.14}$$

where

$$\Gamma_Q(a_{k_1\ldots k_n}) = Q_{j_1k_1}\cdots Q_{j_nk_n}a_{j_1\ldots j_n} \quad . \tag{1.15}$$

By comparison of (1.12) and (1.13) we then observe that

$$\Gamma_Q(P_Q\tilde{q}_\alpha) = \tilde{q}_\alpha , \tag{1.16}$$

provided that there exists a unique state and orientation vector $(\rho, \theta, \vec{g}, \tilde{q}_\alpha)$ for each state and orientation. By exploiting the group property of the transformation operator it then follows that

$$P_Q\tilde{q}_\alpha = \Gamma_{Q^T}\tilde{q}_\alpha . \tag{1.17}$$

It is thus demonstrated that, by subjecting a material element to a simple rigid rotation carrying a particle from spacial position x_i to $y_i = Q_{ij}x_j$, the state and orientation fields $\tilde{q}_\alpha$ change to $P_Q\tilde{q}_\alpha$, where

$$P_Q(q_{k_1 \cdot\cdot k_n}) = Q_{k_1 j_1} \cdots Q_{k_n j_n} q_{j_1 \cdot\cdot j_n} . \tag{1.18}$$

Another necessary feature of this general theory is the existence of an additional operator which determines the updated state variable fields at $(t+\varepsilon)$ from the state variable fields at t and the deformation and temperature history in the intervening period, i.e.,

$$\begin{aligned} \rho(t+\varepsilon) &= \rho(t)/J(t+\varepsilon) \\ \theta(t+\varepsilon) &= \theta(t+\varepsilon) \\ \vec{g}(t+\varepsilon) &= (\underset{\sim}{F}^{-1})^T \vec{G}(t+\varepsilon) \\ \tilde{q}_\alpha(t+\varepsilon) &= \Pi[\rho, \theta, \vec{g}, \tilde{q}_\alpha, \underset{\sim}{F}(t+\varepsilon);\ \underset{\sim}{C}^t_\varepsilon, \theta^t_\varepsilon, \vec{G}^t_\varepsilon] . \end{aligned} \tag{1.19}$$

The additional dependence on the deformation gradient is required, as witnessed in equations $(1.19)_1$, and $(1.19)_3$, in order to relate the state variables defined over the configuration at time t to their updated counterparts defined over the new configuration at $(t+\varepsilon)$. This dependence is, of course, subject to the requirements of material frame indifference which demands that a post rotation of the material element at $(t+\varepsilon)$ rotate the state variable fields in the sense that

$$[P_Q\tilde{q}_\alpha](t+\varepsilon) \;=\; \Pi[\rho,\theta,\vec{g},\tilde{q}_\alpha,Q\underset{\sim}{F}(t+\varepsilon);\; \underset{\sim}{C}^t_\varepsilon,\theta^t_\varepsilon,\vec{G}^t_\varepsilon] \quad . \tag{1.20}$$

Exploiting this, notice that if we take $Q = \underset{\sim}{R}^T(t+\varepsilon)$ where $\underset{\sim}{R}$ represents the orthogonal part of the deformation gradient obtained through the polar decomposition $\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U}$, then (1.20) reduces to

$$\begin{aligned}[P_{R^T}\tilde{q}_\alpha](t+\varepsilon) = [\Gamma_R\tilde{q}_\alpha](t+\varepsilon) &= \Pi[\rho,\theta,\vec{g},\tilde{q}_\alpha,\underset{\sim}{U}(t+\varepsilon);\; \underset{\sim}{C}^t_\varepsilon,\theta^t_\varepsilon,G^t_\varepsilon] \\ &\equiv \Lambda[\rho,\theta,\vec{g},\tilde{q}_\alpha;\; \underset{\sim}{C}^t_\varepsilon,\theta^t_\varepsilon,\vec{G}^t_\varepsilon] \quad ,\end{aligned} \tag{1.21}$$

where the dependence on the right stretch tensor in the second equality is absorbed through the expression

$$\underset{\sim}{U} \;=\; \sqrt{\underset{\sim}{C}} \quad . \tag{1.22}$$

If Λ is then assumed to be continuous and Fréchet differentiable in each of its function arguments* it can be shown that, at time t,

$$\overline{\Gamma_R\tilde{q}_\alpha}^{\,\cdot} \;=\; \tilde{\gamma}_\alpha + \tilde{\kappa}_{\alpha ij}D_{ij} + \tilde{\mu}_\alpha\dot{\theta} + \tilde{\nu}_{\alpha k}\dot{G}_k \quad , \tag{1.23}$$

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* This result depends on Λ being a continuous, Fréchet differentiable operator mapping the Cartesian product space of bounded, continuous stimuli $\{\underset{\sim}{C}^t_\delta,\theta^t_\delta,\vec{G}^t_\delta\}$, for some $\delta > 0$, into the Cartesian product space of bounded continuous state variable response $\{(\tilde{q}_\alpha)^t_\delta\}$. Both the domain and range spaces are considered to be endowed with the product topology obtained from the supermum norm

$$||\tilde{f}^t_\delta||_o \;=\; \sup_{t\,<\,\tau\,<\,(t+\delta)} |\sqrt{\tilde{f}(\tau)\cdot\tilde{f}(\tau)}| \quad .$$

In addition it is required that Λ map the subspace of continuously differentiable stimuli into the subspace of continuously differentiable state variable response.

