ON FACTORS OF RANK ONE SUBSHIFTS

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Dissertation Prepared for the Degree of

DOCTOR OF PHILOSOPHY

UNIVERSITY OF NORTH TEXAS

May 2018

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Ziegler, Caleb. *On Factors of Rank One Subshifts*. Doctor of Philosophy (Mathematics), May 2018, 86 pp., 16 numbered references.

Rank one subshifts are dynamical systems generated by a regular combinatorial process based on sequences of positive integers called the cut and spacer parameters. Despite the simple process that generates them, rank one subshifts comprise a generic set and are the source of many counterexamples. As a result, measure theoretic rank one subshifts, called rank one transformations, have been extensively studied and investigations into rank one subshifts been the basis of much recent work. We will answer several open problems about rank one subshifts. We completely classify the maximal equicontinuous factor for rank one subshifts, so that this factor can be computed from the parameters. We use these methods to classify when large classes of rank one subshifts have mixing properties. Also, we completely classify the situation when a rank one subshift can be a factor of another rank one subshift.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Su Gao, for his mentorship and support in on this dissertation. His guidance and support were integral to this project and more generally, to my maturation as a mathematician. I cannot thank him enough for his investment in me and this project.

I would also like to thank Aaron Hill for many useful conversations on rank one subshifts. I would like to thank my committee members, Lior Fishman and Stephen Jackson, for working with me and making it possible for me to complete this project.

Finally, I would like to thank my family, friends, and fellow graduate students for their support through my graduate school career.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Rank one subshifts are the topological analogue of rank one transformations from the measure-theoretic context. These rank one transformations have been extensively studied since their introduction by Chacon [5] in 1965. These rank one transformations emerged in constructions of specific counterexamples, but the use and study of rank one transformations exploded as many open questions in ergodic theory and dynamical systems were answered with the use of rank one transformations. In 1997, the proliferation of results about rank one transformations prompted Ferenczi to write a survey [11] collecting many of the results involving rank one transformations. Some of the major questions surrounding rank one transformations include questions about what kind of factors rank one transformations can have and when rank one transformations have mixing properties.

Rank one transformations are those which are constructed via a sequence of cut and spacer parameters, which determine the transformation via a system of Rohklin towers. These constructions naturally lend themselves to interpretations within symbolic dynamics, which leads to the idea of rank one subshifts. Rank one subshifts are the topological analogues to rank one transformations, and were first studied by Gao and Hill[13], who determined when two systems are isomorphic based on the cut and spacer parameters. Many questions regarding rank one subshifts remain open, and of particular interest, are those questions which inspired research in the measure theoretic case, such as the factor problem and when such systems have mixing properties.

My research attempts to answer many of the main questions regarding rank one transformations in the context of rank one subshifts. Particularly, looking at the factor structure of rank one subshifts and attempting to categorise mixing properties.

It is expected that many of the results for rank one transformations would carry over to rank one subshifts. However, proofs in the measure theoretic case make use of the fact that the action on a measure zero set can be ignored. This is not the case for topological dynamical systems, so proving these facts for rank one subshifts requires different techniques than those for rank one transformations. In particular, I generalise many of the techniques introduced by Gao and Hill in [13] and combine these techniques with other standard theorems in topological dynamical systems, such as those from reference books [3] or [14].

In looking to address the factor structure, the maximal equicontinuous factor is an important invariant in dynamical systems and a logical place to start when studying factors of rank one subshifts. The maximal equicontinuous factor was guaranteed to exist by Ellis and Gottschalk[10] and this has become an important invariant for many dynamical systems. Determining the maximal equicontinuous factor for a class of systems has been an active area of research in topological dynamical systems, with papers like [16] determining it for Toeplitz shifts and [9] describing a large class of systems for which odometers are the maximal equicontinuous factor. However, the question of the maximal equicontinuous factor was open for rank one subshifts. One of the main theorems of this paper is the following classification of the maximal equicontinuous factor.

THEOREM 1.1. *Let (X, T) be a rank one subshift generated by the v_n. Let q_n be the cutting parameters and $a_{n,i}$ be the spacer paramters.*

If (X, T) has unbounded spacer parameter, then the maximal equicontinuous factor is trivial, the one point system.

If (X, T) has bounded spacer parameter, then the maximal equicontinuous factor is the largest finite factor of the system. In particular, this finite factor has size p, where p is the largest integer so that there is some n, so that for all $m \geq n$, and all $i \leq q_m$, $p|(|v_n| + a_{m,i})$.

Once the maximal equicontinuous factor is settled, we naturally move to the question of mixing properties. First, we consider weakly mixing. Questions about weakly mixing transformations date back to the earliest uses of rank one transformations, as the original intent of Chacon[5] was giving an example of a weakly mixing, but not mixing transformation. Constructing individual rank one transformations which are weakly mixing was an active area of research resulting in such papers as [2] and [4]. The question of when a rank one transformation is weakly mixing in the canonically bounded case was answered by Gao and

Hill in [12], which gives a characterisation of weakly mixing in terms of the canonical spacer parameters. My results answer the classification problem for the weakly mixing property for a large class of rank one subshifts. In particular, we get the following classification which relates to the result on the maximal equicontinuous factor.

THEOREM 1.2. *Let (X, T) be a rank one subshift generated by v_n which has bounded spacer parameter. Let q_n be the cutting parameters and $a_{n,i}$ be the spacer paramters. Then, the following are equivalent:*

(1) *(X, T) is weakly mixing.*

(2) *(X, T) has partition proximality for one point.*

(3) *(X, T) has trivial maximal equicontinuous factor.*

(4) *For all $p \geq 1$ and any $n \in \mathbb{N}$, so that there is some $m \geq n$, and some $i \leq q_m$, $p \not| (|v_n| + a_{m,i})$.*

Furthermore, I show several conditions for when a rank one subshift with unbounded spacer parameter is weakly mixing and give examples for why a classification is similar to that of Part 4 of Theorem 1.2 is unlikely.

The question of weakly mixing also leads into the question of when a topological dynamical system is mixing. For rank one transformations, avoiding the mixing property was part of the motivation for the original rank one transformation studied by Chacon in [5]. In [15], Ornstein constructed stochastic examples of mixing rank one transformations and Adams [1] showed explicit examples of mixing rank one transformations. The mixing problem was further studied by Creutz and Silva[6][7], culminating with a complete classification of when a rank one transformation can be mixing based on the spacer parameters.

My research attempts to analyse the mixing problem for rank one subshifts. Using the techniques developed in the analysis of the maximal equicontinuous factor, I completely address the mixing problem for rank one subshifts with bounded spacer parameter via the following theorem.

THEOREM 1.3. *Let (X, T) be a rank one subshift with bounded spacer parameter. Then*

(X, T) *is never mixing.*

Similarly, I address several conditions for which a rank one subshift with unbounded spacer parameter is mixing, and give examples which show that the mixing property is sensitive to small perturbations of the parameters and hence is unlikely to admit a simple computable classification.

The factor problem for rank one transformations was addressed by del Junco [8] when he showed that a factor of a rank one transformation must be rank one. Although it is not known whether a factor of a rank one subshift must be rank one, I investigate the case of when a rank one subshift has a factor which is also a rank one subshift. My results completely classify the case when a rank one subshift is a factor of another rank one subshift. This classification takes the form of the following dichotomy.

THEOREM 1.4. *Let* (X, T) *and* (Y, S) *be rank one subshifts. Suppose* (Y, S) *is a factor of* (X, T)*. Then either* Y *is finite or* (Y, S) *is isomorphic to* (X, T)*.*

This result shows that if the del Junco result transfers to the case of rank one subshifts, then the factor structure of rank one subshifts is completely determined.

The structure of this paper is as follows. We introduce rank one subshifts in Chapter 2 and standardise notation. We also state many of the previously known results from the literature. Some of these results are given proofs in our notation.

In chapter 3, we state several technical lemmas. These technical lemmas can be split into two types, those which are translations of dynamical properties into conditions on the neighbourhoods based on the rank one schema, and those which are results on specific strings which occur within the infinite rank one word V. These technical lemmas will be used to prove the main results for the paper.

In chapter 4, we prove results about the maximal equicontinuous factor of a rank one subshift. We do this by combining the two different types of technical lemmas to show exactly what properties we need the spacer parameters to satisfy in order to classify the maximal equicontinuous factor and then some additional combinatorial properties to show that the condition from Theorem 1.1 is equivalent to the condition on the spacer parameters.

This chapter culminates in the complete classification of the maximal equicontinuous factor of rank one subshifts in Theorem 1.1.

In chapter 5, we address the different mixing properties of rank one subshifts. In some cases, the mixing properties follow quickly from the technical lemmas in Chapter 3 together with the analysis of the maximal equicontinuous factor from Chapter 4. In this way, we completely classify the mixing properties for rank one subshifts with bounded spacer parameter in Theorems 1.2 and 1.3. In addition, we show several situations when mixing properties hold for rank one subshifts with unbounded spacer parameter and give several examples showing such conditions are as strict as possible and that show the sensitivity to parameters of these properties.

In chapter 6, we address the factor problem of what factors a rank one subshift can have. We include the standard result that a factor which is a shift must be a sliding block code. Considering the sliding block code, we can show several properties that the resulting factor must satisfy. Also, we combine these properties with the rank one structure, and use this to give the solution to the factor problem for rank one factors of a rank one subshift as stated in Theorem 1.4.

Finally, in chapter 7, we discuss some open problems and ideas for future research.

CHAPTER 2

DEFINITIONS

For this paper, we will be concerned with topological dynamical systems, in particular, rank one subshifts. We will typically denote a topological dynamical system as a pair (X, T), where X is the topological space and T is the action. We require that X is a compact metric space and that T acts continuously on X. We will primarily use capital letters near the end of the alphabet, e.g. X, Y to denote the topological spaces and capital letters like T, S to denote the action. For points from the topological space, we will primarily use lower case letters, such as x, y.

Definition 2.1. A *subshift* is a topological dynamical system where X is a closed subspace of $2^{\mathbb{Z}}$ where the topology on X is the subspace topology of the product topology on $2^{\mathbb{Z}}$, and T is the shift map. Since $X \subseteq 2^{\mathbb{Z}}$, each element of X will be a bi-infinite string of digits, so for an element $x \in X$, we will denote the digit in the nth position as $x(n)$. The shift map is then defined as $T(x)(n) = x(n+1)$ for all $x \in X$. For elements of $2^{\mathbb{Z}}$ and $l_1 < l_2 \in \mathbb{Z}$, we will often use the notation $x[l_1, l_2]$ to refer to the characters which occur at indices l_1 up to l_2 in x, i.e. $x[l_1, l_2] = x(l_1)x(l_1+1)...x(l_2)$. Note that $|x[l_1, l_2]| = l_2 - l_1 + 1$.

Note that we typically use lowercase letters from the middle of the alphabet, e.g. h through n to denote integer indices.

Note that since X is a closed subspace of $2^{\mathbb{Z}}$, X is compact and Hausdorff. Also, the shift map T is a homeomorphism of the space.

Next, we define the particular family of subshifts that we are interested in. Rank one subshifts are defined from a particular infinite word that is built up in a constructive way from predictable patterns based on finite words.

Definition 2.2. A *rank one schema* is a finite word v_0 (starting and ending with zero) along with positive integers q_n and $a_{n,i} \in \mathbb{N}$ for each $n \in \mathbb{N}$ and each $1 \leq i \leq q_n$, which defines v_n for each positive integer n by starting with v_0 and setting $v_{n+1} = v_n 1^{a_{n,1}} v_n 1^{n,2} v_n ... v_n 1^{a_{n,q_n}} v_n$.

We call the q_n the *cutting parameter* and the $a_{n,i}$ the *spacer parameter*.

We define the *infinite rank one word* V to be the limit of the v_n, i.e. for each $k \in \mathbb{N}$ if $|v_n| > k$, then $V(k) = v_n(k)$.

Without loss of generality, we can assume that $v_0 = 0$. If we want to mimic a system u_n where u_0 is not 0, we make $v_1 = u_0$ by putting in the necessary spacers between each occurrence of 0 and then setting $v_{n+1} = u_n$ for all $n > 0$.

Note that each v_{n+1} starts with an occurrence of v_n, so each v_{n+1} is an extension of v_n and the limit will be well-defined.

Also note that we have defined q_n in such a way that q_n+1 is the number of occurrences of v_n constructing v_{n+1}. So q_n is the number of spacers between these v_n.

Definition 2.3. A *rank one subshift* is a subshift (X, T) where X is the set of all bi-infinite strings $x \in 2^{\mathbb{Z}}$ where each finite string in x occurs as a finite string in the infinite rank one word V.

Note that this definition is equivalent to saying that $x \in X$ if for any finite subword α of x, there is an n so that α is a subword of v_n.

Note that since the topology on X comes from the product topology, the $U_{\beta,i} = \{x \in X : x(i)x(i+1)...x(i+|\beta|-1) = \beta\}$ form a base for the topology. The rank one definition says we only have to care about finite subwords from the v_n, i.e. we have that $\{U_{\alpha,i} : i \in \mathbb{Z}, \alpha \text{ is a finite subword of some } v_n\}$ is a base for the topology.

This definition coincides with the constructive symbolic definition of a measure-theoretic rank one transformation from [11], so it makes sense that this is an appropriate definition for studying the topological analogue.

Behaviour of the rank one subshift is determined by the spacer and cutting parameters which define the system. In particular, we immediately get very different behaviour based on whether the spacer parameter is eventually bounded or whether we can always find larger spacer parameters. We define the following dichotomy for rank one subshifts.

Definition 2.4. If there is a constant C, so that for all n, j, we have $a_{n,j} \leq C$, then we say the rank one subshift has *bounded spacer parameter*. Otherwise, we say the rank one subshift has *unbounded spacer parameter*.

We often shorten the terminology and refer to a bounded or unbounded rank one subshift.

From [13], we have the following facts about these systems.

Fact 2.5. *A rank one subshift with bounded spacer parameter is a minimal dynamical system.*

A rank one subshift with unbounded spacer parameter has exactly one fixed point $1^{\mathbb{Z}}$*. Every orbit which is not this fixed point is dense.*

It is also useful to note that finite systems are also rank one subshifts. Specifically, we get the following fact about finite systems from [13].

Fact 2.6. *Let* (X, T) *be a rank one subshift. Then* X *is finite iff the spacer parameter is eventually constant.*

This produces one instance of odometers as rank one subshifts. We will actually show that finite systems are the only possible odometers that occur as rank one subshifts.

Depending on the rank one schema, we may have occurrences of v_n within the infinite rank one word which are artefacts of the particular structure and do not come from the rank one structure.

Example 2.7. If $v_0 = 010$, $q_0 = 2$, $a_{0,1} = 0$, $a_{0,2} = 1$, then $v_1 = \underline{010}\,\underline{010}1\underline{010}$. The occurrences of v_0 that come from the rank one structure are underlined.

However, we also see another occurrence of v_0 with $v_1 = 01001\overline{010}10$, which happens to occur when the digits of the word and the spacer line up in the correct way.

We wish to differentiate between occurrences of v_n that occur in the two different ways above. The following notions come from [13].

Definition 2.8. We will call occurrences of v_n which come from the rank one schema *expected occurrences* of v_n and occurrences of v_n which come from fortuitous interactions between the v_n and spacers *unexpected occurrences.*

Note that in the previous example 2.7, the underlined occurrences were expected occurrences and the overlined occurrence was unexpected.

In [13], they proved that for any rank one subshift where X is infinite, expected occurrences are well-defined, i.e. for any $x \in X$ and $n \in \mathbb{N}$, there is a unique way to assign expected occurrences within x, so that every occurrence of 0 within x is contained in exactly one expected occurrence of v_n and none of the expected occurrences of v_n in x overlap.

Because of this, it makes sense to define the neighbourhoods from expected occurrences.

DEFINITION 2.9. We define $E_{v_n,k}$ to be the set of $x \in X$, so that x has an expected occurrence of v_n beginning at index k.

Note that by definition of the shift map, $T(E_{v_n,k}) = E_{v_n,k-1}$ for any n and k.

We have the following facts about the $E_{v_n,k}$ from [13].

FACT 2.10. (1) *For all $x \in X$, $l \in \mathbb{Z}$, if $x(l) = 0$, then for any $n \in \mathbb{N}$, there is some $k \in \mathbb{Z}$, so that $x \in E_{v_n,k}$ and $k \leq l \leq k + |v_n|$, i.e. every occurrence of 0 in x is part of some expected occurrence of v_n.*

(2) *Each $E_{v_n,k}$ is clopen.*

(3) *For any $n \in \mathbb{N}$, there is a constant t and finitely many strings α_j with each $|\alpha_j| \leq t\,|v_n|$, so that for any $x \in X$ and any $k \in \mathbb{Z}$, if $x[k, k + |v_n| - 1] = v_n$, we have that $x \in E_{v_n,k}$ iff $x[k, k + |\alpha_j| - 1] = \alpha_j$ for one of the finitely many α_j.*

Most importantly, the $E_{v_n,k}$ generate the topology of X as they form a subbasis for the bounded spacer parameter case and generate most of a subbasis for the unbounded spacer parameter case.

FACT 2.11. *For any rank one subshift (X, T) and any nonempty open set $U \subseteq X$, there is an $E_{v_n,k} \subseteq U$.*

For a rank one subshift with bounded spacer parameter, the $E_{v_n,k}$ form a subbasis for the topology.

For a rank one subshift with unbounded spacer parameter, the $E_{v_n,k}$ along with the $U_{1^n,k}$ form a subbasis for the topology.

Because the $E_{v_n,k}$ generate the topology, many of our results will come from looking at what we can say about the $E_{v_n,k}$ with regards to a factor map.

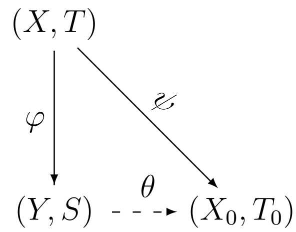
DEFINITION 2.12. Given a topological dynamical system (X, T), we say another topological dynamical system (Y, S) is a *factor* of (X, T) if there is a continuous, onto map $\varphi : X \to Y$, so that $\varphi \circ T = S \circ \varphi$. We call φ the *factor map*.

One particular collection of factors that we are interested is the collection of equicontinuous factors. Equicontinuous systems are systems which have somewhat predictable behaviour in that if two points are close enough together in an equicontinuous system, then they stay close together indefinitely.

DEFINITION 2.13. Let (X, T) be a topological dynamical system. We say (X, T) is *equicontinuous* if for any $\epsilon > 0$, there is some $\delta > 0$ so that for any $x_1, x_2 \in X$, if $d(x_1, x_2) < \delta$, then for any $l \in \mathbb{Z}$ $d(T^l(x_1), T^l(x_2)) < \delta$.

Combining the previous two definitions, an equicontinuous factor is a factor which is equicontinuous as a topological dynamical system. The following is a major invariant for topological dynamical systems.

DEFINITION 2.14. Let (X, T) be a topological dynamical system. We say that (Y, S) is the *maximal equicontinuous factor* of (X, T) if (Y, S) is a factor of (X, T), (Y, S) is equicontinuous, and for any (X_0, T_0) which is an equicontinuous factor of (X, T), (X_0, T_0) is also a factor of (Y, S). In particular, if φ is the factor map from (X, T) to (Y, S) and ψ is the factor map from (X, T) to (X_0, T_0), then there is a factor map θ from (Y, S) so (X_0, T_0) so that the following diagram commutes.



In [10], the maximal equicontinuous factor was always guaranteed to exist.

We will look at factors of rank one subshifts.

2.1. Basic Results Regarding Factors

For this section, the main method will be looking at the what we can say about the $E_{v_n,k}$ neighbourhoods under the factor structure.

The first result of note is on the finite factors of rank one subshifts.

Proposition 2.15. *An unbounded rank one subshift* (X, T) *always has only the trivial finite factor.*

Proof. Let (X, T) be an unbounded rank one subshift with finite factor witnessed by φ.