where the various coefficients are tensor functions of the state variables $(\rho,\theta,\vec{g},\tilde{q}_\alpha)$. By making use of the definition of the co-rotational derivative, $\frac{\mathcal{D}}{\mathcal{D}t}$, and the expression $(1.2)_3$ for the referential temperature gradient, (1.23) reduces to

$$\frac{\mathcal{D}}{\mathcal{D}t}\tilde{q}_\alpha = \tilde{\gamma}_\alpha + \tilde{\lambda}_{\alpha ij}D_{ij} + \tilde{\mu}_\alpha\dot{\theta} + \tilde{\nu}_{\alpha k}\frac{\mathcal{D}}{\mathcal{D}t}g_k \quad , \tag{1.24}$$

where

$$\tilde{\lambda}_{\alpha ij} = \tilde{\kappa}_{\alpha ij} - \tilde{\nu}_{\alpha i}g_j \quad . \tag{1.25}$$

We note that the linearity in the stimulus rate variables is lost if the operator Λ is assumed to be weakly (Gateaux) rather than strongly (Fréchet) differentiable. Also, higher order rates may appear depending on the precise nature of the range and domain spaces of the differentiable operator†

† For instance, if Λ were a continuous, differentiable operator from the space of bounded, continuously differentiable (C^1) stimuli, endowed with the higher order supremum norm

$$||\tilde{f}^t_\delta||_1 = ||\tilde{f}^t_\delta||_0 + \sup_{t<\tau<(t+\delta)} |\sqrt{\dot{\tilde{f}}(\tau)\cdot\dot{\tilde{f}}(\tau)}| \quad ,$$

into the C^1 subspace of bounded continuous state variable response, then second order derivatives of the stimulus variables would appear.

2. KINEMATICS

We assume that the continuing flow or deformation is described in terms of a velocity field $\vec{v}$ defined pointwise over the spacial domain currently occupied by the body. As is customary, the second order tensor $\underset{\sim}{L}$ represents the spacial gradient of the velocity field, $\underset{\sim}{D}$, it symmetric part or rate of deformation, and $\underset{\sim}{\omega}$, its antisymmetric part or rate of rotation. We also recall the definition of the material derivative

$$\dot{\tilde{b}} = \frac{\partial}{\partial t}\tilde{b} + \vec{v}\cdot\vec{\nabla}\tilde{b} \tag{2.1}$$

representing the time rate of change as preceived by an observer translating with a material particle, and the co-rotational derivative

$$\frac{\mathcal{D}}{\mathcal{D}t} b_{k_1..k_n} = \dot{b}_{k_1..k_n} + b_{rk_2..k_n}\omega_{rk_1} + \ldots + b_{k_1..k_{n-1}r}\omega_{rk_n}, \tag{2.2}$$

which represents the time rate of change as seen by an observer participating in both the translational and rotational part of the material flow.

The remembered element configurations are now taken into account by introducing additional kinematic measures in the following way: Looking at the deformed material element in its current configuration we consider the material infinitesimals or directors associated with the orthonormal triad of Cartesian coordinate basis vectors $(\hat{e}_1,\hat{e}_2,\hat{e}_3)$. Now, if the material under consideration exhibits initial elastic response, instantaneous recovery would accompany the sudden removal of supporting load. As a consequence, the material directors associated with $(\hat{e}_1,\hat{e}_2,\hat{e}_3)$ would snap to a new spacial configuration, say $(\vec{f}_1,\vec{f}_2,\vec{f}_3)$. The geometry of the recovered element relative to the observed configuration, irrespective of orientation, would then be fixed by the positive definite, symmetric non-elastic deformation tensor

$$c^n_{ij} \equiv \vec{f}_{(i)} \cdot \vec{f}_{(j)} \quad . \tag{2.3}$$

A convenient measure for this instantaneously recoverable deformation, known as the elastic strain, is the elastic strain tensor defined by

$$2\varepsilon^e_{ij} \equiv \hat{e}_{(i)} \cdot \hat{e}_{(j)} - \vec{f}_{(i)} \cdot \vec{f}_{(j)} = \delta_{ij} - c^n_{ij} \quad . \tag{2.4}$$

It is also assumed that, at each instant, the material element has memory of a rest configuration whose directional characteristics are known and unchanging. To take this into account we suppose that the material directors in the current configuration associated with the set $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ correspond to the triad $(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ in this preferred rest configuration. This final set of vectors clearly fixes the deformation and orientation of the material element relative to its preferred rest configuration. It should also be clear that if this rest configuration is isotropic then only the geometry of this triad, determined by the plastic deformation tensor

$$c^p_{ij} = \vec{r}_{(i)} \cdot \vec{r}_{(j)} \quad , \tag{2.5}$$

and not its orientation, is significant. As mentioned previously this is the case that we shall pursue. In terms of the deformation tensors $\underset{\sim}{c}^n$ and $\underset{\sim}{c}^p$ the anelastic strain tensor, which is clearly a measure of the recoverable strain which remains after sudden removal to load, is defined as

$$2\varepsilon^a_{ij} = \vec{f}_{(i)} \cdot \vec{f}_{(j)} - \vec{r}_{(i)} \cdot \vec{r}_{(j)} = c^n_{ij} - c^p_{ij} \quad . \tag{2.6}$$

We now derive the constraints which guarantee that dilatation is a purely elastic effect. In order to insure that this is the case we shall require that the respective mass densities associated with the two remembered element configurations be identical and constant in time. This assumption incorporates both the incompressibility of plastic flow as well as the instantaneous recovery of dilatation with removal of load.

In this derivation we first observe that, at any instant, both the current mass density ρ and the rest or initial mass density ρ_o are known, the former being given while the latter is obtained through integration of the continuity equation

$$\dot{\rho} + \rho D_{kk} = 0 \quad . \tag{2.7}$$

By hypothesis ρ_o will be the mass density associated with both remembered configurations so that

$$\frac{\rho}{\rho_o} = \frac{dV^n}{dV} = \frac{dV^p}{dV} = \frac{\vec{f}_1 \cdot \vec{f}_2 \times \vec{f}_3}{\hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3} = \frac{\vec{r}_1 \cdot \vec{r}_2 \times \vec{r}_3}{\hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3} \quad . \tag{2.8}$$

By making use of the identity

$$\vec{a}_1 \cdot \vec{a}_2 \times \vec{a}_3 = \sqrt{\det\{\vec{a}_{(i)} \cdot \vec{a}_{(j)}\}} \tag{2.9}$$

the desired condition

$$\rho/\rho_o = \sqrt{\det(\underset{\sim}{c}^n)} = \sqrt{\det(\underset{\sim}{c}^p)} \tag{2.10}$$

is obtained. The differential form of this results from the substitution of (2.10) into (2.7). Recalling Cramer's Rule and the fact that ρ_o is constant we obtain

$$\left.\begin{aligned} \underset{\sim}{b}^n \cdot \dot{\underset{\sim}{c}}^n &= \underset{\sim}{b}^n \cdot \frac{\mathcal{D}}{\mathcal{D}t} \underset{\sim}{c}^n \\ \underset{\sim}{b}^p \cdot \dot{\underset{\sim}{c}}^p &= \underset{\sim}{b}^p \cdot \frac{\mathcal{D}}{\mathcal{D}t} \underset{\sim}{c}^p \end{aligned}\right\} = -2\mathrm{tr}(\underset{\sim}{D}) \quad , \tag{2.11}$$

where

$$\underset{\sim}{b}^n = [\underset{\sim}{c}^n]^{-1} \quad ; \quad \underset{\sim}{b}^p = [\underset{\sim}{c}^p]^{-1} \quad . \tag{2.12}$$

It is easily seen that (2.11) is equivalent to

$$\underset{\sim}{b}^n \cdot \frac{\delta}{\delta t} \underset{\sim}{c}^n = \underset{\sim}{b}^p \cdot \frac{\delta}{\delta t} \underset{\sim}{c}^p = 0 \quad , \tag{2.13}$$

where

$$\frac{\delta}{\delta t} \underset{\sim}{c} = \frac{\mathcal{D}}{\mathcal{D}t} \underset{\sim}{c} + \underset{\sim}{c}\underset{\sim}{D} + \underset{\sim}{D}\underset{\sim}{c} \tag{2.14}$$

represents the so called convected derivative of Oldroyd [5].

As a final kinematic preliminary we derive the necessary and sufficient conditions for vanishing of non-elastic and plastic flow. Let $\underset{\sim}{A}^n$ be the linear operator which maps the vector $\vec{\chi}$ associated with a material infinitesimal or director in the current configuration to its corresponding vector representation $\vec{\xi}$ in the unstressed configuration, i.e.,

$$\underset{\sim}{A}^n \cdot \vec{\chi} = \vec{\xi} \quad . \tag{2.15}$$

Note that this operator has the property that

$$\underset{\sim}{A}^n \cdot \hat{e}_{(i)} = \vec{f}_{(i)} \quad ; \qquad i = 1,2,3 \quad , \tag{2.16}$$

and hence it is seen that

$$\xi_i = A^n_{ij} \chi_j \quad , \tag{2.17}$$

where

$$\vec{\chi} = \chi_i \hat{e}_{(i)} \quad , \qquad \vec{\xi} = \xi_i \hat{e}_{(i)} \tag{2.18}$$

and

$$A^n_{ij} = \hat{e}_{(i)} \cdot \vec{f}_{(j)} \quad . \tag{2.19}$$

Now, the unstressed element configuration is said to be stationary if and only if each deforming material director has a time invariant representation in the unstressed configuration. Stated more precisely, the unstressed element configuration is said to be stationary if and only if every time varying vector $\vec{\chi}$ associated with a deforming material director, $\dot{\chi}_i = L_{ij}\chi_j$, is mapped by $\underset{\sim}{A}^n$ into a constant vector, $\dot{\xi}_i = 0$. Upon differentiation of (2.17) we see that this condition is equivalent to

$$\dot{\vec{f}}_{(i)} = -L_{ij}\vec{f}_{(j)} \quad ; \qquad i = 1,2,3 \quad . \tag{2.20}$$

Similarly, the rest configuration is said to be stationary if and only if

$$\dot{\vec{r}}_{(i)} = -L_{ij}\vec{r}_{(j)} \quad ; \qquad i = 1,2,3 \quad . \tag{2.21}$$

The corresponding conditions

$$\frac{\delta}{\delta t}\,\underset{\sim}{c}^{n} = 0$$

$$\frac{\delta}{\delta t}\,\underset{\sim}{c}^{p} = 0 \qquad (2.22)$$

on the non-elastic and plastic deformation tensors are then obtained by differentiating (2.3) and (2.5).