By Fact 2.5, $1^{\mathbb{Z}}$ is a fixed point within (X, T). Therefore, $\varphi(1^{\mathbb{Z}})$ is a fixed point in the finite factor. But the only finite factor with a fixed point is 1. □

Proposition 2.16. *A bounded rank one subshift* (X, T) *has* $\mathbb{Z}/p\mathbb{Z}$ *as a factor iff there is some* $n \in \mathbb{N}$, *such that for all* $m \geq n$, *and all* $1 \leq i \leq q_m$, $p \mid |v_n| + |a_{m,i}|$.

Proof. The reverse direction comes from defining a factor map according to the parity of distances.

For the forward direction, if (X, T) has a finite factor witnessed by φ, then $\{0\}$ is an open set in this factor. By Fact 2.11, we can find a n, k so that $E_{v_n,k} \subseteq \varphi^{-1}(\{0\})$.

Let $x \in E_{v_n,k}$. So $\varphi(x) = 0$. Consider any other $j \in \mathbb{Z}$ so that $x \in E_{v_n,j}$. By definition of the shift map, $T^{j-k}(x) \in E_{v_n,k}$. So $\varphi(T^{j-k}(x)) = 0$. But $\varphi(T^{j-k}(x)) = S^{j-k}(\varphi(x))$. Since S is the action in the finite factor, $S^{j-k}(\varphi(x)) = (j-k) + \varphi(x) \bmod p$. But $\varphi(T^{j-k}(x)) = 0 = \varphi(x)$, hence $j - k \equiv 0 \bmod p$. But this is true for any k so that $x \in E_{v_n,k}$. Furthermore, since (X,T) is a bounded rank one subshift, we can apply Fact 2.5 and we get that for any $x \in X$, there is some $h \in \mathbb{Z}$, so $T^h(x) \in E_{v_n,k}$. Therefore, for any $x \in X$, the distances between starting points for v_n are all divisible by p. Since v_n's are separated by spacers of the form $a_{m,i}$ where $m \geq n$, the proposition follows. □

PROPOSITION 2.17. *Let (X,T) be a rank one subshift with bounded spacer parameter. Then, there is a maximum p so that $(\mathbb{Z}/p\mathbb{Z}, x \mapsto x+1)$ can be a factor.*

PROOF. Note that by Proposition 2.16, we have that (X,T) has $\mathbb{Z}/p\mathbb{Z}$ as a factor iff there is some n so that $p|\,|v_n|+a_{m,i}$ for all $m \geq n$ and all $1 \leq i \leq q_m$.

Let C be the bound for the spacer parameter. Suppose toward a contradiction that there is no such largest p. Then there is some $p > C$ satisfying that $p|\,|v_n|+a_{m,i}$. But since (X,T) has bounded spacer parameter by C, $a_{m,i} \leq C < p$ for all m, i. But since each $a_{m,i} < p$, we have that $|v_n|+a_{m,i}$ can only take one value. But then $a_{m,i}$ must be eventually constant and so we get a finite system. But this is a contradiction since a finite system cannot have arbitrarily large factors. □

We note the following corollary to emphasise the distinction between topological rank one subshifts and measure theoretic rank one transformations. This shows that infinite odometers are not even factors of rank one subshifts.

COROLLARY 2.18. *Any non-trivial inverse limit $\mathbb{Z}/p_1\mathbb{Z} \to \mathbb{Z}/p_2\mathbb{Z} \to ...$ where $p_1|p_2|...$ cannot be a factor of a symbolic rank one subshift.*

In particular, infinite odometers are not rank one subshifts and are not even factors of rank one subshifts.

Note that in the measure theoretic context, such a nontrivial inverse limit actually is a rank one transformation, whereas in the setting of topological dynamical systems, such a system cannot even be a factor of a rank one subshift.

2.2. Special Points in Rank One Subshifts

We look at particular points that we will show are always contained in particular rank one subshifts. These points witness that the $E_{v_n,k}$ are not a basis and are useful as example points to check intuition and various orbit properties.

For this section, we will let V be the infinite rank one word, i.e. for any $i \in \mathbb{N}$, $V(i) = v_n(i)$ for any n where $|v_n| > i$. Similarly, we will denote V^* as the infinite word generated by the v_n as a limit in the other direction. So V^* has indices which are non-positive integers and for any $i \in \mathbb{N}$, $V^*(-i) = v_n(|v_n| - i - 1)$ for any n so that $|v_n| > i$.

2.2.1. Points for Unbounded Spacer Parameter

FACT 2.19. *Let (X, T) be a rank one subshift with unbounded spacer parameter. Let V be the rank one word, i.e. $V \in 2^{\mathbb{N}}$ with each v_n as an initial segment of V.*

Then the following words are contained in X.

- $1^{\mathbb{Z}}$
- $...111V$ *i.e. the word with an infinite sequence of ones to the left and V to the right*
- $V^*111...$ *i.e. the word with an infinite sequence of ones to the right and V to the left*

PROOF. By definition of X, a point x is contained in X iff every finite subword of x is a subword of V. Note that for $1^{\mathbb{Z}}$, every finite subword is of the form 1^l for some l and since X has unbounded spacer parameter, by finding where the spacer parameter is larger than l, we get that 1^l is a subword of V.

Now consider a finite subword of $...111V$. Note that if the subword is entirely contained in the V part, we are done, and if the subword is entirely contained in the infinite sequence of 1's, we are done by the same argument as above. So we can assume that the subword is of the form $1^l\alpha$, where α is an initial segment of V and $l \geq 0$. Since each v_n is an initial segment of V, we can find n large enough that α is an initial segment of v_n. In this case, α will be an initial segment of any v_m with $m \geq n$. Because of this, we can increase n so that for some $i \leq q_n$, $a_{n,i} \geq l$, which we can do since the spacer parameter is unbounded and so will always eventually increase as n gets large.

Then $1^l\alpha$ is a subword of v_{n+1}, since v_{n+1} has the form $v_n 1^{a_{n,1}} v_n 1^{a_{n,2}} v_n ... v_n 1^{a_{n,q_n}} v_n$ and 1^l is a subword of $1^{a_{n,i}}$ and α will be the initial segment of the immediately subsequent occurrence of v_n.

For a finite subword of $V^*111...$, we can assume that any finite word will be of the form $\alpha 1^l$ where α will be a final segment of V^* and $l \geq 0$. By the same argument as above, we can find an n so that α is a final segment of v_n and there is a spacer of length at least l at the level n. Then similarly to above, v_{n+1} has $\alpha 1^l$ as a subword. □

2.2.2. Other Points

Now we look at points that can occur in any rank one subshift with appropriate parameters. In particular, for this part, we do not specify whether the spacer parameter is bounded or unbounded unless specifically mentioned.

Fact 2.20. *Let a be the length of a spacer that occurs infinitely often in the levels of V, i.e. for all $N \in \mathbb{N}$, there is some $n \geq N$ and some i with $1 \leq i \leq q_n$ so that $a = a_{n,i}$.*

Then $V^*1^aV \in X$.

Proof. Let v_n have spacers of length a infinitely often. We claim that $x = V^*1^aV$ has every finite subword α a subword of some v_n, and hence $x \in X$. Without loss of generality, we can assume that α is of the form $\alpha_1 1^a \alpha_2$, where α_1 is a final segment of V^* and α_2 is an initial segment of V. We can make this assumption because if α is a subword of V or V^*, then the statement follows immediately, and if α is of the form $\alpha_3 1^d$ or $1^d \alpha_4$ for $d < a$, we can extend α to α' which includes all the 1^a and since α is a subword of α', we can just show that the α' is a subword of some v_n. But this α' is of the form $\alpha_1 1^a \alpha_2$, since we do not require both α_1 and α_2 to be non-empty.

Consider α a subword of x, with $\alpha = \alpha_1 1^a \alpha_2$. Since α_1 is a final segment of V^*, there is some m_1 so that α_1 is a final segment of v_{m_1}, and similarly, since α_2 is an initial segment of V, there is m_2 with α_2 an initial segment of v_{m_2}. Then, since a occurs as a spacer infinitely often, we can find some $n \geq \max\{m_1, m_2\}$, so that $v_{n+1} = v_n 1^{a_{n,1}} v_n 1^{a_{n,2}} v_n ... v_n 1^{a_{n,q_n}} v_n$ and $a_{n,i} = a$ for some $1 \leq i \leq q_n$. We claim α is a subword of v_{n+1}. Since $n \geq \max\{m_1, m_2\}$, we have that v_n is built from occurrences of v_{m_1} and is built from occurrences of v_{m_2} and so we have that α_1 will be a final segment of v_n and α_2 will be an initial segment of v_n. Then considering the spacer so that $a_{n,i} = a$, we see that the part of v_{n+1} which is of the form $v_n 1^{a_{n,i}} v_n$ will contain α, hence α is a substring of v_{n+1}. Therefore, $x \in X$ $\square$

This condition can hold regardless of whether the spacer parameter is bounded as long as there is some spacer which occurs infinitely often. However, we get the following corollary for systems with bounded spacer parameter.

Corollary 2.21. *Let (X, T) be an infinite system with bounded spacer parameter. Then we can find a point of the form $V^* 1^a V$ in X.*

In particular, there are at least two values $a, a' \in \mathbb{N}$ so that $V^* 1^a V, V^* 1^{a'} V \in X$.

Proof. By Fact 2.20, it is enough to show that at least two different spacers occur infinitely often. First, note that since the spacer parameter is bounded, there are only finitely many values that the spacer parameter can take. But there are infinitely many spacers, for every $n \in \mathbb{N}$, $q_n \geq 1$. Therefore, at least one spacer occurs infinitely often. This proves the first part.

Next, suppose by contradiction, that there is only one spacer a that occurs infinitely often. Then, there is some $N \in \mathbb{N}$ so that for all $n \geq N$, all the $a_{n,i} = a$, as each of the finitely many possible values which occur finitely often will be exhausted by some n. So we see that this system has eventually constant spacer parameter. But then by Fact 2.6, the system is finite, which contradicts that we assumed the system was infinite. Therefore, there must be at least two different values which occur infinitely often as spacers. $\square$

2.3. Bounds on Deciding Expectness

In this section, we will reprove and extend several results on deciding expectness from [13]. In particular, we will reprove and extend part 3 of Fact 2.10, as we will show that we do not require the v_n to occur at the beginning of the string to determine expectedness. The proofs in this section use the same techniques and methods as [13]. [1]

DEFINITION 2.22. Let V be an infinite rank one word. So V has the form $V = 01^{a_1}01^{a_2}01^{a_3}0...$ Let i_n for $n \in \mathbb{N}$ enumerate the positions of 0's within V. So $i_0 = 0$, $i_{n+1} = i_n + a_{n+1} + 1$. Then the function $L : \mathbb{N}\backslash\{0\} \to \mathbb{N}$ maps $L(n) = a_n$ or equivalently, $L(n) = i_n - i_{n-1} - 1$.

Since V is built up in a predictable way from the v_n, we can get predictable behaviour from the function L based on the v_n.

FACT 2.23. *Let r_n be the number of occurrences of* 0 *in some v_n. Then if $c \equiv c'$ mod r_n and $c \not\equiv 0$ mod r_n, then $L(c) = L(c')$, i.e. L is constant on each non-zero congruence class mod r_n.*

PROOF. V is of the form $v_n 1^{b_1} v_n 1^{b_2} v_n...$ for some b_i for $i \in \mathbb{N}$. Since V is a rank one word, v_n starts and ends with an occurrence of 0. Furthermore, by definition, each v_n contains exactly r_n occurrences of 0. Since we enumerated the 0's of V with $\mathbb{N}$ and started from 0, the last occurrence of 0 in each v_n will always be the $ir_n - 1$th occurrence of 0 in V for some $i \in \mathbb{N}$. Therefore, $L(ir_n)$ will always refer to the gap between the last occurrence of one v_n and the first occurrence of the next v_n. So if $c \not\equiv 0$ mod r_n, then $c \equiv k$ mod r_n means that $L(c)$ will be referring to the gap between the $k-1$th occurrence of 0 in v_n and the kth occurrence of v_n (where we start counting from 0). But each v_n is the same, so the gap between the $k-1$th occurrence of 0 in v_n and the kth occurrence of v_n is constant, so if $c \not\equiv 0$ mod r_n and $c \equiv c'$ mod r_n, then $L(c) = L(c')$, which proves the fact. □

Next, we show that there is a fundamental difference in the behaviour of L on the non-zero congruence classes mod r_n and its behaviour on the class congruent to 0 mod r_n.

FACT 2.24. *Let (X, T) be an infinite rank one subshift generated by V. Let r_n be the number of* 0*'s occurring in v_n. Then there is some constant t so that for any $h \in \mathbb{N}$, $L(hr_n), L((h+$*

[1]As a logical note, the proofs may invoke results about expectedness. However, in the original paper, the result from part 3 of Fact 2.10 was used as a way of showing expectedness is well-defined. Therefore, the proofs in this section should not be considered as new independent verifications of these facts, as that might necessitate concerns about logical circularity, but are included to illustrate the ideas necessary to prove the generalisation Corollary 2.25.

$1)r_n), ..., L((h+t-1)r_n)$ *are not all equal.*

PROOF. First, we will show that there is no value l so that for arbitrarily large t, we can find $h \in \mathbb{N}$ so that $l = L(hr_n) = L((h+1)r_n) = ... = L((h+t-1)r_n)$. Note that these gaps correspond to the spacers between consecutive expected occurrences of v_n in V. Suppose there is such an l. Then for arbitrarily large t, we can find the string $(v_n 1^l)^t$ in V. But then the periodic string $...v_n 1^l v_n 1^l v_n 1^l ... \in X$. But by Fact 2.5, we have that the only periodic point in an infinite rank one subshift is $1^{\mathbb{Z}}$, so we cannot have such an l.

Next, we show that if l is large enough, then for any $h \in \mathbb{N}$, at least one of $L(hr_n) \neq l$ or $L((h+1)r_n) \neq l$. Consider $v_{n+1} = v_n 1^{a_{n,1}} v_n 1^{a_{n,2}} ... v_n 1^{a_{n,q_n}} v_n$ and let $l > \max\{a_{n,1}, ..., a_{n,q_n}\}$. Since $V = v_{n+1} 1^{b_1} v_{n+1} 1^{b_2} v_{n+1} ...$ for some sequence of b_i with $i \in \mathbb{N}$ and each $v_{n+1} = v_n 1^{a_{n,1}} v_n 1^{a_{n,2}} ... v_n 1^{a_{n,q_n}} v_n$, we see that for any two consecutive gaps between occurrences of v_n, at least one will occur in an occurrence of v_{n+1}. But the $L(hr_n)$ correspond exactly to gaps between occurrences of v_n. So if $L(hr_n) = l$, then by definition, $L(hr_n)$ cannot correspond to a gap within an occurrence of v_{n+1}, so $L((h-1)r_n)$ and $L((h+1)n)$ must correspond to gaps within v_{n+1}. But then $L((h-1)r_n) \neq l$ and $L((h+1)r_n) \neq l$. So if $l > \max\{a_{n,1}, ..., a_{n,q_n}\}$, then at least one of $L(hr_n) \neq l$ or $L((h+1)r_n) \neq l$.

Finally, we can get the necessary t as follows. By the first part of the proof, for each l, we can find some minimal t_l so that for any h, $L(hr_n), L((h+1)r_n), ..., L((h+t_l-1)r_n)$ are not all equal to l. By the second part, we see that for all large enough l, $t_l = 2$. Therefore the set $\{t_l : l \in \mathbb{N}\}$ contains only finitely many values, since for all but finitely many $l \in \mathbb{N}$, $t_l = 2$. Therefore, we can let $t = \max\{t_l : l \in \mathbb{N}\}$ and since $t \geq t_l$ for all $l \in \mathbb{N}$, t works. □

We can use this to show the result from Part 3 of Fact 2.10, which says that we can determine whether an occurrence of v_n is expected by looking at a fixed length longer string following it.

RECALL. *(Part 3 of Fact 2.10) For any $n \in \mathbb{N}$, there is a constant t and set of finitely many strings J, so that for each $\alpha \in J$, $|\alpha| = t\,|v_n|$, and for any $x \in X$ and any $k \in \mathbb{Z}$, if $x[k, k+|v_n|-1] = v_n$, we have that $x \in E_{v_n,k}$ iff $x[l, l+|\alpha|-1] \in J$.*

PROOF. The idea behind the proof is that we will use Fact 2.23 and Fact 2.24 which say that L is non-constant only on the congruence class of 0 mod r_n and that this non-constant class corresponds to spacers between expected occurrences of v_n. So given an occurrence of v_n within x, we look at a long enough string so that Fact 2.24 guarantees we will witness that the corresponding L is non-constant on some class. This class will be the distinguished one corresponding to 0 mod r_n and we can determine the expectedness of the v_n.

Fix n. Let $x \in X$ have an occurrence of v_n starting at k, so $x[k, k + |v_n| - 1] = v_n$. Let r_n be the number of 0's in v_n. From Fact 2.24, we can find some $t_0 \in \mathbb{N}$ so that for any $h \in \mathbb{Z}$, $L(hr_n), L((h+1)r_n), ..., L((h+t_0-1)r_n)$ are not all equal. Let $t = r_n(t_0+1)$. Note that there are only finitely many substrings of V with length $t\,|v_n|$, so J will be a subset of such string and this will guarantee that J is finite.

Let α be a substring of V with length $t\,|v_n|$. We will define the function L_α by $L_\alpha(i)$ is the distance between the $i-1$th and the ith occurrence of 0 in α (counting from 0). So letting $\alpha = 01^{a_1}01^{a_2}...01^{a_s}$, then $L_\alpha(i) = a_i$. Note that since α is a substring of V, this definition means that there is some $M \in \mathbb{N}$ so that $L_\alpha(i) = L(M+i)$ for all i in the domain of L_α. Fact 2.23 says that L is constant on all congruence classes $i \not\equiv 0$ mod r_n, and since L_α agrees with L up to a shift M, we have that L_α will be constant on $r_n - 1$ congruence classes mod r_n.

Consider $\alpha = 01^{a_1}01^{a_2}...01^{a_s}$.

If there is some i with $1 \leq i \leq s$, so that $|v_n| \leq a_i$, then we cannot have 1^{a_i} is a spacer within v_n. Therefore, the 0 immediately after the occurrence of 1^{a_i} must be the start of an expected occurrence of v_n and every r_n 0's away will also be the start of an expected occurrence of v_n. Hence every 0 which corresponds to the start of an expected occurrence will be in the congruence class of i mod r_n. So in this case, we see that the occurrence of v_n at the start of α is expected exactly when this $i \equiv 0$ mod r_n.

Now suppose there is no i with $1 \leq i \leq s$ so that $|v_n| \leq a_i$. Then note that each $a_i < |v_n|$ so each string 01^{a_i} has length at most $|v_n|$, hence $|\alpha| \leq s\,|v_n|$. But $|\alpha| = t\,|v_n|$, so $t \leq s$. But $t = r_n(t_0+1)$, so $s \geq r_n(t_0+1)$. But note that s is the number of 0's in α. So

L_α is defined on at least t_0 consecutive members of each congruence class mod r_n.

Note that $L_\alpha(i) = L(M+i)$ for some $M \in \mathbb{N}$, so we can find some $c, d \in \mathbb{N}$, $0 \leq d < r_n$, so that $M = cr_n + d$. Letting $j = r_n - d$, we see that $j, j+r_n, j+2r_n, ..., j+t_0r_n$ are all at most $(t_0+1)r_n \leq s$, so L_α is defined on all of $j, j+r_n, j+2r_n, ..., j+t_0r_n$. But since $L_\alpha(i) = L(M+i)$ for all i, we see $L_\alpha(j) = L(M+j) = L(cr_n+d+j) = L((c+1)r_n)$. So letting $h = c+1$, $L_\alpha(j) = L(hr_n)$ and therefore, $L_\alpha(j), L_\alpha(j+r_n), ..., L_\alpha(j+t_0r_n)$ are $L(hr_n), L((h+1)r_n), ..., L((h+t_0-1)r_n)$. But t_0 was defined so that for any $h \in \mathbb{N}$, $L(hr_n), L((h+1)r_n), ..., L((h+t_0-1)r_n)$ are not all equal, so $L_\alpha(j), L_\alpha(j+r_n), ..., L_\alpha(j+t_0r_n)$ are not all equal. Therefore L_α is not constant on some congruence class mod r_n and this congruence class corresponds to the 0's at the start of expected v_n's. Therefore, the occurrence of v_n at the start of such an α will be expected iff L_α is not constant on the congruence class of 0.

Now we will show which strings to put into J. Let α be a substring of V and $|\alpha| = t\,|v_n|$. Either α contains an occurrence of $1^{|v_n|}$ or not. If it does, we can find i with $1 \leq i \leq s$ so that $a_i \geq |v_n|$, and we will put $\alpha \in J$ iff $i \equiv 0 \mod r_n$. Otherwise, if α does not contain an occurrence of $1^{|v_n|}$, we know that L_α is not constant on some congruence class j mod r, and we put $\alpha \in J$ iff $j \equiv 0 \mod r_n$. By the above analysis, we see that these cases are exactly when the occurrence of v_n at the start of α is expected. □

We can use the same proof to extend the previous result to any long enough string containing v_n rather than just a string starting with v_n.

Corollary 2.25. *For any $n \in \mathbb{N}$, there is a t, so that for any $x \in X$ if x contains an occurrence of v_n and α is a substring of x with length $t\,|v_n|$ which contains the occurrence of v_n, then we can determine whether or not the occurrence of v_n is expected by looking only at α.*

Specifically, if α is a substring of x with length $t\,|v_n|$ and α contains an occurrence of v_n starting at the kth character of α, we can find a finite set of strings J_k so that the occurrence of v_n starting at index k is expected (in x) iff $\alpha \in J_k$.

Proof. We will use the same methods as in the proof of the previous fact. Let n be fixed and let x contain an occurrence of v_n. Let t' be the constant from Part 3 of Fact 2.10. We will set $t = t' + 1$.

Let α be a substring of x with length $t\,|v_n|$ which contains the occurrence of v_n. Let k be the index within α where the v_n starts.

To show the existence of the necessary J_k, we will show how to determine whether the v_n within α is expected. The resulting J_k will be the set of all substrings of V with length $t\,|v_n|$ which satisfy the given properties for expectedness at index k.

Note that α will be of the form $1^{a_0}01^{a_1}01^{a_2}0...01^{a_s}$. Similarly to the previous proof, we define $L_\alpha(i) = a_i$ and we look at how L_α behaves on congruence classes mod r_n. Note that L_α will be constant on at least $r_n - 1$ many of the congruence classes mod r_n and if L_α is non-constant on some class represented by j, then we know that the occurrences of 0 immediately after the 1^{a_i} with $i \equiv j \bmod r_n$ are the starting points of expected occurrences of v_n. Since v_n starts with 0, we have that index k will be immediately after some 1^{a_i}.

Similarly to the previous part, either α contains an occurrence of $1^{|v_n|}$ or it does not. If α contains an occurrence of $1^{|v_n|}$, this corresponds to a spacer with length at least $|v_n|$ which is a spacer that cannot be contained in an expected occurrence of v_n. Let a_j be the spacer with length at least $|v_n|$. Then, similarly to above, we have that occurrences of 0's which are in the same congruence class mod r_n as j will be the start of expected occurrences of v_n. So we have that the v_n starting at index k will be expected iff $i \equiv j \bmod r_n$, where $a_j \geq 1^{|v_n|}$.

Otherwise, since $\alpha = 1^{a_0}01^{a_1}01^{a_2}0...01^{a_s}$ has length $t\,|v_n| = (t'+1)\,|v_n|$. we see that $01^{a_1}01^{a_2}0...01^{a_s}$ will have length at least $t'\,|v_n|$, so by the argument from the previous proof, we can find a congruence class j mod r_n so that L_α will be non-constant on that congruence class and expected occurrences of v_n will begin at the 0's from that congruence class. So the v_n starting at index k will be expected iff $i \equiv j \bmod r_n$ where j is a representative of the nonconstant congruence class mod r_n for L_α.

So we will define J_k as follows. Each string α in J_k will be a substring of V with length $t\,|v_n|$ with an occurrence of v_n starting at index k which is immediately after the 1^{a_i} for $0 \leq i \leq s-1$.

If such a string α has $a_j \geq |v_n|$, then we add it to J_k iff $j \equiv i \bmod r_n$. Otherwise, by length considerations, we know that α must have exactly one class $j \bmod r_n$ on which L_α is not constant, and we add α to J_k iff $j \equiv i \bmod r_n$. $\square$

CHAPTER 3

TECHNICAL LEMMAS

One of the main ways that we can tackle problems about rank one subshifts is by transferring dynamical properties into topological ones. Because the $E_{v_n,k}$ determine the topology of a rank one subshift, we can translate various dynamical properties for rank one subshifts into properties about the $E_{v_n,k}$. In particular, many topological properties can be translated into statements about intersections between various $E_{v_n,k}$'s, which can be further reduced to questions about particular words appearing in the rank one word V. This chapter will focus on these technical results which allow us to translate from abstract dynamical properties into statements about words in V.

3.1. Dynamical Properties as $E_{v_n,k}$ intersections

First, we translate dynamical properties into conditions on various $E_{v_n,k}$'s. We will start with the topological weakly mixing property.

Definition 3.1. A topological dynamical system (X, T) is *weakly mixing* if for any nonempty open sets U, V, W, Z, there is some $l \in \mathbb{Z}$ so that $T^l(U) \cap V \neq \emptyset$ and $T^l(W) \cap Z \neq \emptyset$.

Using Fact 2.11, we know that the $E_{v_n,k}$ generate the topology and we can use this to translate the statement about general open sets into statements about the $E_{v_n,k}$. Furthermore, because of the predictable behaviour of the shift map on the indices for the $E_{v_n,k}$, we can use this to get more freedom on the particular indices for the $E_{v_n,k}$.

Fact 3.2. *Let (X, T) be a rank one subshift. Then (X, T) is weakly mixing iff for any $n \in \mathbb{N}$ and any $k_1, k_2 \in \mathbb{Z}$, there exists some $l \in \mathbb{N}$, so that $E_{v_n,-l} \cap E_{v_n,k_1} \neq \emptyset$ and $E_{v_n,-l} \cap E_{v_n,k_2} \neq \emptyset$.*

Proof. For the backward direction, let U, V, W, Z be nonempty open. By Fact 2.11, there are $E_{v_{n_U},k_U} \subseteq U$, $E_{v_{n_V},k_V} \subseteq V$, $E_{v_{n_W},k_W} \subseteq W$, and $E_{v_{n_Z},k_Z} \subseteq Z$. Since each v_n is an initial segment of each v_m for $m \geq n$, we have for any $n \in \mathbb{N}, m \geq n$, and $k \in \mathbb{Z}$, $E_{v_m,k} \subseteq E_{v_n,k}$.

Therefore, taking n to be the maximum, we can replace n_U, n_V, n_W, and n_Z, with a fixed n and get that $E_{v_n,k_U} \subseteq U$, $E_{v_n,k_V} \subseteq V$, $E_{v_n,k_W} \subseteq W$, and $E_{v_n,k_Z} \subseteq Z$.

So it is enough to find some $l \in \mathbb{N}$ so that $T^l(E_{v_n,k_U}) \cap E_{v_n,k_V} \neq \emptyset$ and $T^l(E_{v_n,k_W}) \cap E_{v_n,k_Z} \neq \emptyset$. But note that $T^l(E_{v_n,k}) = E_{v_n,k-l}$ and that if $x \in E_{v_n,h_1} \cap E_{v_n,h_2}$, then for any $m \in \mathbb{Z}$, $T^m(x) \in E_{v_n,h_1-m} \cap E_{v_n,h_2-m}$. Hence, we can shift the neighbourhoods, and assume $k_U = k_W = 0$ and let $k_1 = k_V - k_U$ and $k_2 = k_W - k_Z$.

Then the statement becomes there is an $l \in \mathbb{N}$ so that $T^l(E_{v_n,0}) \cap E_{v_n,k_1} \neq \emptyset$ and $T^l(E_{v_n,0}) \cap E_{v_n,k_2} \neq \emptyset$. But $T^l(E_{v_n,0}) = E_{v_n,-l}$, so the statement becomes $E_{v_n,-l} \cap E_{v_n,k_1} \neq \emptyset$ and $E_{v_n,-l} \cap E_{v_n,k_2} \neq \emptyset$. But this holds by assumption. Hence, we have that $T^l(U) \cap V \neq \emptyset$ and $T^l(W) \cap Z \neq \emptyset$, and so the system is weakly mixing.

Now suppose that the forward assumption does not hold, i.e. there exists some $n \in \mathbb{N}$ and some k_1, k_2, so that for all $l \in \mathbb{N}$, either $E_{v_n,-l} \cap E_{v_n,k_1} = \emptyset$ or $E_{v_n,-l} \cap E_{v_n,k_2} = \emptyset$. But since $E_{v_n,-l} = T^l(E_{v_n,0})$, by letting $U = W = E_{v_n,0}$, $V = E_{v_n,k_1}$, and $Z = E_{v_n,k_2}$, we get a counterexample to weakly mixing. □

We can apply the same process for topological mixing.

Definition 3.3. Let (X, T) be a topological dynamical system. (X, T) is *mixing* for any non-empty open sets U, V, there is an $L \in \mathbb{N}$, so that for any $l \geq L$, $T^l(U) \cap V \neq \emptyset$.

Fact 3.4. *Let (X, T) be a rank one subshift. Then (X, T) is mixing iff for any n and any $k \in \mathbb{Z}$, we can find an $L \in \mathbb{N}$ so that for any $l \geq L$, $E_{v_n,-l} \cap E_{v_n,k} \neq \emptyset$.*

The proof of this follows the same method as the proof of Fact 3.2.

Proof. For the backward direction, let U, V be non-empty open sets. By Fact 2.11 and following the same process as in the previous proof to get a single n, there are $E_{v_n,k_U} \subseteq U$ and $E_{v_n,k_V} \subseteq V$. So it is enough to find an L, so that for any $l \geq L$, $T^l(E_{v_n,k_U}) \cap E_{v_n,k_V} \neq \emptyset$.

Similarly to the previous proof, if $x \in T^l(E_{v_n,k_U}) \cap E_{v_n,k_V}$, then $T^{k_U}(x) \in T^l(E_{v_n,0}) \cap E_{v_n,k_V-k_U}$. So we can assume $k_U = 0$ and will denote $k_V - k_U$ by k.

So it is enough to show that there is some L so that for any $l \geq L$, $T^l(E_{v_n,0}) \cap E_{v_n,k} \neq \emptyset$. $T^l(E_{v_n,0}) = E_{v_n,-l}$, so this follows by the assumption. Therefore, if for any n and any $k \in \mathbb{Z}$, we can find $L \in \mathbb{N}$ so that for any $l \in L$, $E_{v_n,-l} \cap E_{v_n,k} \neq \emptyset$, then (X,T) is mixing.

For the reverse direction, as in the previous proof, let n and k be so that for any $L \in \mathbb{N}$ we can find $l \geq L$, $E_{v_n,-l} \cap E_{v_n,k} = \emptyset$. Note that $E_{v_n,-l} = T^l(E_{v_n,0})$. But then letting $U = E_{v_n,0}$ and $V = E_{v_n,k}$, we get a counterexample to mixing. □

Next, we introduce a new dynamical property that allows classification of the maximal equicontinuous factor.

Definition 3.5. Let (X,T) be a topological dynamical system. We say that X has *partition proximality* if there are finitely many points $x_1, ..., x_p$ so that for any $x \in X$, there is some x_i, $1 \leq i \leq p$, so that for any $\delta > 0$, we can find $z_1, z_2 \in X$ and some $l, l_1, l_2 \in \mathbb{Z}$, with $d(T^{l_1}(x), T^{l_1}(z_1)) < \delta$, $d(T^{l_2}(x_i), T^{l_2}(z_2)) < \delta$, and $d(T^l(z_1), T^l(z_2)) < \delta$. We call the finitely many $x_1, ..., x_p$ *reference points.*

This property is of interest because of the following proposition.

Proposition 3.6. *Let (X,T) be a topological dynamical system with partition proximality and let (Y,S) be equicontinuous. Then if (Y,S) is a factor of (X,T), then Y is finite. In particular, $|Y| \leq p$ where p is the number of reference points.*

In the proof, we only use that the spaces X and Y are compact, and that the factor map is continuous, so this proposition is general and applies to topological dynamical systems which are not just rank one subshifts.

Proof. Let (X,T) have partition proximality witnessed by x_i for $1 \leq i \leq p$ and (Y,S) equicontinuous. Let (Y,S) be a factor of (X,T) witnessed by φ.

We claim that for all $x \in X$, there is some i with $1 \leq i \leq p$ so that $\varphi(x) = \varphi(x_i)$, in particular, the i from the partition proximality definition. Suppose otherwise. Then $d_Y(\varphi(x), \varphi(x_i)) = \epsilon$ for some $\epsilon > 0$.

Since (Y,S) is equicontinuous, there is some $\delta_y > 0$, so that for any $y_1, y_2 \in Y$ if $d_Y(y_1, y_2) < \delta_y$, then for all $l \in \mathbb{Z}$, $d_Y(S^l(y_1), S^l(y_2)) < \frac{\epsilon}{3}$.

Since X and Y are compact metric spaces, we have that φ will be uniformly continuous. That means there is some $\delta > 0$, so that for any $x_1, x_2 \in X$, if $d_X(x_1, x_2) < \delta$, then $d_Y(\varphi(x_1), \varphi(x_2)) < \delta_y$.

Since (X, T) has partition proximality, we can find $z_1, z_2 \in X$, $l, l_1, l_2 \in \mathbb{Z}$, so that $d_X(T^{l_1}(x), T^{l_1}(z_1)) < \delta$, $d_X(T^{l_2}(x_i), T^{l_2}(z_2)) < \delta$, and $d_X(T^l(z_1), T^l(z_2)) < \delta$. Using the triangle inequality, we have $\epsilon = d_Y(\varphi(x), \varphi(x_i)) \leq d_Y(\varphi(x), \varphi(z_1)) + d_Y(\varphi(z_1), \varphi(z_2)) + d_Y(\varphi(z_2), \varphi(x_i))$.

Since $d_X(T^l(z_1), T^l(z_2)) < \delta$, we have $d_Y(\varphi(T^l(z_1)), \varphi(T^l(z_2))) < \delta_y$, and so by equicontinuity, $d_Y(S^{-l}(\varphi(T^l(z_1))), S^{-l}(\varphi(T^l(z_2)))) = d_Y(\varphi(z_1), \varphi(z_2)) < \frac{\epsilon}{3}$. Repeating this argument for $d_X(T^{l_1}(x), T^{l_1}(z_1))$ and $d_X(T^{l_2}(x_i), T^{l_2}(z_2))$, we see $d_Y(\varphi(x), \varphi(z_1)) < \frac{\epsilon}{3}$ and $d_Y(\varphi(z_2), \varphi(x_i)) < \frac{\epsilon}{3}$.

But then continuing our string of inequalities, we have that $\epsilon < d_Y(\varphi(x), \varphi(z_1)) + d_Y(\varphi(z_1), \varphi(z_2)) + d_Y(\varphi(z_2), \varphi(x_i)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$, which is impossible.

Therefore, for all $x \in X$, there is some i with $1 \leq i \leq p$ so that $\varphi(x) = \varphi(x_i)$ and so we see that since φ is onto, Y must be finite. ☐

So if we can show that a topological dynamical system has partition proximality, then we know that it can only have a finite factor as a maximal equicontinuous factor.

Similarly to the mixing properties above, we can determine a characterisation of partition proximality for systems with bounded spacer parameter based on the $E_{v_n,k}$.

Lemma 3.7. *Let (X, T) be a rank one subshift with spacer parameter bounded by C. Fix $p \in \mathbb{N}$. (X, T) has partition proximality for p reference points iff for any large enough n and any $x \in X$, we can find some i where $1 \leq i \leq p$, and some $k_1, k_2, k_3 \in \mathbb{Z}$ so that $x \in E_{v_n,k_1}$, $x_i \in E_{v_n,k_2}$, and both $E_{v_n,k_1} \cap E_{v_n,k_3} \neq \emptyset$ and $E_{v_n,k_2} \cap E_{v_n,k_3} \neq \emptyset$.*

Proof. Suppose (X, T) has partition proximality for $x_1, ..., x_p$. Fix $x \in X$ arbitrary and n large enough. Let x_i be the point guaranteed by partition proximality for x. By part 3 of Fact 2.10, we can find some t and finitely many strings α_j with length $\leq t\,|v_n|$, so that for any point $y \in X$ and $l \in \mathbb{Z}$ which satisfies $y[l, l + |v_n| - 1] = v_n$, $y \in E_{v_n,k}$ iff there is some j, so that $y[l, l+|\alpha_j|-1] = \alpha_j$. Let $\delta > 0$ be small enough so that $d(y_1, y_2) < \delta$ implies that y_1, y_2 agree on a string of length at least $(t+1)|v_n| + C$. In particular, $\delta = 2^{-\lceil \frac{(t+1)|v_n|+C}{2} \rceil}$ would work.

Suppose $y_1, y_2 \in X$ agree on a string of length $(t+1)|v_n| + C = |v_n| + C + t|v_n|$ starting at index k. Then since $\{E_{v_n,k+h} : 0 \leq h \leq |v_n| + C\}$ are a cover of X, we have that $y_1 \in E_{v_n,k+h}$ for some $0 \leq h \leq |v_n| + C$. Since y_2 agrees with y_1, y_2 has an occurrence of v_n starting at index $k+h$. By part 3 of Fact 2.10, since $y_1 \in E_{v_n,k+h}$, there is some fixed α_j with $|\alpha_j| \leq t|v_n|$ so that $y_1[k+h, k+h+|\alpha_j|-1] = \alpha_j$. But $k+h+|\alpha_j|-1 \leq k+h+t|v_n|-1 \leq k+|v_n|+C+t|v_n|-1$, so y_2 contains the string α_j starting at index $k+h$, hence $y_2 \in E_{v_n,k+h}$. Therefore, we see that if any two points agree on a string of length $(t+1)|v_n| + C$, then there is some k so that both points are in $E_{v_n,k}$.

From the definition of partition proximality, we get $z_1, z_2 \in X$ and $l_1, l_2, l \in \mathbb{Z}$. Since $d(T^{l_1}(x), T^{l_1}(z_1)) < \delta$, $T^{l_1}(x), T^{l_1}(z_1)$ agree on a string of length $(t+1)|v_n| + C$, hence $T^{l_1}(x), T^{l_1}(z_1) \in E_{v_n,k_x}$ for some k_x. But since shifts work predictably with the $E_{v_n,k}$, we have $x, z_1 \in E_{v_n,k_x+l_1}$. Let $k_1 = k_x + l_1$, so $x, z_1 \in E_{v_n,k_1}$. By a similar argument, we have some k_2, k_3, so that $x_i, z_2 \in E_{v_n,k_2}$, and $z_1, z_2 \in E_{v_n,k_3}$. But this finishes that direction of the proof, since $z_1 \in E_{v_n,k_1} \cap E_{v_n,k_3}$ and $z_2 \in E_{v_n,k_2} \cap E_{v_n,k_3}$.

Now for the reverse direction. Let $x \in X$ be arbitrary and let $\delta > 0$.