In general it is easily established that if $\vec{x}_1$ and $\vec{x}_2$ are the time varying vector representations of two deforming material directors and $\vec{\xi}_1$ and $\vec{\xi}_2$, $\vec{\zeta}_1$ and $\vec{\zeta}_2$ are their respective images in the unstressed and rest configurations, then

$$\overline{\vec{x}_1\cdot\vec{x}_2}^{\,\bullet} = \vec{x}_1\cdot 2\underset{\sim}{D}\cdot\vec{x}_2$$

$$\overline{\vec{\xi}_1\cdot\vec{\xi}_2}^{\,\bullet} = \vec{x}_1\cdot 2\underset{\sim}{D}^{n}\cdot\vec{x}_2 \qquad (2.23)$$

$$\overline{\vec{\zeta}_1\cdot\vec{\zeta}_2}^{\,\bullet} = \vec{x}_1\cdot 2\underset{\sim}{D}^{p}\cdot\vec{x}_2 \;,$$

where

$$2\underset{\sim}{D} = \frac{\delta}{\delta t}\,\underset{\sim}{I} = \underset{\sim}{L} + \underset{\sim}{L}^{T}$$

$$2\underset{\sim}{D}^{n} \equiv \frac{\delta}{\delta t}\,\underset{\sim}{c}^{n} \qquad (2.24)$$

$$2\underset{\sim}{D}^{p} \equiv \frac{\delta}{\delta t}\,\underset{\sim}{c}^{p} \;.$$

The quantities $\underset{\sim}{D}^{n}$ and $\underset{\sim}{D}^{p}$ are thus denoted as the non-elastic and plastic deformation rate tensors. We note also, after taking the convected derivative of equations (2.4) and (2.6), that

$$\underset{\sim}{D} = \frac{\delta}{\delta t}\, \underset{\sim}{\varepsilon}^{e} + \underset{\sim}{D}^{n}$$

and (2.25)

$$\underset{\sim}{D}^{n} = \frac{\delta}{\delta t}\, \underset{\sim}{\varepsilon}^{a} + \underset{\sim}{D}^{p} \quad .$$

These expressions replace the analogous kinematic expressions in the theory of Hart.

3. GENERAL THEORY

The general theory for materials of this type is based on the assumption that the local state and orientation is determined by the set $\{\theta,\vec{g},q_\alpha,\underset{\sim}{c}^n,\vec{r}_{(i)}\}$, where $\{q_\alpha\}_{\alpha=1}^N$ represents some set of scalar state variables and the dependence on ρ has been absorbed through (2.10). We recall from Section 1 that this implies the existence of operators Σ and Π of the form

$$
\begin{aligned}
[\underset{\sim}{T},\psi,\eta,\vec{H}](t+\varepsilon) &= \Sigma[\theta,\vec{g},q_\alpha,\underset{\sim}{c}^n,\vec{r}_{(i)};\ \underset{\sim}{C}^t_\varepsilon,\theta^t_\varepsilon,\vec{G}^t_\varepsilon] \\
[q_\alpha,\underset{\sim}{c}^n,\vec{r}_{(i)}](t+\varepsilon) &= \Pi[\theta,\vec{g},q_\alpha,\underset{\sim}{c}^n,\vec{r}_{(i)},\ \underset{\sim}{F}(t+\varepsilon);\ C^t_\varepsilon,\theta^t_\varepsilon,\vec{G}^t_\varepsilon]
\end{aligned}
\tag{3.1}
$$

for the response and updated state and orientation variables. As noted in Section 2, the dependence on the orientation vectors $\vec{r}_{(i)}$ is replaced by the plastic deformation tensor, $c^p_{ij} = \vec{r}_{(i)}\cdot\vec{r}_{(j)}$, in the isotropic theory. In addition, if viscoelastic properties are assumed to be unaltered by continued plastic flow then the relative distortions between the remembered configurations, together with the set $(\theta,\vec{g})$, should suffice to fix the response functional $(3.1)_1$. As a consequence, the scalar state variables $\{q_\alpha\}$ are excluded as arguments of Σ. With these simplifications, together with the additional assumptions that the material responds continuously to continuous stimuli and that Π is continuous and Fréchet differentiable, the operators in (3.1) reduce to the set of equations

$$
\begin{aligned}
(\underset{\sim}{\sigma},\psi,\eta,\vec{h}) &= R(\theta,\vec{g},\underset{\sim}{c}^n,\underset{\sim}{c}^p) \\
\dot{q}_\alpha &= \gamma_\alpha + \lambda_{\alpha ij}D_{ij} + \mu_\alpha\dot{\theta} + \nu_{\alpha k}\frac{\mathcal{D}}{\mathcal{D}t}g_k \\
\frac{\mathcal{D}}{\mathcal{D}t}\underset{\sim}{c}^n &= \underset{\sim}{\gamma}^n + \underset{\sim}{\lambda}^n_{ij}D_{ij} + \underset{\sim}{\mu}^n\dot{\theta} + \underset{\sim}{\nu}^n_k\frac{\mathcal{D}}{\mathcal{D}t}g_k \\
\frac{\mathcal{D}}{\mathcal{D}t}\underset{\sim}{c}^p &= \underset{\sim}{\gamma}^p + \underset{\sim}{\lambda}^p_{ij}D_{ij} + \underset{\sim}{\mu}^p\dot{\theta} + \underset{\sim}{\nu}^p_k\frac{\mathcal{D}}{\mathcal{D}t}g_k \quad ,
\end{aligned}
\tag{3.2}
$$

where the tensor coefficients in the rate equations are functions of the full set of state variables $(\theta, \vec{g}, q_\alpha, \underset{\sim}{c}^n, \underset{\sim}{c}^p)$.