Let D be a ball of radius $\frac{\delta}{3}$. By Fact 2.11, there is a $E_{v_m,k} \subseteq D$. Hence $\operatorname{diam}(E_{v_n,k}) < \delta$ for any $n \geq m$. Furthermore, for any k', there is an l' so that $\operatorname{diam}(T^{l'}(E_{v_n,k'})) < \delta$ by letting $l' = k' - k$. Let $n \geq m$ be large enough as required by the statement of the lemma.

By the assumption, there is a k_1 so that $x \in E_{v_n,k_1}$, k_2 with $x_i \in E_{v_n,k_2}$ and k_3 so that $E_{v_n,k_1} \cap E_{v_n,k_3} \neq \emptyset$ and $E_{v_n,k_2} \cap E_{v_n,k_3} \neq \emptyset$. Let $z_1 \in E_{v_n,k_1} \cap E_{v_n,k_3}$ and $z_2 \in E_{v_n,k_2} \cap E_{v_n,k_3}$. Since $x, z_1 \in E_{v_n,k_1}$, we have $T^{k_1-k}(x), T^{k_1-k}(z_1) \in E_{v_n,k}$, so letting $l_1 = k_1 - k$, we have $d(T^{l_1}(x), T^{l_1}(z_1)) < \delta$. Similarly, letting $l_2 = k_2 - k$ and $l = k_3 - k$, we get $d(T^{l_2}(x_i), T^{l_2}(z_2)) < \delta$ and $d(T^{l}(z_1), T^{l}(z_2)) < \delta$. Therefore, (X, T) has partition proximality witnessed by $x_1, ..., x_p$. □

3.2. Analysis of v_n-Blocks

As we saw in section 3.1, we can prove various dynamical properties about rank one subshifts by considering various intersections between different $E_{v_n,k}$'s. Because the $E_{v_n,k}$'s are determined by the starting positions, it is often useful to abstract away the notation needed for the spacer and refer to v_n-spacer pairs as a unit, as strings made from these "blocks" are exactly distances between starting points of different $E_{v_n,k}$'s.

Definition 3.8. Fix n. We define an *v_n-block* to be an expected occurrence of v_n along with the immediately subsequent maximal sequence of ones. We will refer to the sequence of ones as the *spacer*.

Note that if the v_n-block comes from an element $x \in X$, then the subsequent spacer can be determined by looking at a long enough string. If the spacer (as a substring of x) is contained in some expected v_m, then we can find a minimal one and determine exactly what $a_{m,i}$ the spacer comes from. If the spacer is not contained in any v_m, then we will assume $m = \infty$. Note that $m \geq n$, but not necessarily equal to n.

We say a finite word α *consists of v_n-blocks* if α is a subword of V and has the form $v_n 1^{a_1} v_n 1^{a_2} v_n ... v_n 1^{a_q}$ for some $q \in \mathbb{N}$, where each of the v_n's occurring in that string are expected occurrences of v_n and each of the 1^{a_i} are maximal. So the set of strings which consist of v_n-blocks are finite substrings of V which start with an expected occurrence of v_n and end the index immediately before another expected occurrence of v_n.

When α consists of v_n-blocks, we can find an occurrence of α within V, so that α starts with an expected occurrence of v_n in V and α ends immediately before an expected occurrence of v_n in V.

Note that when a finite word α consists of v_n-blocks, it can be uniquely split into consecutive v_n-blocks, and it contains the entirely of each component v_n-block.

Note that in the previous definition, we are only considering expected occurrences, so when we refer to an occurrence of the words that compose the rank one system, we are referring only to expected occurrences and may omit the "expected".

Because each α which consists of v_n-blocks exists as a substring within V, we can find some minimal m so that α occurs as a substring of an expected v_m within V. Within this v_m, each spacer canonically comes from some $a_{i,j}$ where $i \leq m-1$ and $1 \leq j \leq q_m$. We will often want to refer to this canonical value for the spacer.

Definition 3.9. Let α consist of v_n-blocks. As noted above, there is some minimal m so that α occurs as a substring of an expected v_m within V, which determines a canonical $a_{i,j}$ for each spacer. We will call this canonical $a_{i,j}$ the *value* of the spacer. We will refer to the i as the *level* of the spacer.

The primary motivation for considering words that consist of v_n-blocks is in the following lemma.

Lemma 3.10. *Let (X,T) be a rank one subshift generated by V. Fix n and let α consist of v_n-blocks.*

Let $k_1 \neq k_2 \in \mathbb{Z}$ be so that $|k_1 - k_2| = |\alpha|$. Then $E_{v_n,k_1} \cap E_{v_n,k_2} \neq \emptyset$.

In particular, for any $x \in X$ so that $x \neq 1^{\mathbb{Z}}$, we can find some $l \in \mathbb{Z}$ so that $T^l(x) \in E_{v_n,k_1} \cap E_{v_n,k_2}$.

Proof. Let α consist of v_n-blocks. By above, we can find some m so that α occurs as a substring of an expected occurrence of v_m with V so that each of the occurrences of v_n that witness that α consists of v_n-blocks are expected within V.

Let $x \in X$ with $x \neq 1^{\mathbb{Z}}$. Since $x \neq 1^{\mathbb{Z}}$, x contains an occurrence of 0. Since (X,T) is a rank one subshift, this occurrence of 0 is contained in an expected occurrence of v_m. By definition of m, we see that this x contains an occurrence of α so that the occurrences of v_n that witness that α consists of v_n-blocks are expected within x

Let $|k_1 - k_2| = |\alpha|$. Without loss of generality, assume $k_1 < k_2$. Then we shift x by some $l \in \mathbb{Z}$ so that $T^l(x)$ has the occurrence of α starting at index k_1. Since α consists of v_n-blocks and the occurrences of v_n witnessing this are all expected, we have $T^l(x) \in E_{v_n,k_1}$, as α starts with an expected v_n. Similarly, since α ends immediately before an expected v_n, $T^l(x) \in E_{v_n,k_1+|\alpha|}$. But $|\alpha| = k_2 - k_1$, so $T^l(x) \in E_{v_n,k_2}$. Therefore, $T^l(x) \in E_{v_n,k_1} \cap E_{v_n,k_2}$. $\square$

Often it will be useful to reference the number of occurrences of v_n with some larger v_m. Recall that we defined q_n to be the number of spacers involved when constructing v_{n+1} from v_n. So the number of occurrences of v_n within v_{n+1} is $q_n + 1$.

DEFINITION 3.11. Fix n and let $m \geq n$. Let q_n^m denote the number of expected occurrences of v_n within v_m.

From this definition, it follows immediately that $q_n^n = 1$ for any n. Also, q_n^m is easy to compute inductively from our cut parameters, as $q_n^{m+1} = (q_{m+1} + 1) \cdot q_n^m$.

We look closely at the distribution of v_n-blocks in showing a combinatorial property which gives a condition for classifying the maximal equicontinuous factor.

Toward this end, we list several results based on the distribution of blocks.

LEMMA 3.12. *Fix v_n. Suppose that a finite substring α of V consists of v_n-blocks. Let $m > n$. Let β_1 and β_2 be v_n-blocks within α such that there are exactly $q_n^m - 1$ v_n-blocks between β_1 and β_2. Let a_{i_1,j_1} and a_{i_2,j_2} be the values of the spacers for β_1 and β_2 respectively.*

Then $i_1 < m$ iff $i_2 < m$, and in this case, $i_1 = i_2$ and $j_1 = j_2$.

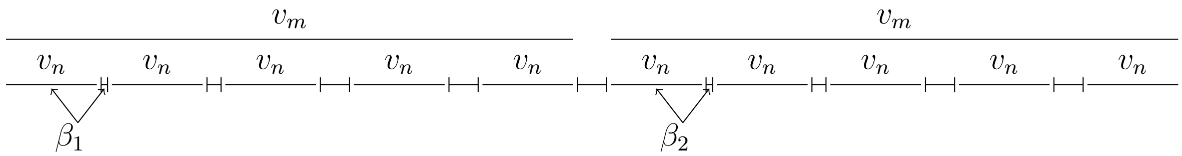
Otherwise, $i_1 \geq m$ and $i_2 \geq m$.

PROOF. Let n, m and β_1 be as above. By part 1 of Fact 2.10, we have that the v_n in β_1 is contained in some expected occurrence of v_m. There will be exactly one spacer before this v_m and then another occurrence of v_m preceding it, similarly exactly one spacer after and another occurrence of v_m.

Suppose β_1 is the first occurrence of v_n within the v_m. If β_2 is after β_1, then the blocks between β_1 and β_2 are the $q_n^m - 1$ v_n-blocks immediately after β_1. Then the $q_n^m - 1$ v_n-blocks after β_1 comprise the rest of the v_m containing β_1 and so β_2 is the first v_n-blocks of the next v_m.

Similarly, if β_2 is before β_1, then the $q_n^m - 1$ v_n-blocks before β_1 are exactly the final $q_n^m - 1$ v_n-blocks of the v_m preceding β_1 and so β_2 will be the first v_n-block of the v_m preceding β_1.

This is the picture for $q_n^m = 5$. Note that β_1 and β_2 both occur at the same relative position within v_m and hence have the same spacer.



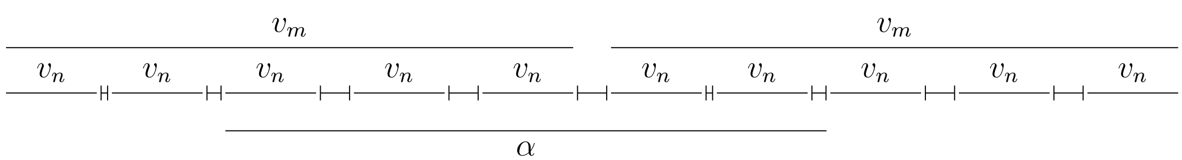
Therefore, in either case, β_2 will be the first v_n-block of v_m. But therefore, the two have the same corresponding spacer (in particular $i_1 = i_2 = n$ and $j_1 = j_2 = 1$) and so the proposition holds for the first block of v_m.

For the other blocks, consider inductively shifting over by one block. As v_m is uniquely defined from the v_n-blocks, any v_n-block completely contained within v_m is defined by the spacers and will have $i_1 = i_2 < m$ and $j_1 = j_2$. Otherwise, if β_1 is the final v_n-block from the expected occurrence of v_m, then the spacer comes from a higher level and so $i_1, i_2 \geq m$. □

Proposition 3.13. *Fix n. Let α be a finite string consisting of v_n-blocks. Suppose there is some $m > n$, so α contains exactly q_n^m many v_n-blocks.*

Then for each i with $n \leq i < m$, for all j with $1 \leq j \leq q_i$, the number of spacers of the form $1^{a_{i,j}}$ are the same as in v_m (when we include the subsequent spacer). Also, there is exactly one occurrence of $1^{a_{i,\cdot}}$ where $i \geq m$.

This is the picture for v_m consisting of 5 v_n-blocks. Note that α contains one of each spacer from v_m along with exactly one not contained in v_m.



Proof. We show this by induction on k where $m = n + k$.

The base case is $k = 1$. Note that $q_n^m = q_n^{n+1} = q_n + 1$, so α contains exactly $q_n + 1$ occurrences of v_n.

Note that each v_n occurring within α occurs in a unique expected occurrence of v_{n+1}. Since v_{n+1} contains exactly $q_n + 1$ occurrences as well, we see that α contains occurrences from at most two adjacent occurrences of v_{n+1}. If α contains only occurrences from one v_{n+1}, then we are done.

Otherwise, note that by Lemma 3.12, we have that each v_n-block within α and one of the two v_{n+1}'s is separated by exactly q_n v_n-blocks from a block not within α in the other occurrence of v_{n+1}. Hence for each position within v_{n+1}, there is exactly one v_n-block within α corresponding to that position. Therefore, the base case holds.

Now suppose the statement holds for $k-1$, i.e. for all strings which contain exactly q_n^{m-1} v_n-blocks. Suppose α contains exactly q_n^m v_n-blocks, where $m = n+k$. Note that v_m consists of $q_m + 1$ occurrences of v_{m-1}. Since q_n^{m-1} is the number of v_n-blocks in v_{m-1}, we have that $q_n^m = (q_m + 1) \cdot q_n^{m-1}$ and so α contains $(q_m + 1) \cdot q_n^{m-1}$ v_n-blocks.

By the induction hypothesis, subdividing α into $q_m + 1$ pieces which consist of q_n^{m-1} v_n-blocks, we get $q_m + 1$ strings in correspondence with $q_m + 1$ consecutive copies of v_{m-1}. By the argument for the base case, we have that any $q_m + 1$ consecutive copies of v_{m-1} will be in correspondence with v_m and satisfy the proposition. Finally, we note that α comes from shifting v_n-blocks from a string consisting of $q_m + 1$ consecutive occurrences of v_{m-1}, and applying Lemma 3.12, we get that the v_n-blocks correspond and hence, the statement holds. □

Corollary 3.14. *Let α be a string consisting of v_n-blocks. Suppose there is some $m \geq n$ so that the number of v_n-blocks in α is q_n^m. Then $|\alpha|$ and $|v_m|$ differ by the length of a single spacer.*

Proof. Note that Proposition 3.13 applies and we have that α contains exactly one spacer $a_{i,j}$ with $i \geq m$, and this is the only spacer that can differs from those in v_m. □

As mentioned before, strings consisting of v_n-blocks are of interest because they correspond to distances $k_1 - k_2$ so that $E_{v_n,k_1} \cap E_{v_n,k_2}$ are non-empty. Because of this, it is useful to build such strings in a way that allows choices that control the lengths between different choices. Below, we give a construction which will give us some control.

Lemma 3.15. *Let $m \geq n$ and suppose α_0 be a finite string satisfying the following inductive conditions (on m):*

- *α_0 consists of v_n-blocks*
- *α_0 contains exactly one distinguished spacer with level i_0 so that $i_0 \geq m$*
- *α_0 contains at most q_n^m v_n-blocks.*

If α_0 satisfies these conditions, then denoting the distinguished spacer by b, α_0 is of the form $\alpha_f{}^\frown b^\frown \alpha_i$.

Let $A = (a_{m,1}, a_{m,2}, ..., a_{m,q_m})$ be the sequence of spacer parameters between occurrences of v_m in v_{m+1}. Let γ be any non-empty, contiguous proper subsequence of the spacer parameters A.

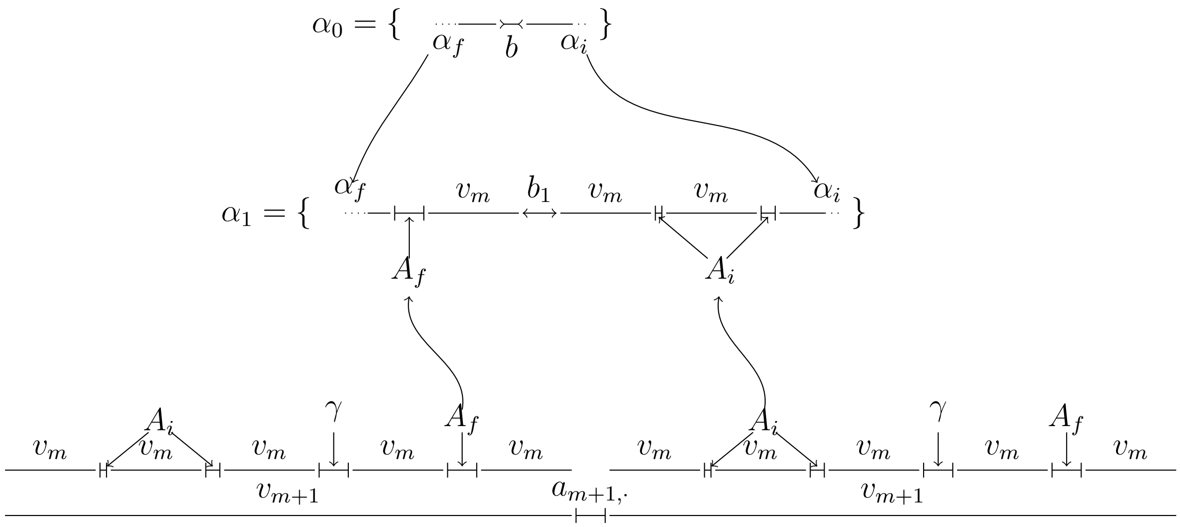
Then, we can construct α_1 from α_0 and $A \backslash \gamma$, in the sense that α_1 has α_f as an initial segment, α_1 has α_i as a final segment, each spacer within $A \backslash \gamma$ occurs within α_1, and α_1 contains exactly $|A \backslash \gamma|$ occurrences of v_m between α_f and α_i. Furthermore, α_1 satisfies the following inductive properties:

- *α_1 consists of v_n-blocks*
- *α_1 contains exactly one distinguished spacer with level i_1 so that $i_1 \geq m+1$*
- *α_1 contains at most q_n^{m+1} v_n-blocks.*

Note that the choice of γ is arbitrary and allows us to control the length of the resulting α_1. We will often use this lemma to construct two different strings, and at each stage, choose different γ's which control the difference in length between the two strings.

We give a diagram of Lemma 3.15.

We are given an α_0 with a distinguished spacer denoted b



We use b to determine α_f and α_i, which by assumption are respectively, final and initial segments of v_m. From γ, we determine A_f and A_i as subsequences of the rest of the spacers, and put the pieces together according to the arrows in the diagram. From the diagram, if b_1 can be put in as a_{m+1}, it is clear where α_1 fits as a subword of V.

Proof. The proof is basically the same as the picture in the construction above.

Let α_0 and $\gamma \subseteq A$ be as above. We will split α_0 and $A \backslash \gamma$ into appropriate pieces to define the construction.

Note that α_0 consists of v_n-blocks occurring before b, a v_n-block containing the distinguished spacer b, and v_n-blocks occurring after b. We will denote the part of α_0 before the distinguished spacer b by α_f and the part of α_0 after b by α_i. Note that α_i consists of v_n-blocks and can be empty, while α_f has a (possibly empty) string consisting of v_n-blocks followed by exactly one occurrence of v_n at the end. This is the decomposition $\alpha_0 = \alpha_f{}^\frown b^\frown \alpha_i$ mentioned in the statement of the lemma above.

The names come from noticing that α_i is a sequence of v_n-blocks occurring after a spacer coming from a level at least m, and α_i is a proper substring of α_0, which contains at most q_n^m many v_n-blocks, so α_i is an initial segment of v_m. Similarly, α_f ends with the occurrence of v_n immediately prior to a spacer from level at least m and the portion of α_f before that final occurrence consists of fewer than q_n^m many v_n-blocks, so α_f is a final segment of v_m.

We will also label the leftover parts of A after removing γ. We will call the part of A before γ, A_f and the part of A after γ, A_i. Similarly, to above, it is possible for one of A_f or A_i to be empty.

We define α_1 as follows. We let α_1 start with α_f. As noted above, this is a final segment of v_m. So after α_f, α_1 will contain a spacer corresponding to a level m or above.

If A_f is empty, then we note that the construction so far is a final segment of v_m, hence a final segment of v_{m+1}, so we will put in the spacer for b_1 which will be the distinguished spacer from a level larger than $m+1$. Note that this spacer can be arbitrary as long as it is from a level at least $m+1$.

If A_f is not empty, then after the currently constructed string we will put in the first spacer from A_f, then occurrences of v_m to create v_m-blocks for each spacer left in A_f, and finally one extra occurrence of v_m. Note that after this step, the construction is a final segment of v_m, followed by spacers and v_m's in the order necessary to be a final segment of v_{m+1}, so the construction is a final segment of v_{m+1}. Then we add the spacer for b_1.

Note that the construction so far, in either case, consists of a final segment of v_{m+1} followed by exactly one spacer of level at least $m+1$. Also note that after α_f, α_1 so far contains exactly $|A_f|$ of occurrences of v_m.

Then we add occurrences of v_m to make v_m-blocks for each spacer in A_i and after that add α_i to complete the construction of α_1. Note that A_i consists of spacers from the first part of v_{m+1} and α_i is an initial segment of v_m, so we see that the v_m-blocks with spacers from A_i together with α_i added on to the end form an initial segment of v_{m+1}. Also, note that we added exactly $|A_i|$ occurrences of v_m to α_1 after b_1.

Now we verify the properties that α_1 must satisfy. By construction, it is clear that α_f is an initial segment of α_1 and that α_i is a final segment of α_1. Also, we have added each spacer from A_f and A_i to α_1, and since $A\backslash\gamma$ was decomposed into A_f and A_i, we have added the necessary spacers. Furthermore, since we added exactly $|A_f|$ occurrences of v_m to α_1 before b_1, and exactly $|A_i|$ occurrences of v_m to α_1 after b_1, we added a total of $|A \backslash \gamma|$ occurrences of v_m to α_1.

Now we will verify the inductive properties.

Note that since α_0 consists of v_n-blocks, α_f will start at an expected occurrence of v_n, and α_i will end immediately before an expected occurrence of v_n, so to show that α_1 consists of v_n-blocks, it is enough to show that α_1 occurs as a finite subword of V.

The construction of α_1 consists of a final segment of v_{m+1} followed by a distinguished spacer for level at least $m+1$ followed by an initial segment of v_{m+1}, so α_1 will be a substring of any v_M where M is large enough to contain the distinguished spacer. Therefore, we can find an occurrence of α_1 within V so that the expected occurrences of v_n in α_1 line up with the expected occurrences of v_n in V, and hence α_1 consists of v_n-blocks.

It is clear from the construction that the only spacer in α_1 from a level $m+1$ or higher was b_1.

Finally, since γ was non-empty and α_0 contains at most q_n^m v_n-blocks, α_1 consists of at most q_m v_m-blocks together with the at most q_n^m v_n-blocks coming from α_0. Hence the number of v_n-blocks in α_1 is at most $q_m(q_n^m) + q_n^m = (q_m + 1)q_n^m = q_n^{m+1}$, which is the necessary bound.

□

We have the following corollary of the previous construction. We are allowed to specify a set string before α_0 which will be maintained through the construction.

Corollary 3.16. *Fix $N \in \mathbb{N}$. Let $m \geq n$ and suppose α_0 be a finite string satisfying the following inductive conditions (on m):*

- *α_0 consists of v_n-blocks*
- *α_0 contains exactly one distinguished spacer with level i_0 so that $i_0 \geq m$*
- *α_0 contains at most q_n^m v_n-blocks.*

From the previous lemma, if we denote the distinguished spacer by b, α_0 can be de-

composed into $\alpha_f{}^\frown b^\frown \alpha_i$*. Furthermore, assume that* m *is large enough that* $|v_m| \geq N + |\alpha_f|$*. Fix some occurrence of* α_0 *within* V *and let* η *be the string of* N *characters occurring before* α_0*.*

Let $A = (a_{m,1}, a_{m,2}, ..., a_{m,q_m})$ *be the sequence of spacer parameters between occurrences of* v_m *in* v_{m+1}*. Let* γ *be any non-empty, contiguous proper subsequence of the spacer parameters* A*.*

Then, we can construct α_1 *from* α_0 *and* $A\backslash\gamma$*, in the sense that* α_1 *has* α_f *as an initial segment,* α_1 *has* α_i *as a final segment, each spacer within* $A\backslash\gamma$ *occurs within* α_1*, and* α_1 *contains exactly* $|A\backslash\gamma|$ *occurrences of* v_m *between* α_f *and* α_i*. Furthermore,* α_1 *satisfies the following inductive properties:*

- α_1 *consists of* v_n*-blocks*
- α_1 *contains exactly one distinguished spacer with level* i_1 *so that* $i_1 \geq m+1$
- α_1 *contains at most* q_n^{m+1} v_n*-blocks.*

Finally, we also have that η *will be the string of* N *characters occurring before* α_1*.*

Note that this corollary is exactly Lemma 3.15 with the conditions involving the N character string added. Therefore, except for the conditions on the N character string, Lemma 3.15 gives us the necessary α_1.

Proof. First, we claim that the condition that m needs to be large enough that $|v_m| \geq N + |\alpha_f|$ is not difficult to satisfy. In particular, if we have a string satisfying the conditions from Lemma 3.15 which does not satisfy that $|v_m| \geq N + |\alpha_f|$, then we can get a new string that does satisfy the length condition by replacing only the distinguished spacer.

Let α_0' be a string satisfying the conditions from Lemma 3.15 for $m' \geq n$ and denote the distinguished spacer by b'. From the previous proof, we can decompose α_0' into $\alpha_f{}^\frown b'{}^\frown \alpha_i$ so that α_f will be a final segment of $v_{m'}$ and α_i will be an initial segment of $v_{m'}$. But then for any $m \geq m'$, α_f is also a final segment of v_m and α_i is an initial segment of v_m, hence we can replace b' by a spacer b from a level m, where m is large enough that $|v_m| \geq N + |\alpha_f|$. This will be the α_0 satisfying the induction hypothesis. Note since b comes from a level at least m and α_f is a final segment of v_m, then N character string η will be contained in the occurrence of v_m before b.

Now use the construction from Lemma 3.15. Note that all the inductive conditions for α_1 will be satisfied and it is enough to show that η is the N character string occurring before α_1. But the construction put a spacer from at least level m immediately after α_f and α_f is a final segment of v_m, so within V, there will be an occurrence of v_m ending with α_f overlapping with α_1. But note that η was defined as the N character string immediately preceding α_f within v_m. Therefore, we see that the N character string preceding α_1 is the N character string preceding α_f within v_m, so η is the N character string preceding α_1. $\square$

We will use this construction iteratively to build strings with appropriate length gaps. Since the strings always consist of v_n-blocks, we see that these strings will witness intersections between different $E_{v_n,k}$'s.

Next, we have a few conditions on length specifically for subshifts with bounded spacer parameter.

Lemma 3.17. *Let (X,T) be a rank one subshift with spacer parameter bounded by C.*

Fix n so that $|v_n| \gg C$. Let α be a string that consists of v_n-blocks and suppose $|\alpha| = |v_m| + d$ for some $m \geq n$ and some $0 \leq d < |v_n|$. Then α contains exactly q_n^m many v_n-blocks.

Proof. Note that when we refer to $|v_m|$ or $|v_n|$, we do not include the subsequent spacer, so v_m and v_n themselves do not consist of v_n-blocks.

Let α be a string consisting of v_n-blocks with $|\alpha| = |v_m| + d$ for some $m \geq n$ and $0 \leq d < |v_n|$ and suppose by contradiction that α does not contain exactly q_n^m many v_n-blocks.

First, we eliminate the case where $m = n$. Then α consists of v_n-blocks and has length $|v_n| + d$ for some $0 \leq d < |v_n|$. Since $|\alpha| = |v_n| + d < 2|v_n|$, we have that there must be fewer than two occurrences of v_n in α and since α is not empty, we get α consists of exactly one v_n-block.

Now suppose $m > n$. First, assume that α contains fewer than q_n^m many v_n-blocks.

Then, we can extend α to a string β consisting of v_n-blocks which contains exactly q_n^m v_n-blocks. Note that since $|\alpha| = |v_m| + d$ for some $0 \le d < |v_n|$, $|\beta| \ge |v_m| + |v_n|$.

Since β contains exactly q_n^m v_n-blocks, by Proposition 3.13, we see that both v_m and β contain the same number of occurrences of v_n and the same occurrences of $1^{a_{i,j}}$ from spacers where $n \le i < m$. Furthermore, β contains exactly one occurrence of $1^{a_{i,j}}$ where $i \ge m$ and this does not occur in v_m. Hence β differs from v_m by exactly one spacer so $||\beta| - |v_m|| \le C$. But $|\beta| \ge |v_m| + |v_n|$, so $||\beta| - |v_m|| \ge |v_n|$. Therefore, $|v_n| \le C$. But this contradicts that $|v_n| > C$. Hence α cannot contain fewer than q_n^m v_n-blocks.

Now suppose α contains more than q_n^m v_n-blocks. Then we can shrink α to a string β' which contains exactly q_n^m v_n blocks. Since we removed at least one occurrence of v_n from α to get β' and $|\alpha| = |v_m| + d$ where $0 \le d < |v_n|$, we get that $|\beta'| \le |v_m| - d'$ where $d' > 0$.

Again, we can use Proposition 3.13 to see that β' differs from v_m by exactly one spacer. Furthermore, since v_m does not consist of v_n-blocks, i.e. does not include the subsequent spacer, we see that β' contains the extra spacer of the form $a_{i,\cdot}$ where $i \ge m$, and hence $|\beta'| \ge |v_m|$. But putting this inequality together with $|\beta'| \le |v_m| - d'$, we see that $|v_m| \le |v_m| - d'$ so $d' \le 0$, which contradicts that $d' > 0$. Therefore, α cannot contain more than q_n^m v_n-blocks, hence the only possibility is that α contains exactly q_n^m many v_n-blocks. □

COROLLARY 3.18. *Let (X, T) be a rank one subshift with spacer parameter bounded by C.*

Fix n large enough so that $v_n \gg C$ and let $0 \le d \le |v_n|$. Let $N = |v_m|$ for some $m \ge n$. Then we can have $E_{v_n,0} \cap E_{v_n,N+d} \ne \emptyset$ only when $d \le C$.

PROOF. Let n, m, d be as above. Since $|v_n| \gg C$, we have that each v_n-block will have length much larger than C. If $x \in E_{v_n,0} \cap E_{v_n,N+d}$, then x must have expected occurrences of v_n and hence v_n-blocks which start at 0 and $N + d$, and so the string $x[0, N + d]$ must be built from v_n-blocks. By 3.17, we have that $x[0, N + d]$ must contain exactly q_n^m many v_n-blocks and by 3.13, $|v_m|$ and $N + d$ can differ by no more than C, hence $d \le C$. □

CHAPTER 4

CLASSIFYING THE MAXIMAL EQUICONTINUOUS FACTOR

In this section, we will prove Theorem 1.1 using the results from the previous chapter.

First, we start by showing the result for systems with unbounded spacer parameter. For this proof, we do not need nearly as many technical results as we can exploit the existence of the fixed point to constrain the possible equicontinuous factors.

4.1. Maximal Equicontinuous Factor for Unbounded Spacer Parameter

Recall from Fact 2.19, that the following points are always present in a rank one subshift with unbounded spacer parameter:

Recall. *Let (X, T) be a rank one subshift with unbounded spacer parameter and V be the infinite rank one word for (X, T). Then $1^{\mathbb{Z}} \in X$ and all shifts of $...111V$ are in X.*

We can use these particular points to directly observe partition proximality for systems with unbounded spacer parameter.

Proposition 4.1. *Let (X, T) be an unbounded rank one subshift. Then (X, T) has partition proximality with one reference point.*

Proof. Let $x \in X$ and let $\delta > 0$. Then we can find a neighbourhood U_1 with $x \in U_1$ and $\operatorname{diam}(U_1) < \delta$. We can assume U_1 will be of the form $U_{\alpha,k}$ for some α a subword of some large enough v_n and $k \in \mathbb{Z}$. Let $x_1 = 1^{\mathbb{Z}}$, which we know is in X by Fact 2.19.

Let $z_1 \in X$ be of the form, $...111V$ with $z_1 \in U_1$. Since U_1 is of the form $U_{\alpha,k}$, we can shift the V so that appropriate v_n which contains α is situated so that α starts at index k, and take that as z_1. Then, we shift z_1 by l into the infinite sequence of 1's long enough so that $T^l(z_1)$ has a long enough sequence of 1's around index 0, and since $z_2 = 1^{\mathbb{Z}}$, we get $d(T^l(z_1), T^l(z_2)) < \delta$, so z_1, z_2 guarantee that (X, T) has partition proximality with one reference point. □

Note that in this proof, we were able to use $l_1 = l_2 = 0$ and avoid mentioning them explicitly.

Out of this proof, we get the following corollary.

COROLLARY 4.2. *If (X, T) is a rank one subshift with unbounded spacer parameter, the maximal equicontinuous factor is the trivial one point transformation.*

PROOF. By Proposition 4.1, we have that (X, T) has partition proximality and in particular, with exactly one reference point. By Proposition 3.6, any equicontinuous factor of (X, T) must have at most one point. The trivial one point transformation is the only topological dynamical system with one point, and hence, it is the maximal equicontinuous factor. □

4.2. On Bounded Spacer Parameter

For this section, we will work with rank one subshifts (X, T) with a bounded spacer parameter with bound given by C.

By Lemma 3.7, we can put partition proximality in terms of the $E_{v_n,k}$. Since the $E_{v_n,k}$ behave predictably under shift and in particular, $E_{v_n,k_1} \cap E_{v_n,k_2} \neq \emptyset$ iff $E_{v_n,0} \cap E_{v_n,k_2-k_1} \neq \emptyset$, we can restate the condition for partition proximality entirely in terms of neighbourhoods.

PROPOSITION 4.3. *Suppose (X, T) is a rank one subshift with spacer parameter bounded by C. Fix $p \geq 1$. Suppose for sufficiently large n, and every $h \in \mathbb{N}$ so that $ph \leq |v_n| + C$, there are strings α, $\beta \subseteq V$ so that α and β consist of v_n-blocks and $|\alpha| - |\beta| = ph$. Then (X, T) has partition proximality with at most p reference points.*

PROOF. Assume that the above property holds for some $p \geq 1$. Since (X, T) has spacer bounded, we can apply Corollary 2.21,so there is at least one a so that all points of the form V^*1^aV are in X. We will fix one such a. Define $x_1, ..., x_p$ so that x_i is the point of the form V^*1^aV so that the occurrence of V starts at index i. Let n be large enough. Note that for each i with $1 \leq i \leq p$, $x_i \in E_{v_n,i}$ as the infinite word V starts at index i. By Lemma 3.7, it is enough to show for any $x \in X$, we can find some i where $1 \leq i \leq p$, and some $k_1, k_2, k_3 \in \mathbb{Z}$ so that $x \in E_{v_n,k_1}$, $x_i \in E_{v_n,k_2}$, and both $E_{v_n,k_1} \cap E_{v_n,k_3} \neq \emptyset$ and $E_{v_n,k_2} \cap E_{v_n,k_3} \neq \emptyset$.

Since the spacer parameter is bounded by C, we have that $\{E_{v_n,k} : 0 \leq k \leq |v_n| + C\}$ cover X, so $x \in E_{v_n,k_1}$ for some $0 \leq k_1 \leq |v_n| + C$. There is some i so that $k_1 \equiv i \mod p$. Let x_i be this i and let $k_2 = i$, so $x_i \in E_{v_n,k_2}$. So we just need to find some k_3 so that both $E_{v_n,k_1} \cap E_{v_n,k_3} \neq \emptyset$ and $E_{v_n,k_2} \cap E_{v_n,k_3} \neq \emptyset$.

Since $k_1 \equiv i \mod p$, there is some $h \in \mathbb{N}$ so that $|k_1 - i| = ph$. If $h = 0$, then $k_1 = k_2$ and letting $k_3 = k_1$ we get the necessary k_3 trivially. So we will assume $h > 0$. By the assumption, there are α and β subwords of V which consist of v_n-blocks, so that $|\alpha| - |\beta| = ph$. Without loss of generality, assume that $k_1 - i = ph$. Then, note that $k_1 + |\beta| = k_2 + |\alpha|$. We will set $k_3 = k_1 + |\beta| = k_2 + |\alpha|$. Since β consists of v_n-blocks and $k_3 = k_1 + |\beta|$, by Lemma 3.10, we get that there is a $z_1 \in E_{v_n,k_1} \cap E_{v_n,k_3}$.

Similarly, since α consists of v_n-blocks and $k_3 = k_2 + |\alpha|$, by 3.10, we get that there is a $z_2 \in E_{v_n,k_2} \cap E_{v_n,k_3}$. Therefore, the choice of k_3 witnesses that the necessary intersections are non-empty and we get partition proximality. (If instead $i - k_1 = ph$, then we can just switch α and β in the proof.) □

This will be the main tool for showing partition proximality for bounded spacers. To start, we will note a corollary about when the maximal equicontinuous factor is trivial. Knowing when the maximal equicontinuous factor is trivial will also be helpful for showing mixing properties.

Corollary 4.4. *Suppose (X,T) is a rank one subshift with spacer parameter bounded by C. Suppose for sufficiently large n, and every k, with $0 \leq k \leq |v_n| + C$, there are strings α, β so that α and β consist of v_n-blocks and $|\alpha| - |\beta| = k$. Then (X,T) has partition proximality with exactly one reference point.*

Proof. This is Proposition 4.3 with $p = 1$. □

We combine this property with Lemma 3.15 to get the following result.

Corollary 4.5. *Let (X,T) be a rank one subshift so that for infinitely many m, there are $a_{m,i}, a_{m,j}$ with $|a_{m,i} - a_{m,j}| = 1$. Then, for any $n, h \in \mathbb{N}$, we can find substrings α, β of V*

consisting of v_n-blocks where $|\alpha| - |\beta| = h$.

Hence, the maximal equicontinuous factor is the trivial one point system.

Note that this includes the Chacon system.

PROOF. We inductively use Lemma 3.15. Let α_0, β_0 be arbitrary equal strings satisfying the induction hypotheses for Lemma 3.15. At each level $m = n + l$, if there is a pair $a_{m,i} - a_{m,j} = 1$, then we omit $a_{m,j}$ for α_l and omit $a_{m,i}$ for β_l and otherwise, choose the same sequence for α_l, β_l. Whenever there is the difference, this increases the gap between α_l and β_l by one. Since this occurs infinitely often, for any $h \in \mathbb{N}$, we repeat until we have hit h pairs that differ by 1, so we get the necessary α and β. By Corollary 4.4, this also gives us that the maximal equicontinuous factor is trivial. $\square$

Note that the previous corollary required that the different spacers came from the same level of the system. However, this is something of an artefact of the notation as we can relabel to combine levels together and it will not change the resulting system.

EXAMPLE 4.6. Let (X, T) have cutting parameters q_n and spacer parameters $a_{n,i}$. We will show how to define new parameters q_n^* and $a_{n,i}^*$ which combine the first and second levels $(n = 0, 1)$, but generate the same system.

Let $q_0^* = (q_0 + 1)(q_1 + 1) - 1$. For the $a_{0,i}^*$, first consider the congruence class of i mod $q_0 + 1$. If $i \not\equiv 0 \bmod q_0 + 1$, then we will let $a_{0,i}^* = a_{0,i}$. There are $q_0(q_1 + 1)$ spacers with values $i \not\equiv 0 \bmod q_0 + 1$, so this leaves the remaining $(q_0 + 1)(q_1 + 1) - 1 - q_0(q_1 + 1) = q_1 + 1 - 1 = q_1$ spacers with values $a_{0,(q_0+1)j}^*$ for $1 \leq j \leq q_1$. So we let $a_{0,(q_0+1)j}^* = a_{1,j}$.

Then, we define the rest of the $q_n^* = q_{n+1}$ and $a_{n,i}^* = a_{n+1,i}$ for $n \geq 1$ and $1 \leq i \leq q_n^*$.

We claim that this generates the same system. Let v_n denote the rank one words generated by the q_n and $a_{n,i}$ and v_n^* be the words generated by the q_n^* and $a_{n,i}$. Since we defined $q_n^* = q_{n+1}$, $a_{n,i}^* = a_{n+1,i}$ for $n \geq 1$, it is enough to show that $v_1^* = v_2$, since if this is the case, the rest of the words are built up in the same way across the different schemas.