If we now demand the non-negativity of the internal entropy production,

$$-\rho\eta\dot{\theta} - \rho\dot{\psi} + \underset{\sim}{\sigma}\cdot\underset{\sim}{D} + \theta^{-1}\vec{g}\cdot\vec{h} \geq 0 \quad , \tag{3.3}$$

for all conceivable stimuli, then it is necessary to require that the inequality

$$\begin{aligned} -\rho[\eta + \frac{\partial\psi}{\partial\theta} + \frac{\partial\psi}{\partial\underset{\sim}{c}^n}\cdot\underset{\sim}{\mu}^n + \frac{\partial\psi}{\partial\underset{\sim}{c}^p}\cdot\underset{\sim}{\mu}^p]\dot{\theta} + [\sigma_{ij} - \rho(\frac{\partial\psi}{\partial\underset{\sim}{c}^n}\cdot\underset{\sim}{\lambda}^n_{ij} + \frac{\partial\psi}{\partial\underset{\sim}{c}^p}\cdot\underset{\sim}{\lambda}^p_{ij})]D_{ij} \\ -\rho[\frac{\partial\psi}{\partial g_k} + \frac{\partial\psi}{\partial\underset{\sim}{c}^n}\cdot\underset{\sim}{\nu}^n_k + \frac{\partial\psi}{\partial\underset{\sim}{c}^p}\cdot\underset{\sim}{\nu}^p_k]\frac{\mathcal{D}}{\mathcal{D}t}g_k \\ -\rho[\frac{\partial\psi}{\partial\underset{\sim}{c}^n}\cdot\underset{\sim}{\gamma}^n + \frac{\partial\psi}{\partial\underset{\sim}{c}^p}\cdot\underset{\sim}{\gamma}^p] + \theta^{-1}\vec{g}\cdot\vec{h} \geq 0 \end{aligned} \tag{3.4}$$

hold for arbitrary specification of $\dot{\theta}$, $\underset{\sim}{D}$ and $\frac{\mathcal{D}}{\mathcal{D}t}\vec{g}$. By standard arguments it is thus seen that the constitutive expressions (3.2) must be specified so as to satisfy the constraints

$$\begin{aligned} \eta &= -[\frac{\partial\psi}{\partial\theta} + \underset{\sim}{\tau}^n\cdot\underset{\sim}{\mu}^n + \underset{\sim}{\tau}^p\cdot\underset{\sim}{\mu}^p] \\ \sigma_{ij} &= \rho[\underset{\sim}{\tau}^n\cdot\underset{\sim}{\lambda}^n_{ij} + \underset{\sim}{\tau}^p\cdot\underset{\sim}{\lambda}^p_{ij}] \\ \frac{\partial\psi}{\partial g_k} + \underset{\sim}{\tau}^n\cdot\underset{\sim}{\nu}^n_k + \underset{\sim}{\tau}^p\cdot\underset{\sim}{\nu}^p_k &= 0 \\ \rho\theta[\underset{\sim}{\tau}^n\cdot\underset{\sim}{\gamma}^n + \underset{\sim}{\tau}^p\cdot\underset{\sim}{\gamma}^p] - \vec{g}\cdot\vec{h} &\leq 0 \quad , \end{aligned} \tag{3.5}$$

in which we have introduced the thermodynamic tensions

$$\begin{aligned} \tilde{\tau}^n &\equiv \frac{\partial\psi}{\partial \tilde{c}^n} \\ \tilde{\tau}^p &\equiv \frac{\partial\psi}{\partial \tilde{c}^p} \quad . \end{aligned} \tag{3.6}$$

The incompressibility conditions (2.11) impose the further constraints

$$\left.\begin{aligned} \tilde{b}^n \cdot [\tilde{\gamma}^n + \tilde{\lambda}^n_{ij} D_{ij} + \tilde{\mu}^n \dot{\theta} + \tilde{\nu}^n_k \frac{\mathcal{D}}{\mathcal{D}t} g_k] \\ \tilde{b}^p \cdot [\tilde{\gamma}^p + \tilde{\lambda}^p_{ij} D_{ij} + \tilde{\mu}^p \dot{\theta} + \tilde{\nu}^p_k \frac{\mathcal{D}}{\mathcal{D}t} g_k] \end{aligned}\right\} = -2\mathrm{tr}(\tilde{D}) \quad . \tag{3.7}$$

In order that these conditions hold for arbitrary specification of $\tilde{D}$, θ, $\frac{\mathcal{D}\vec{g}}{\mathcal{D}t}$ it is necessary and sufficient to require that

$$\begin{aligned} \tilde{b}^n \cdot \tilde{\gamma}^n &= \tilde{b}^p \cdot \tilde{\gamma}^p = 0 \\ \tilde{b}^n \cdot \tilde{\mu}^n &= \tilde{b}^p \cdot \tilde{\mu}^p = 0 \\ \tilde{b}^n \cdot \tilde{\nu}^n_k &= \tilde{b}^p \cdot \tilde{\nu}^p_k = 0 \\ \tilde{b}^n \cdot \tilde{\lambda}^n_{ij} D_{ij} &= \tilde{b}^p \cdot \tilde{\lambda}^p_{ij} D_{ij} = -2\mathrm{tr}(\tilde{D}) \quad . \end{aligned} \tag{3.8}$$