So consider v_2. $v_2 = v_1 1^{a_{1,1}} v_1 1^{a_{1,2}} v_1 ... v_1 1^{a_{1,q_1}} v_1$. But $v_1 = v_0 1^{a_{0,1}} v_0 1^{a_{0,2}} v_0 ... v_0 1^{a_{0,q_0}} v_0$. So we see that v_2 consists of occurrences of v_0 followed by spacers $1^{a_{0,i}}$ between them up, in order according to i. This continues until the $q_0 + 1$ occurrence of v_0, after which there is a spacer $1^{a_{1,0}}$. This pattern repeats with $q_0 + 1$ occurrences of v_0 separated by the spacers of the form $1^{a_{0,i}}$ in sequence (according to i), followed by the next spacer $1^{a_{1,j}}$, until there have been q_1 spacers with values $a_{1,j}$, after which there are $q_0 + 1$ occurrences of v_0 separated by spacers $1^{a_{0,i}}$.

However, we see that this is exactly how v_1^* is defined, as v_1^* consists of the string with $q_0 + 1$ occurrences of v_0 separated by the $1^{a_{0,i}}$ in sequence (according to i), repeated q_1 times, with the $1^{a_{1,j}}$ in between each sequence of the $q_0 + 1$ occurrences of v_0.

In this manner, we can combine and relabel consecutive levels by repeating this process for each additional level we want to combine. Therefore, we do not need to require that differing spacers occur on the same level, as if they occur on different levels, we can simply collapse the relevant levels. This gives us the following corollary.

Corollary 4.7. *Let (X, T) be a rank one subshift so that for arbitrarily large i, i', there are $a_{i,j}, a_{i',j'}$ with $|a_{i,j} - a_{i',j'}| = 1$. Then, for any $h \in \mathbb{N}$, we can find substrings α, β of V consisting of v_n-blocks where $|\alpha| - |\beta| = h$.*

Also, we get that the maximal equicontinuous factor is the trivial one point system.

Back to the more general context, Corollary 4.7 gives a condition in terms of the spacer parameter for when the system has trivial maximal equicontinuous factor. The current statement for the general case, Proposition 4.3, has a condition for when a rank one subshift has maximal equicontinuous factor at most p, but it is not clear how to determine this from an arbitrary spacer parameter and it does not completely settle the question of the maximal equicontinuous factor. The next section will be devoted to proving that we can show the necessary gaps and get exactly the finite factor from the condition in Proposition 2.16.

4.3. Completely Classifying the Maximal Equicontinuous Factor

We will assume the following about our system of $a_{i,j}$. First, we can assume that (X, T) is not periodic, since that would mean we have finitely many points in the system and (X, T) would be its own maximal equicontinuous factor. Since (X, T) has bounded spacer parameter, the $a_{i,j}$ are all $\leq C$. So there are finitely many possible values and hence at least two (by non-periodicity) occur infinitely often. We will assume that our m is large enough that only the values that occur infinitely often occur in our set $\{a_{m,1}, a_{m,2}, ..., a_{m,q_m}\}$. Furthermore, by relabelling and telescoping, we can assume that each level m contains all spacer lengths that occur infinitely often.

To prove the theorem, we first establish combinatorial facts about a particular type of gcd.

DEFINITION 4.8. Let $\{a_1, ..., a_l\}$ be a finite set. We define the *up-down gcd* to be the minimum value $d \geq 1$ achievable by a sum of the form $\sum_{i=1}^{I}(a_{i,+} - a_{i,-})$ for some $I \in \mathbb{N}$, where each $a_{i,+}, a_{i,-} \in \{a_1, ..., a_l\}$.

Intuitively, the up-down gcd is the gcd obtainable by a restricted Euclidean algorithm, where at each stage you are required to add exactly one element and subtract exactly one element from the given set. This immediately suggests that the gcd of a set will divide the up-down gcd.

FACT 4.9. *Let $\{a_1, ..., a_l\}$ be a finite set, let p be the gcd, and let d be the up-down gcd. Then $p|d$.*

PROOF. Since p is the gcd, $p|a_i$ for every i. But then for any sum of the form $\sum_{i=1}^{I}(a_{i,+}-a_{i,-})$, we have that $p|a_{i,+}$ and $p|a_{i,-}$ for every i, hence $p|(a_{i,+} - a_{i,-})$ for each i, and therefore p divides the entire sum. But since $d = \sum_{i=1}^{I}(a_{i,+} - a_{i,-})$ for some I and values of $a_{i,+}, a_{i,-}$, we have p divides d. $\square$

FACT 4.10. *Let $\{a_1, ..., a_l\}$ be a finite set and let $\{b_1, ..., b_h\}$ be the set of differences from $\{a_1, .., a_l\}$, i.e. the b_i are all values of the form $\left|a_j - a'_j\right|$ for $1 \leq j \neq j' \leq l$. Then the up-down gcd of $\{a_1, ..., a_l\}$ is the gcd of $\{b_1, ..., b_h\}$.*

PROOF. Since the $\{b_1, ..., b_k\}$ are each of the form $|a_j - a_{j'}|$ for $1 \leq j \neq j' \leq l$, any integer combination of the b_i can be written as $\sum_{i=1}^{I}(a_{i,+} - a_{i,-})$, and hence the gcd of $\{b_1, ..., b_k\}$

can be written as a sum of that form. Hence the minimum value of such a sum will be at most the gcd of $\{b_1, ..., b_k\}$.

Next, note that since $\{b_1, ..., b_k\}$ is the set of differences, every term of the form $a_{i,+} - a_{i,-}$ is equal to or the negative of some b_j for any i. So the gcd of $\{b_1, ..., b_k\}$ will divide every term of the form $a_{i,+} - a_{i,-}$, and hence will divide the sum. Therefore, the minimum value of such a sum can be no more than the gcd of $\{b_1, ..., b_k\}$. □

LEMMA 4.11. *Let $\{a_1, ..., a_l\}$ be the finite set of lengths for the spacer parameters that occur at each level and let d be the up-down gcd. Using Lemma 3.15, for any $h \in \mathbb{N}$, we can find α and β whose length differ by hd.*

PROOF. Note that by telescoping and going up to a high enough level, we can assume that the $\{a_1, ..., a_l\}$ are the only lengths of spacer parameters occurring at each level m and that at least one of each occurs in each level. Because of this uniformity in the levels, if we can show how to go from the same length to a d length difference, then we can simply repeat the process h times and get the necessary gap in lengths.

By Fact 4.10, we have that d is the gcd of the differences from $\{a_1, ..., a_l\}$, so by the Euclidean algorithm, we can find $a_{i,+}, a_{i,-} \in \{a_1, ..., a_l\}$ so that $d = \sum_{i=1}^{j}(a_{i,+} - a_{i,-})$.

Fix $\alpha_0 = \beta_0$ consisting of v_n-blocks and satisfying the hypotheses for Lemma 3.15. To define α_i, β_i for $i \geq 1$, note that taking all $(n+i)$-blocks except one omits a proper contiguous subset of the $a_{n+i,\cdot}$ and hence will be a valid choice for γ. The we can use Lemma 3.15. We let α_i be generated from α_{i-1} by choosing γ to be a spacer with length $a_{i,-}$ and β_i be generated from β_{i-1} by choosing γ to be a spacer with length $a_{i,+}$. By 3.12, we have that strings containing all the of $(n+i)$-blocks for the v_{n+i+1} will have the same length (by choosing the same $n+i+1$ level spacer parameter), and so when we omit a block containing spacer of length $a_{i,-}$ from α_i and length $a_{i,+}$ from β_i, we see that $|\alpha_i| - |\beta_i| = |\alpha_{i-1}| - |\beta_{i-1}| - (a_{i,i} - a_{i,+})$. Furthermore, by Lemma 3.15, we have that the new α_i, β_i will satisfy the hypotheses of Lemma 3.15, so we can continue inductively.

Doing this for each $i \leq j$, we see that $|\alpha_j| - |\beta_j| = |\alpha_0| - |\beta_0| - \sum_{i=1}^{j}(a_{i,-} - a_{i,+}) =$

$|\alpha_0| - |\beta_0| + \sum_{i=1}^{j}(a_{i,+} - a_{i,-}) = |\alpha_0| - |\beta_0| + d$. But since we started with $|\alpha_0| = |\beta_0|$, we get that $|\alpha_j| - |\beta_j| = d$.

Furthermore, to get a gap of hd inductively, we can start with α_0 and β_0 so that $|\alpha_0| - |\beta_0| = (h-1)d$ and follow these steps. Then $|\alpha_j| - |\beta_j| = |\alpha_0| - |\beta_0| + d = (h-1)d + d = hd$. □

Next, we will use the stronger construction from Corollary 3.16 to get the optimal gap for each spacer set.

PROPOSITION 4.12. *Let (X, T) be rank one subshift with p the largest integer so that there is some n, so that for all $m \geq n$, and all $1 \leq i \leq q_m$, $p|(|v_n| + a_{m,i})$. Let the lengths of spacers in each level have up-down gcd d. Then for any $h \in \mathbb{N}$, we can find α and β consisting of v_n-blocks with gap hp for $hp \leq |v_n| + C$ and we cannot get smaller gaps.*

PROOF. First note that if p satisfies the above property for some n, then it satisfies it for all $n' \geq n$. This is because $v_{n'}$ is made up of expected occurrences of v_n followed by spacers of the form $1^{a_{m,i}}$ where $n \leq m < n'$ and then a single expected occurrence of v_n at the end before the level n' spacer. Since $p|\,|v_n| + a_{m,i}$ for each $m \geq n$, we have that $p|\,|v_{n'}| + a_{m,i}$, since that string is composed of component strings which all have length divisible by p. Therefore, for every $n' \geq n$ every index which is a starting point of expected occurrences of $v_{n'}$ will be in the same congruence class modulo p.

Now suppose p is the largest such integer. Then as noted, every index which is a starting point of expected occurrences of $v_{n'}$ will be in the same congruence class modulo p. Suppose we could get a smaller gap, so there is some positive integer p' with $p' < p$ so that we can find α and β' consisting of v_n-blocks with $|\alpha| - |\beta'| = p'$. But since p divides the length of all v_n-blocks, we have that $p|\,|\alpha|$ and $p|\,|\beta'|$. Therefore, $p|\,|\alpha| - |\beta'|$, so $p|p'$. But this is a contradiction since we assumed $p' < p$.

First, note that by Corollary 3.16, we will fix the initial $\frac{d}{p}(|v_n| + C)$ characters to the left of α and β' for our inductive construction and these will be the same regardless of the number of inductive steps necessary to achieve the gap.

We will show that we can get a gap of size hp for any $h \in \mathbb{N}$ with $0 \leq hp \leq |v_n| + C$. First, we examine the spacer parameters and determine some divisibility properties that will help us determine what strings to use to get the necessary gap of size hp.

By Lemma 4.11, we can get all gaps of the form $h'd$ for $h' \in \mathbb{N}$ and in particular for $h'd \leq \frac{d}{p}|v_n| + C$. Also, by Fact 4.10, we have that d is the gcd of the gaps between lengths of spacer parameters, so in particular, we can write the lengths of spacer parameters in the form $\{a+l_1d, a+l_2d, ..., a+l_{q_n}d\}$. Since p is the gcd of the set $\{|v_n|+a_{n,1}, |v_n|+a_{n,2}, ..., |v_n|+a_{n,q_n}\}$, and $p|d$, we can write $\{|v_n|+a_{n,1}, |v_n|+a_{n,2}, ..., |v_n|+a_{n,q_n}\}$ as $\{lp+l_1d, lp+l_2d, ..., lp+l_{q_n}d\}$.

Note that $\gcd(l, \frac{d}{p}) = 1$. Otherwise, if there was some $j > 1$ with $j|l$, $j|\frac{d}{p}$, then jp would divide each $lp + l_id$ as $jp|lp$ and $jp|l_i\frac{d}{p} \cdot p$ regardless of the value of l_i, and therefore, jp would be the gcd of $\{lp + l_1d, lp + l_2d, ..., lp + l_{q_n}d\}$. But this contradicts that p was the gcd of this set.

Therefore, $\gcd(l, \frac{d}{p}) = 1$, so for all h, there is some h' so that $h'\frac{d}{p} = jl + h$ for some integer j. Also, this j is less than $\frac{d}{p}$, since otherwise, we could subtract off $\frac{d}{p}l$ from each side and get a smaller h' that works. By multiplying the previous equation by p, we get a $j < \frac{d}{p}$ so that $h'd = jpl + hp$.

Now suppose we want to get a gap of size hp where $0 \leq hp \leq |v_n| + C$. From the above, we can find a $j < \frac{d}{p}$ so that $jpl + hp = h'd$. We will fix the initial $\frac{d}{p}(|v_n| + C)$ characters to the left of β' will be fixed. Since $j < \frac{d}{p}$ and each v_n-block will have length at most $|v_n| + C$ due to the bound on the spacer parameter, we have that this fixed string before β' will contain the j many v_n-blocks before β'.

Since we can write each $|v_n| + a_{n,i}$ as $lp + l_id$, the j v_n-blocks to the left of β' will have total length $\sum_{g=1}^{j}(pl + l_{i_g}d)$, where each i_g satisfies $1 \leq i_g \leq q_n$. Simplifying this, we get $\sum_{g=1}^{j}(pl + l_{i_g}d) = jpl + \sum_{g=1}^{j} l_{i_g}d$. We will use this quantity to choose appropriate length α and β' to get the necessary gap.

Using Corollary 3.16 and preserving these j many v_n-blocks to the left of β', we find α and β' so that $|\alpha| - |\beta'| = (h' + \sum_{g=1}^{j} l_{i_g})d$. We can find the necessary strings by Lemma 4.11. We claim that using α and the j many v_n-blocks to the left of β' together with β', which together will form the relevant β, we get the necessary gap. By definition of α and β', $|\alpha|-|\beta'| = (h'+\sum_{g=1}^{j} l_{i_g})d = h'd+\sum_{g=1}^{j} l_{i_g}d$. But $h'd = jpl+hp$, so the gap between α and β' is $jpl + hp + \sum_{g=1}^{j} l_{i_g}d$. Comparing the length of β, we include the the j many v_n-blocks to the left of β', so $|\alpha|-|beta|$ will be the previous gap reduced by the length of these blocks, which is $jpl + \sum_{g=1}^{j} l_{i_g}d$, so the gap becomes $jpl + hp + \sum_{g=1}^{j} l_{i_g}d - (jpl + \sum_{g=1}^{j} l_{i_g}d) = hp$. Therefore, we can achieve the gap of hp. □

Combining Proposition 4.3 with Proposition 4.12, we get the following theorem.

THEOREM 4.13. *Let (X,T) have bounded spacer parameter. Then (X,T) has finite maximal equicontinuous factor. Furthermore, the size of this factor can be determined from the parameters for the system, specifically it has size p, where p is the largest integer so that there is some n, so that for all $m \geq n$, and all $i \leq q_m$, $p|(|v_n| + a_{m,i})$.*

Together Theorem 4.13 and Corollary 4.2 show Theorem 1.1, which we restate here.

THEOREM 1.1. *Let (X,T) be a rank one subshift.*

If (X,T) has unbounded spacer parameter, then the maximal equicontinuous factor will be trivial, the one point system.

If (X,T) has bounded spacer parameter, then the maximal equicontinuous factor will be the largest finite factor of the system. In particular, this finite factor has size p, where p is the largest integer so that there is some n, so that for all $m \geq n$, and all $i \leq q_m$, $p|(|v_n| + a_{m,i})$.

CHAPTER 5

MIXING PROPERTIES OF RANK ONE SUBSHIFTS

In this chapter, we will use the machinery and results developed in previous chapters to prove results about the weakly mixing and mixing properties of rank one subshifts. The main goals for this chapter are a complete classification of the weakly mixing and mixing properties for rank one subshifts with bounded spacer parameter, which are Theorem 1.2 and Theorem 1.3 respectively. We will also discuss results about the case of rank one subshifts with unbounded spacer parameter and show why a similar classification as a simple computable result from the spacer parameters is likely impossible to obtain.

5.1. Weakly Mixing Results

First, we will look at the weakly mixing property for rank one subshifts. For bounded rank one subshifts, we can get a complete classification based on the work from the previous chapter and well-known equivalences from previous work.

From a general reference on minimal systems (e.g. [3] Theorem 9.13), we get the following theorem.

THEOREM 5.1. *Let (X, T) be a minimal topological dynamical system with X a metric space which admits an invariant measure λ. Then the following are equivalent:*

- *(X, T) is weakly mixing*
- *(X, T) has no nontrivial equicontinuous factor*

In the previous section, we showed exactly when a rank one subshift has no nontrivial equicontinuous factor. In the case where the spacer parameter is bounded, (X, T) is minimal and admits an invariant measure, so Theorem 5.1 holds.

Therefore, we get the following as an immediate corollary.

COROLLARY 5.2. *Let (X, T) be a rank one subshift with bounded spacer parameter. Then (X, T) is weakly mixing iff $p = 1$ is the largest positive integer p satisfying that for some $n \in \mathbb{N}$, all $m \geq n$, and all $1 \leq i \leq q_m$, we have $p \mid |v_n| + a_{m,i}$.*

This corollary shows that for rank one subshifts with bounded spacer parameter, whether the system is weakly mixing can be computed in a simple way from parameters for the rank one schema.

Since this property is the same as what we showed for having trivial maximal equicontinuous factor, we see that this gives the classification of weakly mixing from the introduction.

THEOREM 1.2. *Let (X, T) be a rank one subshift generated by v_n which has bounded spacer parameter. Let q_n be the cutting parameters and $a_{n,i}$ be the spacer paramters. Then, the following are equivalent:*

(1) *(X, T) is weakly mixing.*
(2) *(X, T) has partition proximality for one point.*
(3) *(X, T) has trivial maximal equicontinuous factor.*
(4) *For all $p \geq 1$ and any $n \in \mathbb{N}$, so that there is some $m \geq n$, and some $i \leq q_m$, $p \not| (|v_n| + a_{m,i})$.*

A system with unbounded spacer parameter is not minimal, and hence we will need to look at the weakly mixing property more directly. To address this case, we recall the translation of the weakly mixing property to a condition on intersections of $E_{v_n,k}$'s.

RECALL. (Fact 3.2) *Let (X, T) be a rank one subshift. Then (X, T) is weakly mixing iff for any $n \in \mathbb{N}$ and any $k_1, k_2 \in \mathbb{Z}$, there exists some $l \in \mathbb{N}$, so that $E_{v_n,-l} \cap E_{v_n,k_1} \neq \emptyset$ and $E_{v_n,-l} \cap E_{v_n,k_2} \neq \emptyset$.*

As in our work on the maximal equicontinuous factor, we can restate this property as a condition on gaps between words in V.

PROPOSITION 5.3. *Let (X, T) be a rank one subshift and V be the rank one word. Then (X, T) is weakly mixing iff for any $n \in \mathbb{N}$, and any $h \in \mathbb{N}$, we can find finite words α, β in V which consist of v_n-blocks, so that $|\alpha| - |\beta| = h$.*

PROOF. Suppose (X, T) is weakly mixing and fix $n \in \mathbb{N}$ and $h \in \mathbb{N}$. Let $k_1 = h$ and $k_2 = 0$ so $k_1 - k_2 = h$. By Fact 3.2, we can find an $l \in \mathbb{N}$ so that $E_{v_n,-l} \cap E_{v_n,k_1} \neq \emptyset$ and

$E_{v_n,-l} \cap E_{v_n,k_2} \neq \emptyset$. Let $x_1 \in E_{v_n,-l} \cap E_{v_n,k_1}$ and $x_2 \in E_{v_n,-l} \cap E_{v_n,k_2}$.

Let $\alpha = x_1[-l, k_1 - 1]$, i.e. α is the word in x_1 starting at index $-l$ and ending just before index k_1. Since $x_1 \in E_{v_n,-l}$, α starts with an expected occurrence of v_n and since $x_1 \in E_{v_n,k_1}$, α ends just before another expected occurrence of v_n, and hence α consists of v_n-blocks.