4. SIMPLEST THEORY

We now consider a special case of this general theory which incorporates both plastic and non-elastic (viscous) flow as strictly dissipative processes. In order to include this feature we shall demand that the plastic and non-elastic dissipation inequalities

$$\begin{aligned} &\underset{\sim}{\tau}^n \cdot \underset{\sim}{\dot{\gamma}}^n \leq 0 \\ &\underset{\sim}{\tau}^p \cdot \underset{\sim}{\gamma}^p \leq 0 \end{aligned} \tag{4.1}$$

hold independently - the respective equalities being valid whenever and provided that the stationary flow conditions $(2.22)_1$ or $(2.22)_2$ are met. After writing the rate equation $(3.2)_3$ in the alternate form

$$2D^n_{ij} = \frac{\delta}{\delta t} c^n_{ij} = \gamma^n_{ij} + [\lambda^n_{ijmn} + c^n_{im}\delta_{jn} + c^n_{jm}\delta_{in}]D_{mn} + \mu^n_{ij}\dot{\theta} + \nu^n_{ijk}\frac{\mathcal{D}}{\mathcal{D}t}g_k , \tag{4.2}$$

it is easily seen that the equalities

$$\begin{aligned} \underset{\sim}{\gamma}^n &= \underset{\sim}{\mu}^n = \underset{\sim}{\nu}^n_k = 0 \\ \lambda^n_{ijmn} &= -[c^n_{im}\delta_{jn} + c^n_{jm}\delta_{in}] \end{aligned} \tag{4.3}$$

must hold for any specification of state variables for which $\underset{\sim}{\tau}^n \cdot \underset{\sim}{\gamma}^n = 0$. If, on the other hand, the state variables are specified so that $\underset{\sim}{\tau}^n \cdot \underset{\sim}{\gamma}^n < 0$ then it suffices to require, since the stimulus rates may still be specified arbitrarily and independently, that the symmetric second order tensor $\underset{\sim}{\gamma}^n$ lie outside of the subspace spanned by the ten symmetric tensors $\{(\underset{\sim}{\lambda}^n_{mn} + \underset{\sim}{\beta}^n_{mn}), \underset{\sim}{\mu}^n, \underset{\sim}{\nu}^n_k\}$, where $\beta^n_{ijmn} = c^n_{im}\delta_{jn} + c^n_{jm}\delta_{in}$. Indentical conclusions are reached for the coefficients $\underset{\sim}{\gamma}^p, \underset{\sim}{\lambda}^p_{ij}, \underset{\sim}{\mu}^p, \underset{\sim}{\nu}^p_k$ appearing in the plastic rate equation $(3.2)_4$.

The simplest way to guarantee that these conditions, as well as the incompressibility requirements (3.8), are met is to choose the coefficients $\underset{\sim}{\mu}^n$, $\underset{\sim}{\mu}^p$, $\underset{\sim}{\nu}_k^n$, $\underset{\sim}{\nu}_k^p$ to be zero, and

$$\begin{aligned} \lambda^n_{ijmn} &= -[c^n_{im}\delta_{jn} + c^n_{jm}\delta_{in}] \\ \lambda^p_{ijmn} &= -[c^p_{im}\delta_{jn} + c^p_{jm}\delta_{in}] \end{aligned} \tag{4.4}$$

identically. For this choice we obtain the much simplified set of constitutive equations

$$\begin{aligned} \psi &= \hat{\psi}(\theta, \underset{\sim}{c}^p, \underset{\sim}{c}^n) \\ \eta &= -\frac{\partial\hat{\psi}}{\partial\theta} \\ \sigma_{ij} &= -2\rho[\tau^n_{r(i}c^n_{j)r} + \tau^p_{r(i}c^p_{j)r}] \\ \vec{h} &= \vec{h}(\theta, \vec{g}, \underset{\sim}{c}^n, \underset{\sim}{c}^p) \\ \dot{q}_\alpha &= \gamma_\alpha + \lambda_{\alpha ij}D_{ij} + \mu_\alpha\dot{\theta} + \nu_{\alpha k}\frac{\mathcal{D}}{\mathcal{D}t}g_k \\ 2\underset{\sim}{D}^n &= \underset{\sim}{\gamma}^n \\ 2\underset{\sim}{D}^p &= \underset{\sim}{\gamma}^p \ , \end{aligned} \tag{4.5*}$$

where the various unspecified coefficients are functions of the full set of state variables $(\theta, \vec{g}, q_\alpha, \underset{\sim}{c}^n, \underset{\sim}{c}^p)$. The specification of these functions must still be consistent with the dissipation constraints (4.1) and $(3.5)_4$, and the incompressibility constraints $(3.8)_1$.

A particularly simple choice for the functions $\underset{\sim}{\gamma}^n$ and $\underset{\sim}{\gamma}^p$, consistent with these requirements, is

---
*Standard notation for the symmetric part is used in the third equation.

$$\gamma^n_{ij} = -k_1[c^n_{im}c^n_{jn} - \frac{1}{3}c^n_{ij}c^n_{mn}]\tau^n_{mn}$$

$$\gamma^p_{ij} = -k_2[c^p_{im}c^p_{jn} - \frac{1}{3}c^p_{ij}c^p_{mn}]\tau^p_{mn} , \tag{4.6}$$

where k_1 and k_2 are strictly positive functions of the full set of state variables. With this choice we identify the thermodynamic tensions (3.6) as the driving forces for plastic and non-elastic flow.

A complete theory is then set with the identification of the scalar state variables q_α, the fixing of their corresponding rate laws $(4.5)_5$, and the choice of the strictly positive scalar functions k_1 and k_2 and the free energy function $(4.5)_1$. We shall also demand, in keeping with the qualitative features discussed earlier, that the free energy function be chosen so that a fixed temperature minimum is attained at zero strain ($\underset{\sim}{\varepsilon}^e = \underset{\sim}{\varepsilon}^a = 0$) and that Cauchy stress, determined through $(4.5)_3$, vanish with vanishing elastic strain.

5. SMALL STRAIN LIMIT

In the small strain limit the deformation tensors $\underset{\sim}{c}^n$ and $\underset{\sim}{c}^p$ may, by virtue of (2.4) and (2.6), be approximated by the unit tensor $\underset{\sim}{I}$. Thus, to lowest order in strain, the non-elastic and plastic deformation rate equations $(4.5)_6$ and $(4.5)_7$ with (4.6) reduce to

$$\begin{aligned} 2\underset{\sim}{D}^n &\approx -k_1 \underset{\sim}{\tau}'^n \\ 2\underset{\sim}{D}^p &\approx -k_2 \underset{\sim}{\tau}'^p \quad , \end{aligned} \tag{5.1}$$

where the prime denotes the deviatoric part. In addition, the Cauchy stress equation $(4.5)_3$ is approximated by

$$\underset{\sim}{\sigma} \approx -2\rho_o(\underset{\sim}{\tau}^n + \underset{\sim}{\tau}^p) \quad . \tag{5.2}$$

These approximate equations reduce to the familier forms

$$\begin{aligned} \underset{\sim}{\sigma} &\approx \underset{\sim}{\sigma}^n + \underset{\sim}{\sigma}^p \\ \underset{\sim}{D}^n &\approx \alpha_1 \underset{\sim}{\sigma}'^n \quad ; \quad \alpha_1 = k_1/4\rho_o \\ \underset{\sim}{D}^p &\approx \alpha_2 \underset{\sim}{\sigma}'^p \quad ; \quad \alpha_2 = k_2/4\rho_o \end{aligned} \tag{5.3}$$

with the introduction of the new stress components

$$\begin{aligned} \underset{\sim}{\sigma}^n &\equiv -2\rho_o \underset{\sim}{\tau}^n \\ \underset{\sim}{\sigma}^p &\equiv -2\rho_o \underset{\sim}{\tau}^p \quad . \end{aligned} \tag{5.4}$$

In order to expand the free energy function ψ we first introduce the change of variables

$$\tilde{\psi}(\theta,\underset{\sim}{\varepsilon}^{e},\underset{\sim}{\varepsilon}^{a}) \equiv \hat{\psi}[\theta,\underset{\sim}{I}-2\underset{\sim}{\varepsilon}^{e},\underset{\sim}{I}-2(\underset{\sim}{\varepsilon}^{e}+\underset{\sim}{\varepsilon}^{a})] \tag{5.5}$$

and note the identities

$$\begin{aligned}\frac{\partial\psi}{\partial\underset{\sim}{\varepsilon}^{e}} &= -2(\underset{\sim}{\tau}^{n}+\underset{\sim}{\tau}^{p}) \approx \underset{\sim}{\sigma}/\rho_{o}\\ \frac{\partial\psi}{\partial\underset{\sim}{\varepsilon}^{a}} &= -2\underset{\sim}{\tau}^{p} = \underset{\sim}{\sigma}^{p}/\rho_{o}\ .\end{aligned} \tag{5.6}$$

With this it is easily seen that an admissible free energy function is necessarily of the form

$$\psi = \psi_{o}+\frac{1}{2}K^{e}_{ijmn}\varepsilon^{e}_{ij}\varepsilon^{e}_{mn}+\frac{1}{2}K^{a}_{ijmn}\varepsilon^{a}_{ij}\varepsilon^{a}_{mn}+0(\varepsilon^{3})\ , \tag{5.7}$$

where the fourth-order tensors $\underset{\approx}{K}^{e}$ and $\underset{\approx}{K}^{a}$ are positive definite isotropic and all scalars are functions of the scalar state variables. The absence of terms linear in strain and the positive definiteness of the fourth-order coefficient tensors is a consequence of requiring that the zero strain configuration yield a fixed temperature minimum for free energy. The approximation $(5.6)_1$ and the requirement that Cauchy stress must vanish with vanishing elastic strain eliminates the possibility of a second-order cross term and the isotropy of the zero strain or rest configuration fixes the isotropy of the coefficient matrices.

After combining equations (5.6) and (5.7) we then deduce the approximate forms

$$\begin{aligned}\underset{\sim}{\sigma} &\approx \lambda\underset{\sim}{\varepsilon}^{e}+2\mu\underset{\sim}{I}\mathrm{tr}(\underset{\sim}{\varepsilon}^{e})\\ \underset{\sim}{\sigma}^{p} &\approx \lambda^{*}\underset{\sim}{\varepsilon}^{a}+2\mu^{*}\underset{\sim}{I}\mathrm{tr}(\underset{\sim}{\varepsilon}^{a})\end{aligned} \tag{5.8}$$

for any choice of energy function consistent with the stated requirements. A final simplification is achieved by noting that it is possible, as a result of the incompressibility condition (2.10), to establish that the trace of the anelastic strain tensor is itself a higher order term, i.e.,

$$\mathrm{tr}(\underset{\sim}{\varepsilon}^a) = O(\varepsilon^2) \quad . \tag{5.9}$$

Thus, after collecting results

$$\begin{aligned}
\underset{\sim}{\sigma} &= \underset{\sim}{\sigma}^n + \underset{\sim}{\sigma}^p \\
\underset{\sim}{D} &= \frac{\delta}{\delta t}\, \underset{\sim}{\varepsilon}^e + \underset{\sim}{D}^n \\
\underset{\sim}{D}^n &= \frac{\delta}{\delta t}\, \underset{\sim}{\varepsilon}^a + \underset{\sim}{D}^p \\
\underset{\sim}{\sigma} &= \lambda \underset{\sim}{\varepsilon}^e + 2\mu \underset{\sim}{I}\, \mathrm{tr}(\underset{\sim}{\varepsilon}^e) \\
\underset{\sim}{\sigma}^p &= \lambda^* \underset{\sim}{\varepsilon}^a \\
\underset{\sim}{D}^n &= \alpha_1 \underset{\sim}{\acute{\sigma}}^n \\
\underset{\sim}{D}^p &= \alpha_2 \underset{\sim}{\acute{\sigma}}^p \quad ,
\end{aligned} \tag{5.10}$$

we observe the exact analogy between the small strain limit of the simple finite strain theory set forth in the previous section and the equations of Hart.