Similarly, let $\beta = x_2[-l, k_2 - 1]$, i.e. β is the word in x_1 starting at index $-l$ and ending just before index k_2. By the same argument as we stated for α, β consists of v_n-blocks. But then note that $|\beta| = k_2 - 1 + l = l - 1$ and $|\alpha| = k_1 - 1 + l = h + l - 1$, so $|\alpha| - |\beta| = h$.

Now suppose for any $n \in \mathbb{N}$ and any $h \in \mathbb{N}$, we can find finite words α, β in V which consist of v_n-blocks, so that $|\alpha| - |\beta| = h$.

Let $n \in \mathbb{N}$ and $k_1, k_2 \in \mathbb{Z}$. Without loss of generality, assume $k_1 - k_2 > 0$, since if $k_1 = k_2$, the condition is trivial and if $k_1 < k_2$, we can switch the labels. Then let $h = k_1 - k_2$. So we can find finite strings α, β substrings of V which consist of v_n-blocks so that $|\alpha| - |\beta| = h$. Let $l = |\beta| - k_2$. So $|\beta| = k_2 - (-l)$ and β consists of v_n-blocks, therefore by Lemma 3.10, we can find $x_1 \in E_{v_n,k_2} \cap E_{v_n,-l}$.

Also $-l = k_2 - |\beta| = k_1 - (k_1 - k_2) - |\beta| = k_1 - h - |\beta| = k_1 - (|\alpha| - |\beta|) - |\beta| = k_1 - |\alpha|$, so $|\alpha| = k_1 - (-l)$. Since α consists of v_n-blocks, we can invoke Lemma 3.10, so we can find $x_2 \in E_{v_n,k_1} \cap E_{v_n,-l}$.

But then we see that $E_{v_n,-l} \cap E_{v_n,k_1} \neq \emptyset$ and $E_{v_n,-l} \cap E_{v_n,k_2} \neq \emptyset$. Since n, k_1, and k_2 were arbitrary, we have that (X, T) is weakly mixing. □

Therefore, it is enough to show that we can find α and β consisting of v_n-blocks with arbitrary gap between their lengths. The following proposition gives us a situation when this is possible.

Proposition 5.4. *Let (X, T) be a rank one subshift with unbounded spacer parameter. Suppose for arbitrarily large i, i', there are j, j' with $1 \leq j \leq q_i$ and $1 \leq j' \leq q_{i'}$ so that $|a_{i,j} - a_{i',j'}| = 1$. Then (X, T) is weakly mixing.*

Proof. Note that this is the condition from Corollary 4.7. So, for any $h \in \mathbb{N}$, we can find substrings α, β of V consisting of v_n-blocks where $|\alpha| - |\beta| = h$. But this is exactly the condition from Proposition 5.3. $\square$

So this gives a very general case when we can show the weakly mixing property.

Corollary 5.5. *Let (X, T) be a rank one subshift with unbounded spacer parameter. If the spacer parameter forms a subset of $\mathbb{N}$ with measure greater than $\frac{1}{2}$, then (X, T) is weakly mixing.*

Proof. Recall that the measure of A a subset of $\mathbb{N}$ is $\lim_{n\to\infty} \frac{|A \cap \{0, 1, 2, ..., n-1\}|}{n}$, i.e. it is the eventual ratio of the elements in A to the entire set. It is clear that if $A, A' \subseteq \mathbb{N}$ are so that A and A' differ by only finitely many elements and the measure of A' is well-defined, then the measure of A is the same as the measure of A'.

We will work by contrapositive. Let A be set of all spacers from (X, T), so $a \in A$ iff there is some $i \in \mathbb{N}$ and j with $1 \leq j \leq q_i$, so that $a = a_{i,j}$. Suppose (X, T) is not weakly mixing. By Proposition 5.4, there are not arbitrarily many pairs i, i' where there are j, j' with $|a_{i,j} - a_{i',j'}| = 1$. Therefore, we can find some n so that for any $i, i' > n$ and any j, j', we have $|a_{i,j} - a_{i',j'}| \neq 1$.

Let A' be the set of all spacers from (X, T) which occur at a level at least n, i.e. $a \in A'$ iff there is some $i \in \mathbb{N}$, $i \geq n$, and j with $1 \leq j \leq q_i$ so that $a = a_{i,j}$. We claim that there are no $a, b \in A'$ so that $|a - b| = 1$, since otherwise, the i, i' witnessing that $a = a_{i,j}$ and $b = a_{i',j'}$ would contradict that no such pairs occur above level n. But since there are no $a, b \in A'$ so that $|a - b| = 1$, we see that for any $a \in \mathbb{N}$, either $a \notin A'$ or $a + 1 \notin A'$. Therefore, if the measure of A' is well-defined, then A' must have measure at most $\frac{1}{2}$.

Also, note that A' and A differ by at most finitely many elements, since the number of spacers that occur at each level is finite, and there are only finitely many levels strictly below n. Therefore, if A has well-defined measure, then A has measure at most $\frac{1}{2}$. In either case, it is not the case that A has measure greater than $\frac{1}{2}$, which proves the contrapositive. $\square$

Next, we give the following example that shows that this measure requirement is strict, so there are systems where the spacer set contains a set of measure $\frac{1}{2}$, but are not weakly mixing.

Example 5.6. Let $v_0 = 00$. For each n, we let $v_{n+1} = v_n v_n 11 v_n v_n 1111 v_n v_n 111111 v_n v_n ... v_n v_n 1^{2|v_n|} v_n$. So the spacers in v_{n+1} have length $2i$ for each $0 \leq i \leq |v_n|$.

It will be useful later to note that this construction maintains that there are an odd number of occurrences of v_n at each level.

Proposition 5.3 gives that system is not weakly mixing as every spacer parameter has even length and each v_n has even length, so no gaps of odd length are possible.

However, note that the spacer set is exactly the even integers, which has measure $\frac{1}{2}$.

5.2. Results on Mixing

Similarly to above, we look at the mixing property. We recall the translation of the mixing property to a condition on the $E_{v_n,k}$'s.

Recall. (Fact 3.4) *Let (X,T) be a rank one subshift. Then (X,T) is mixing iff for any n and any $k \in \mathbb{Z}$, we can find an $L \in \mathbb{N}$ so that for any $l \geq L$, $E_{v_n,-l} \cap E_{v_n,k} \neq \emptyset$.*

Similarly to above, we can translate this into a property on strings from V that consist of v_n-blocks.

Proposition 5.7. *Let (X,T) be a rank one subshift generated by infinite rank one word V. Then (X,T) is mixing iff for any n, we can find an $H \in \mathbb{N}$ so that for any $h \geq H$, there is a string α which consists of v_n-blocks and is substring of V, so that $|\alpha| = h$.*

Proof. Suppose (X,T) is mixing. By Fact 3.4, for any n and any k, we can find $L \in \mathbb{N}$ so that for any $l \geq L$, $E_{v_n,-l} \cap E_{v_n,k} \neq \emptyset$.

Fix n arbitrary. Let $k = 0$. Then we can find an L as above. Let $H = L$. Let $h \geq H$ be arbitrary. Since $H = L$, $h \geq L$ and letting $l = h$, we get that $E_{v_n,-h} \cap E_{v_n,0} \neq \emptyset$. Let $x \in E_{v_n,-h} \cap E_{v_n,0}$. Let $\alpha = x[-h,-1]$. Note that $|\alpha| = h$. Since α starts at index $-h = -l$ in x and $x \in E_{v_n,-l}$, there is an expected occurrence of v_n starting at index $-h$ in x, so α starts with an expected occurrence of v_n. Also, since α ends at index -1 in x, and $x \in E_{v_n,0}$, α finishes the index before another expected occurrence of v_n and hence contains the entire

v_n-block before index 0. Therefore, α consists of v_n-blocks. So α witnesses that we can find the necessary string of length h. Since $h \geq H$ was arbitrary, we have proved the forward direction.

Now suppose for any n, we can find an $H \in \mathbb{N}$ so that for any $h \geq H$, there is a string α which consists of v_n-blocks and is substring of V, so that $|\alpha| = h$.

Fix n arbitrary and let $k \in \mathbb{Z}$ be arbitrary. By assumption, we can find the relevant H, so let $L = \max\{0, H - k\}$. This ensures that $L \in \mathbb{N}$. Let $l \geq L$. Let h be the length of the string from index $-l$ to index $k - 1$. Since $l \geq L \geq H - k$, we have that is $h = k - (-l) = k + l \geq k + H - k = H$. So we can find α consisting of v_n-blocks with $|\alpha| = h$.

Since α consists of v_n-blocks, and $k - (-l) = h = |\alpha|$, we can invoke Lemma 3.10 and we get $x \in E_{v_n,-l} \cap E_{v_n,k}$, so $E_{v_n,-l} \cap E_{v_n,k} \neq \emptyset$.

Since $l \geq L$ was arbitrary, this shows that we can find such a string for any $l \geq L$ and so we get that the choice of L works for n. Since n was arbitrary, this proves that (X, T) is mixing. ☐

This shows that the mixing property can be defined entirely in terms of what lengths are possible for strings consisting of v_n-blocks.

Now, we completely classify the mixing property on bounded rank one subshifts. Specifically we will show the theorem from the introduction.

THEOREM 1.3. *Let (X, T) be a rank one subshift with bounded spacer parameter. Then (X, T) is never mixing.*

PROOF. Let C be the bound on the spacer parameter.

Let n be large enough so that $|v_n| \gg C$, in particular $|v_n| > 3C + 2$ works. Let $V = E_{v_n,0}$ and let $U = E_{v_n,-(C+1)}$. Let $L \in \mathbb{N}$ be arbitrary. We will show that there is some $l \geq L$, so that $T^l(U) \cap V = \emptyset$.

Let $m \geq n$ be such that $l = |v_m| \geq L$. Consider $T^l(U) \cap V = E_{v_n,-l-(C+1)} \cap E_{v_n,0}$. But $-l - (C + 1) = -(l + C + 1) = -(|v_m| + C + 1)$. So $E_{v_n,-l-(C+1)} \cap E_{v_n,0}$ is empty iff $E_{v_n,0} \cap E_{v_n,|v_m|+C+1}$ is empty. By definition, $C + 1 < |v_n|$. Therefore $|v_m| + C + 1$ is of the form $N + d$, where $N = |v_m|$ for some m and $C + 1 = d < |v_n|$. Therefore, by Corollary 3.18, we have $E_{v_n,0} \cap E_{v_n,|v_m|+C+1} \neq \emptyset$ only when $d < C$. But $d = C + 1$, so we would need $C + 1 < C$, which is impossible. Therefore, $E_{v_n,0} \cap E_{v_n,|v_m|+C+1} = \emptyset$, so $T^l(U) \cap V = \emptyset$, and we see that (X, T) cannot be mixing. □

Notice that we strongly used the bound on the spacer parameter to get the previous result. For systems with unbounded spacer parameter, we do not have such a bound and hence can have very different behaviour. In particular, it is not hard to construct a mixing subshift.

EXAMPLE 5.8. Let $v_0 = 0$, $v_{n+1} = v_n v_n 1 v_n 11 v_n 111 v_n ... v_n 1^{|v_n|} v_n$. So the spacers in v_{n+1} have each length i where $0 \leq i \leq |v_n|$. Then this subshift is mixing. Fix n. By construction, we can find spacer parameters of any length, so letting $N = |v_n|$ for any $m > N$, we can find a spacer parameter of length $m - N$. Then an occurrence of v_n, followed by this spacer parameter witnesses the necessary α.

This generalises to the following proposition.

PROPOSITION 5.9. *Suppose (X, T) is a rank one subshift and the set of spacer parameters contains a tail of $\mathbb{N}$, i.e. if A denotes the set of spacer parameters, we can find some M so that for all $m > M$, $m \in A$.*

Then (X, T) is mixing.

PROOF. Let (X, T) and A be as above. Fix n. Let $L = |v_n| + M$. Then for any $l \geq L$, $l = |v_n| + M + m'$ where $m' > 0$. By the condition on A, there is a spacer parameter of length $M + m'$, so an occurrence of v_n followed by this spacer works. □

We can improve this to a small set of missing spacers as follows.

PROPOSITION 5.10. *Let (X, T) be a rank one subshift and let A be the set of lengths of spacers. Let $(d_i)_{i=1}^{\infty}$ be the set of differences between consecutive elements of $\mathbb{N} \backslash A$, i.e. if $\mathbb{N} \backslash A$ is enumerated in an increasing manner by $\{b_1, b_2, ...\}$, then $d_i = b_{i+1} - b_i$.*

If $\liminf (d_i)_{i \in \mathbb{N}} \to \infty$, then (X, T) is mixing.

Proof. Fix v_n. Let A_n be the set of lengths of spacers that occur only in levels at least n, i.e. $A_n = \{a_{i,j} : i \geq n, 1 \leq j \leq q_i\}$. Note that A_n differs from A by only a finite set, since only finitely many spacers occur at levels below n. So we can find some N so that for all $a \geq N$, $a \in A$ iff $a \in A_n$. Therefore, by only considering elements larger than N, we get that A_n and A agree, and so without loss of generality, we will assume $A = A_n$.

Since $\liminf(d_i)_{i\in\mathbb{N}} \to \infty$, we can find some M so that for all $i > M$, $d_i > |v_n| + a_{n,q_n}$. Let $M' > b_{M+1}$. We note that all of the elements in $\mathbb{N}\backslash A$ which are larger than M' cannot be within $|v_n| + a_{n,q_n}$ of each other, since if $b \in \mathbb{N}\backslash A$ and $b > M'$, then $b = b_i$ for some $i > M$, and by choice of M, the differences must be at least $|v_n| + a_{n,q_n}$.

We will let $L = |v_n| + M'$. Let $l \geq L$ be arbitrary. If $l - |v_n| \in A$, then we can take an occurrence of v_n followed by $l - |v_n|$ 1's. Otherwise, since $l \geq L$, the previous $|v_n| + a_{n,q_n}$ integers before $l - |v_n|$ are all in A. Then we can take an occurrence of v_n at the end of a v_{n+1}, the v_n and a_{n,q_n} before it and the spacer of length $l - 2|v_n| - a_{n,q_n}$. This spacer length is in A because it is within $|v_n| + a_{n,q_n}$ of $l - |v_n|$. Also note that if $n' > n$, a_{n,q_n} will be the value of the level n spacer immediately before a spacer from level n'.

Therefore, the relevant constructed string will consist of a v_n followed by a spacer of length a_{n,q_n}, followed by another v_n and the spacer of length $l - 2|v_n| - a_{n,q_n}$. Hence we get a gap of $|v_n| + a_{n,q_n} + |v_n| + l - 2|v_n| - a_{n,q_n} = l$ and so we can get every gap $l \geq L$. □

Working in a different direction, we can get the following:

Proposition 5.11. *Let (X, T) be a rank one subshift and let A be the set of spacer parameters. Suppose that A contains an arithmetic sequence $ci + d$ for all $i \in \mathbb{N}$, $c > 0, d \geq 0$ and for every n, there are $v_{m_1}, ..., v_{m_c}$ with each $m_j \geq n$ and $|v_{m_j}| \equiv j \bmod c$.*

Then (X, T) is mixing.

Proof. Fix n. Let v_{m_j} for $1 \leq j \leq c$ be as above. Let L be so that all terms $ci + d > \max\{|v_{m_j}| : 1 \leq j \leq c\} + L$ occur as $a_{n',j'}$ where $n' > \max\{m_j : 1 \leq j \leq c\}$. Let $l \geq L$. Then $l - d \equiv |v_{m_j}| \bmod c$ for some j. So $l - d = ci + |v_{m_j}|$ for some i, so $l = ci + d + |v_{m_j}|$. Since $l \geq L$, $ci + d \in A$ and occurs as $a_{n',j'}$ for $n' > m_j$.

Then we get a gap l by looking at an occurrence of v_{m_j} followed by a spacer of size $ci + d$. By the choice of L, such a combination occurs and since $m_j \geq n$, we have that v_{m_j} starts and ends with an occurrence of v_n, so this gives us the necessary gaps between v_n occurrences. □

Note that the proof would have worked with any final segment of some v_m which started with an expected occurrence of v_n where $m > n$, not just entire v_m's. So we get the following corollary.

COROLLARY 5.12. *Let (X, T) be a rank one subshift and let A be the set of spacer parameters. Suppose that A contains an arithmetic sequence $ci + d$ for all $i \in \mathbb{N}$, $c > 0, d \geq 0$ and for every n, there are $w_1, ..., w_c$ where each w_j is a final segment of some v_{m_j} with $m_j \geq n$ and w_j starts with an expected occurrence of v_n (within v_m), and $|w_j| \equiv j$ mod c.*

Then (X, T) is mixing.

However, not all rank one subshifts where A contains an arithmetic sequence satisfy this condition. It is not hard to construct systems with the same spacer set A, where A contains an arithmetic sequence, but the resulting systems differ on whether they can satisfy mixing properties.

EXAMPLE 5.13. Recall the system defined in Example 5.6.

Let $v_0 = 00$. For each n, we let $v_{n+1} = v_n v_n 11 v_n v_n 1111 v_n v_n ... v_n v_n 1^{2|v_n|} v_n$. Note that this construction maintains that there are an odd number of occurrences of v_n at each level.

This system is not weakly mixing, since no gaps which are not even are possible. This system is not mixing for the same reason.

We will define a variant of the system above, which will have vastly different mixing properties. For the new system, use the same spacer and cut parameters as above except let $v_0 = 0$. Since there are an odd number of v_n's in each v_{n+1} and $|v_0|$ is odd, we see that $|v_n|$ will be odd for every n. It is not hard to see that the system is mixing.

For odd gaps, note that taking a large enough n and then a single v_n followed by the large enough spacer will give the necessary gap. For even gaps, take two consecutive v_n's followed by the necessary spacer.

We can generalise this construction to get arbitrarily small measure spacer sets A which nevertheless exhibit this sensitive behaviour.

EXAMPLE 5.14. Fix $p \geq 2$.

For the first system, let $v_0 = 0^p$. For each n, we let $v_{n+1} = v_n v_n 1^p v_n 1^{2p} v_n 1^{3p} v_n ... v_n 1^{(|v_n|-1)p} v_n$. Since $|v_0| = p$ and every spacer has length divisible by p, we have each $|v_n|$ is divisible by p.

Note that all but the final v_n are immediately followed by a spacer of length ip for each $0 \leq i \leq |v_n| - 1$, so there are $|v_n| + 1$ many occurrences of v_n in each v_{n+1}. But since each $|v_n|$ is divisible by p, this construction maintains that the number of occurrences of v_n in v_{n+1} is congruent to 1 modulo p.

This system is not even weakly mixing because v_0 has length divisible by p and every spacer has length divisible by p, so every v_n has length divisible by p. Since v_n-blocks consist of occurrences of v_n together with spacers, every v_n-block will have length divisible by p and so we can never get lengths which are not divisible by p.

For the second system, use the same spacer and cut parameters, but let $v_0 = 0$. We claim that for every n, $|v_n| \equiv 1 \bmod p$. This clearly holds for 0, and inductively if it holds for v_n, the number of occurrences of v_n in v_{n+1} is congruent to 1 modulo p, so the length of all the v_n's in v_{n+1} will sum to a multiple of p plus 1, and all the spacers have length divisible by p, therefore $|v_{n+1}| \equiv 1 \bmod p$.

To get an arbitrary large enough length l, look at $j \equiv l \bmod p$, where $0 < j \leq p$. Take $j-1$ consecutive occurrences of v_n, which by taking them from the front of a v_{n+1}-block, we can guarantee this has length no more than $(p-1)|v_n| + \frac{p(p-1)(p-2)}{2}$. Then to the front of this string, put an occurrence of v_n followed by the spacer of appropriate length to match l.

This will contain j occurrences of v_n, so the v_n's will contribute length which is in congruent class $j \bmod p$. Since the spacer set contains all the multiples of p, we can find the necessary spacer to match l and we get the the system is mixing (and hence also weakly mixing).