6. CONCLUSION

A precise thermomechanical model based on the notion of "selective memory" has been proposed as a theoretical foundation for the inelastic state variable theory of Hart. We note, however, that the general technique, which is based largely on the state variable structure of Onat and the concept of remembered element configurations, has more general applicability. In particular, this technique may be used to generate three-dimensional, finite strain constitutive theories from any one-dimensional viscoelastic or viscoplastic model composed of basic elements. The remarks in Section 1 regarding the discontinuous response of the Kelvin-Voigt model to continuous stimuli become pertinent when the selected model does not exhibit initial elastic response.

It is also noteworthy that the small strain limit of the simple theory proposed in Section 4, although identical in form to the theory of Hart, is significant in its own right. We observe that the equations (5.10) as they stand are fully frame invariant and all fields are defined over the current element configuration and related to the observables $\underset{\sim}{\sigma}$ and $\underset{\sim}{D}$. Moreover, since small elastic and anelastic strain does not imply small deformation, nor are any restrictions imposed on the velocity and velocity gradient fields, these equations apply in certain extended regimes. For instance, it is conceivable that these equations could be used to accurately model certain large deformation metal forming processes.

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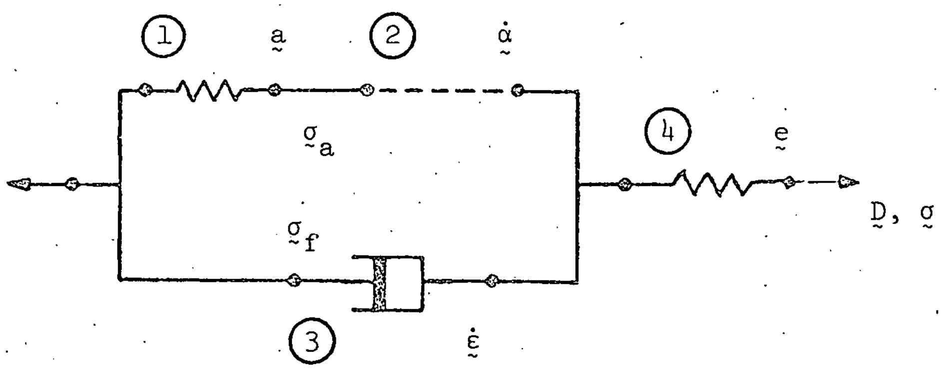

Fig. 1. Schematic model representation of Hart's constitutive relations.

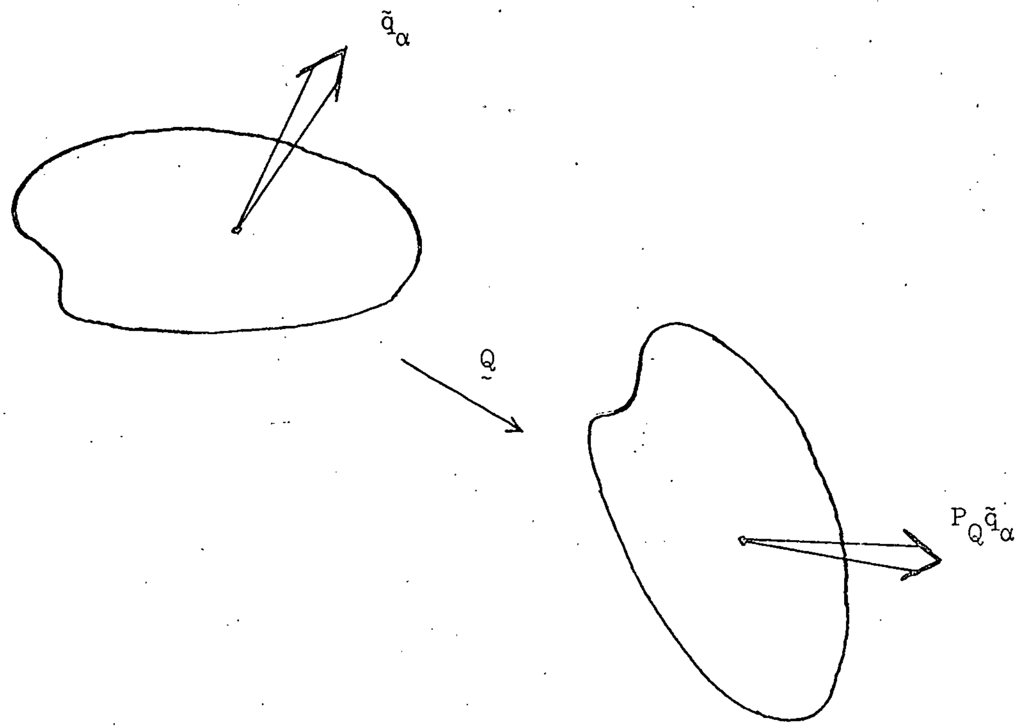

Fig. 2. Effect of rigid rotation on state variable fields.