Hence, we see that the mixing properties for unbounded rank one subshifts are particularly sensitive to the parameters and suggests that a complete categorisation in terms of the spacer parameters and/or the measure of the spacer set may not be feasible.

CHAPTER 6

RANK ONE FACTORS OF RANK ONE SUBSHIFTS

In this section, we will prove results about rank one factors of rank one subshifts. Specifically, we will prove Theorem 1.4 from the introduction.

6.1. Sliding Block Codes

First, we will show some results for factors of rank one subshifts that happen to be shifts. As a general result, we will show that any such factor will be defined by some sliding block code. Such a result can be found in a general reference on symbolic dynamics such as [14].

DEFINITION 6.1. Let $(X,T),(Y,S)$ be subshifts, i.e. $X,Y \subseteq 2^{\mathbb{Z}}$ and T,S are the shift maps on X,Y respectively. Then a *sliding block code* between X and Y is an onto map $\varphi : X \to Y$, which is defined by $N \in \mathbb{N}$, $m \in \mathbb{Z}$ and a finite set of strings s_0 of length N, so that for all $l \in \mathbb{Z}$ and $x \in X$, $\varphi(x)(l) = 0$ iff $x[m+l, m+l+N-1] \in s_0$ and $\varphi(x)(l) = 1$ iff $x[m+l, m+l+N-1] \notin s_0$.

We will call N the *block length* and m the *offset*.

Note that φ acts on strings of length N which are substrings of some $x \in X$. We will call these strings *blocks*.

Intuitively, a sliding block code is a map where the image of x is defined by looking at each substring of x with fixed length and checking whether that finite set of strings codes a 0 or a 1.

We have the following proposition about factors of subshifts.

PROPOSITION 6.2. *Let (X,T) and (Y,S) be subshifts. Then if (Y,S) is a factor of (X,T) witnessed by φ, then φ is a sliding block code.*

PROOF. Let (X,T) and (Y,S) be subshifts with (Y,S) a factor of (X,T) witnessed by $\varphi : X \to Y$. Then by definition of a factor map, φ is continuous and onto and φ commutes with the shift.

Note that since (X, T) and (Y, S) are subshifts, X, Y are subspaces of $2^{\mathbb{Z}}$ with the product topology, so basic open sets are generated by sets of the form $U_{\alpha,k}$ where $\alpha \in 2^{<\omega}$ and $k \in \mathbb{Z}$. For this proof, we will denote the basic open sets of X by $U_{\alpha,k}$ and the basic clopen sets of Y by $V_{\alpha,k}$. Also, note that $2^{\mathbb{Z}}$ is compact and Hausdorff, so X and Y are also compact and Hausdorff.

So consider $V_{0,0} \subseteq Y$. Note that this is a clopen set in Y, so $\varphi^{-1}(V_{0,0})$ is clopen in X. Since $\varphi^{-1}(V_{0,0})$ is open, we can write $\varphi^{-1}(V_{0,0})$ as a union of basic open sets of the form $U_{\alpha,k}$. Since $\varphi^{-1}(V_{0,0})$ is closed and X is compact and Hausdorff, we have that $\varphi^{-1}(V_{0,0})$ is compact and hence there is a finite cover of basic open sets. Let $\bigcup_{i=1}^{M} U_{\alpha_i,k_i}$ be the cover of $\varphi^{-1}(V_{0,0})$.

Note that if x has any α_i starting at k_i, then $\varphi(x)(0) = 0$. Furthermore, there are only finitely many of the α_i and k_i, so we can find $m = \min_i\{k_i\}$ and $m' = \max\{|\alpha_i|+k_i\}$ and if $x \in U_{\alpha_i,k_i}$ for some $1 \leq i \leq M$, then the occurrence of α_i starting at k_i will be contained in the interval $x[m, m'-1]$.

We will define $s_0 = \{\beta \in 2^{m'-m} : \beta[k_i - m, k_i - m + |\alpha_i| - 1] = \alpha_i \text{ for some } i \leq M\}$. We claim that the sliding block code generated by s_0 is the map φ.

First, note that for any $l \in \mathbb{Z}$, $\varphi^{-1}(V_{0,l}) = \varphi^{-1}(S^{-l}(V_{0,0})) = T^{-l}(\varphi^{-1}(V_{0,0})) = T^{-l}(\bigcup_{i=1}^{M} U_{\alpha_i,k_i}) = \bigcup_{i=1}^{M} T^{-l}(U_{\alpha_i,k_i}) = \bigcup_{i=1}^{M} U_{\alpha_i,k_i+l}$. Therefore, any occurrence of 0 in $\varphi(x)$ will come from the shift of one of the strings which generates our sliding block code.

Next, we note that this sliding block code appropriately matches φ. By a similar argument to the previous section, we can find a finite cover $\bigcup_{i=1}^{M'} U_{\beta_i,l_i} = \varphi^{-1}(V_{1,0})$. Note that since $V_{1,0}$ and $V_{0,0}$ are disjoint, we have $\bigcup_{i=1}^{M'} U_{\beta_i,l_i}$ and $\bigcup_{i=1}^{M} U_{\alpha_i,k_i}$ must be disjoint. So we can never have a string from s_0 starting at m and simultaneously have β_i starting at l_i. So if some $x \in X$ has $\varphi(x)(0) = 1$, then we cannot have a string from s_0 starting at m in x.

Also since $V_{1,0}$ and $V_{0,0}$ cover X, we have $\bigcup_{i=1}^{M'} U_{\beta_i,l_i}$ and $\bigcup_{i=1}^{M} U_{\alpha_i,k_i}$ cover the space, so in particular, if there is some $x \in X$ so that no α_i occurs starting at k_i in x, then there must be some β_i starting at l_i in x. Since strings from s_0 starting at m constitute all possible strings which have an occurrence of α_i starting at k_i, we have that for any $x \in X$, if x does not have a string from s_0 starting at m, then x must have some β_i starting at l_i and hence $\varphi(x)(0) = 1$. Combining this with the previous remark, we see that $\varphi(x)(0) = 0$ iff there is a string from s_0 starting at m. By commutativity of the shift, we get the similar property for any $l \in \mathbb{Z}$, that $\varphi(x)(l) = 0$ iff there is a string from s_0 starting at $m + l$. But letting $N = m' - m$, we get that s_0 generates the necessary sliding block code for φ. □

So we see that every factor map from a shift space to another shift space will be a sliding block code. In particular, each factor of a rank one subshift that is itself a subshift will come from a sliding block code. Next, we will look at what the rank one structure can tell us about the sliding block code.

PROPOSITION 6.3. *Let (X, T) be a rank one subshift generated by v_n for $n \in \mathbb{N}$. Let (Y, S) be a factor of (X, T) which is also a subshift. Then there is some $N \in \mathbb{N}$ and $m \in \mathbb{Z}$, so that for any n such that $|v_n| > N$, there is a string α_n with length $|v_n| - N$, so that for any $k \in \mathbb{Z}$ and any $x \in X$, $x \in E_{v_n,k}$ implies that $\varphi(x)$ has an occurrence of α_n starting at index $k - m$.*

Furthermore, if $n \leq n'$, then α_n is an initial segment of $\alpha_{n'}$ and α_n is a final segment of $\alpha_{n'}$.

The basic intuition behind this proposition is that when there is a sliding block code, each v_n determines a string of characters in the factor, and the number of characters determined increases with the length of the v_n. This means that if we know where all the v_n's are in some point of the original system, we have a great deal of information about what the image must be in the factor.

PROOF. Let (X, T) be a rank one subshift and (Y, S) be a factor of (X, T) which is also a shift. By Proposition 6.2, we have that this factor map is witnessed by a sliding block code, hence there is an $N \in \mathbb{N}$, an $m \in \mathbb{Z}$, and a finite set s_0 consisting of strings of length N, so that for all $l \in \mathbb{Z}$ $\varphi(x)(l) = 0$ iff $x[m + l, m + l + N - 1] \in s_0$.

Fix n with $|v_n| > N$. Let $x \in E_{v_n,k}$. Since x has an occurrence of v_n starting at index k, $x[k, k+|v_n|-1]$ is determined. But then $x[k, k+N-1]$, $x[k+1, k+N]$, ... up to $x[k+|v_n|-N, k+|v_n|-1]$ are strings of length N which are determined and hence we can check whether or not these will be in s_0. Therefore, there is a fixed string α_n, induced by these N character substrings of v_n, that will appear in the image under the factor map. By definition of the sliding block code, the strings $x[k, k+N-1]$, $x[k+1, k+N]$, ... up to $x[k+|v_n|-N, k+|v_n|-1]$ will determine $\varphi(x)(k-m)$, $\varphi(x)(k-m+1)$,...,$\varphi(k+|v_n|-N-m)$ respectively, hence $\varphi(x)[k-m, k-m+|v_n|-N] = \alpha_n$. Since k was arbitrary, we get that this holds regardless of the index k. Also, since n was arbitrary, we have that the N and m work for any n.

Now let $n' \geq n$ and let $x \in E_{v_{n'},k}$. By the above, we get that $\varphi(x)$ has an occurrence of $\alpha_{n'}$ starting at index $k-m$ and $|\alpha_{n'}| = |v_{n'}| - N$. Since $v_{n'}$ contains v_n as an initial segment, we see that an occurrence of $x \in E_{v_{n'},k}$ implies that $x \in E_{v_n,k}$, and so we have that $\varphi(x)$ has an occurrence of α_n starting at index $k-m$ and $|\alpha_n| = |v_n| - N$. But therefore, α_n starts at the same index as $\alpha_{n'}$ and $|\alpha_n| \leq |\alpha_{n'}|$, so α_n is an initial segment of $\alpha_{n'}$. Following a similar argument to above, v_n is a final segment of $v_{n'}$, so the occurrence of α_n coming from this v_n at the end of $v_{n'}$ will be a final segment of $\alpha_{n'}$. □

6.2. Rank One Factors of Rank One Subshifts

We will use Proposition 6.3 to analyse factors of rank one subshifts which happen to be rank one.

For this section, we will let (X, T) be a rank one subshift generated by v_n and (Y, S) be a rank one subshift generated by u_n. We will let (Y, S) be a factor of (X, T) witnessed by φ. We will also assume that Y is not finite.

First, we see that rank one factors of bounded rank one subshifts will be bounded and similarly, that rank one factors of unbounded rank one subshifts will be unbounded.

Fact 6.4. *Let $(X, T), (Y, S)$ be as above. Suppose (X, T) has a bounded spacer parameter. Then either (Y, S) has bounded spacer parameter or (Y, S) is trivial and $|Y| = 1$.*

Proof. Assume Y does not have bounded spacer parameter, and hence has unbounded spacer parameter. Then we know $1^{\mathbb{Z}} \in Y$. Since (Y, S) is a factor of (X, T), there is some $x \in X$, so that $\varphi(x) = 1^{\mathbb{Z}}$. Then $\varphi(T^m(x)) = S^m(\varphi(x)) = S^m(1^{\mathbb{Z}}) = 1^{\mathbb{Z}}$ for any $m \in \mathbb{Z}$. But $\{T^m(x) : m \in \mathbb{Z}\}$ is dense in X and hence its image must be dense in Y. But $\varphi(\{T^m(x) : m \in \mathbb{Z}\}) = \{1^{\mathbb{Z}}\}$, so $\{1^{\mathbb{Z}}\}$ is dense in Y. But the only way this is possible is if Y itself is a singleton. $\square$

FACT 6.5. *Let (X, T) and (Y, S) be as above. Suppose (X, T) has unbounded spacer parameter and suppose Y is infinite. Then (Y, S) must have unbounded spacer parameter.*

PROOF. Note that $1^{\mathbb{Z}} \in X$ is a fixed point under T, so $\varphi(1^{\mathbb{Z}})$ must be a fixed point under S. But an infinite rank one subshift has a fixed point iff it has unbounded spacer parameter. $\square$

So we see that if (X, T) has bounded spacer parameter, then the factor (Y, S) must also have bounded spacer, and similarly, if (X, T) has unbounded spacer parameter, then the factor (Y, S) must also have unbounded spacer parameter.

One of the main tools we will use is a method for proving when a factor map is one-to-one.

LEMMA 6.6. *Let (X, T) be a rank one subshift and let (Y, S) be an infinite rank one subshift which is a factor of (X, T) as witnessed by sliding block code φ. Since (X, T) and (Y, S) are rank one subshifts, we will let v_n generate X and u_n generate Y.*

For any n, suppose that for some n', there is a string β and an $m \in \mathbb{Z}$ so that for any $x \in X$, if x has an occurrence of $v_{n'}$ starting at index k, then $\varphi(x)$ has an occurrence of β starting at index $k - m$. Also, suppose β starts and ends with expected occurrences of u_n and that for any $x \in X$, any expected occurrence of u_n in $\varphi(x)$ is contained in exactly one β which comes from an expected occurrence of $v_{n'}$ in x.

Let $x_0 \in E_{v_{n'},k_0}$ and $x_1 \in E_{v_{n'},k_1}$ where $k_0 \neq k_1$ and $|k_0 - k_1| < |v_{n'}|$. Then $\varphi(x_0) \neq \varphi(x_1)$.

PROOF. Let (X, T), (Y, S), φ, and β be as above. Fix n. We will let q be the number of expected occurrences of u_n in β.

We break this proof into two parts. First, we show that we can assign a label from a congruence class modulo q to each occurrence of u_n according to the position within β. We will show that the spacers are determined by these congruence classes and that the spacers determined by $q-1$ of these congruence classes are constant. Then we will use the presence of these constant classes to show that the relevant points must have different images under φ.

We know β starts and ends with expected occurrences of u_n and β contains q many such occurrences. So $\beta = u_n 1^{a_1} u_n 1^{a_2} u_n ... u_n 1^{a_{q-1}} u_n$. Since every occurrence of u_n comes from some β which is induced by an expected occurrence of $v_{n'}$, we see that every $y \in Y$ will be of the form $...\beta 1^{b_{i-1}} \beta 1^{b_i} \beta 1^{b_{i+1}} \beta...$ i.e., each point $y \in Y$ is alternating occurrences of β with strings of 1's. (Note that we do not require the strings of 1's to have positive length). Also, since φ is an onto map, we have that for any $y \in Y$, there is some $x \in X$ so that $\varphi(x) = y$, and in this case, the occurrences of β witnessing that y has the form $...\beta 1^{b_{i-1}} \beta 1^{b_i} \beta 1^{b_{i+1}} \beta...$ come from expected occurrences of $v_{n'}$ in x.

So given $y \in Y$, we can find some $x \in X$ so that $\varphi(x) = y$, and we see that for all $k \in \mathbb{Z}$, $x \in E_{v_{n'},k}$ iff $\varphi(x)$ has an occurrence of β starting at index $k - m$.

Now we can define a labelling on occurrences of u_n. Let $y \in Y$. Then there is some $x \in X$ so that $\varphi(x) = y$ and for any $k \in \mathbb{Z}$, $x \in E_{v_{n'},k}$ implies that there is an occurrence of β starting at index $k-m$ in y. Furthermore, each u_n in y comes from exactly one occurrence of such a β. (For the rest of this proof, we will refer to one of these β which is induced by an expected $v_{n'}$ when we refer to an occurrence of β. We will not be considering occurrences of β which come from unexpected occurrences of $v_{n'}$ or occurrences of β which occur in an "unexpected" manner from overlapping "expected" occurrences.) Therefore, each u_n comes from a unique such occurrence of β and so there is a unique position for each u_n within β. We will label each occurrence of u_n in β according to this position within β, starting from 0 and increasing to $q-1$. In particular, for each i where $1 \leq i \leq q-1$, we have that each u_n labelled i will be immediately preceded by the spacer 1^{a_i}. Furthermore, note that the only u_n's which can have different spacers in front of them are those with the 0 label, as all others have the spacers preceding them in β and since β is a fixed string, the spacer preceding these u_n's will be determined by the label.

Note that given some $x \in X$ so that $\varphi(x) = y$, such a labelling on y is well-defined, as the expected occurrences of $v_{n'}$ in x induce non-overlapping occurrences of β in y so that each u_n is contained in exactly one occurrence of β. (Note that, at this point, we are not ruling out the possibility that two different $x_0 \neq x_1 \in X$ with $\varphi(x_0) = y = \varphi(x_1)$ could induce different non-overlapping occurrences of β that make up the same string y.)

We also note that this labelling is picking a particular u_n which will be 0 and counting u_n's modulo q. This is because the labels come from counting u_n's within non-overlapping occurrences of β, which each contain q many occurrences of u_n, and all occurrences of u_n are accounted for within one of these occurrences of β. Therefore, it makes sense to refer to the labels as congruence classes and note that the spacers preceding occurrences of u_n from congruence classes $i \not\equiv 0 \bmod q$ will be constant.

Now let $x_0 \neq x_1 \in X$ be so that there are $k_0, k_1 \in \mathbb{Z}$ with $|k_0 - k_1| < |v_{n'}|$ such that $x_0 \in E_{v_{n'},k_0}$ and $x_1 \in E_{v_{n'},k_1}$.

Now suppose by contradiction, $\varphi(x_0) = y = \varphi(x_1)$. We will consider the congruence classes on y induced by x_0 and x_1. Note that these are not assumed to be the same, hence we will differentiate between the two types of congruence classes. We will denote the congruence class labelled i from x_0 by i_{x_0} and the congruence class labelled i from x_1 as i_{x_1}.

Since $x_0 \in E_{v_{n'},k_0}$, $\varphi(x_0)$ has an occurrence of β starting at index $k_0 - m$, hence a u_n labelled 0_{x_0} starts at $k_0 - m$. Since each occurrence of β in $\varphi(x_0)$ comes from an occurrence of $v_{n'}$ in x_0 and occurrences of $v_{n'}$ cannot overlap in x_0, we have that each β must start at least $|v_{n'}|$ characters away from each other. Therefore, any two occurrences of u_n which are in the same x_0 congruence class must start at least $|v_{n'}|$ characters away. In particular, any u_n which is in congruence class 0_{x_0} must start at least $|v_{n'}|$ characters away from $k_0 - m$.

Since $x_1 \in E_{v_{n'},k_1}$, $\varphi(x_1)$ has an occurrence of β starting at index $k_1 - m$, so the occurrence of u_n starting at $k_1 - m$ will be in congruence class 0_{x_1}. Since $|k_0 - k_1| < |v_{n'}|$, these $k_0 - m$ and $k_1 - m$ are fewer than $|v_{n'}|$ characters away from each other. Recall that the occurrence of u_n starting at $k_0 - m$ was in congruence class 0_{x_0}. Since $k_1 - m$ and $k_0 - m$ are fewer than $|v_{n'}|$ characters away from each other, the u_n starting at $k_1 - m$ and the u_n starting at $k_0 - m$ cannot be in the same x_0 congruence class.

Let i_{x_0} be the congruence class for the occurrence of u_n starting at $k_1 - m$. Note that this i_{x_0} cannot be 0_{x_0}. Since this u_n is not in the congruence class 0_{x_0}, we have that the spacer preceding this u_n must be 1^{a_i}. Furthermore, the spacer preceding any u_n from class i_{x_0} must be 1^{a_i}. Therefore, the spacer preceding this class will be constant.

But because the u_n starting at $k_1 - m$ is in congruence class 0_{x_1}, all other congruence classes in $y = \varphi(x_1)$ will have constant preceding spacers. Therefore, the definition of the congruence classes from x_1 witnesses that y is of the form $...\beta 1^{a_i} \beta 1^{a_i} \beta 1^{a_i} \beta...$ which is a periodic point which contains an occurrence of 0. But (Y, S) is a rank one subshift, so the only way there can be such a periodic point in (Y, S) is if Y is finite. But we assumed Y was infinite, so this is impossible. Therefore, we cannot have $\varphi(x_0) = \varphi(x_1)$. □

This lemma is particularly useful for characterising rank one factors of rank one subshifts in light of the following topological fact.

FACT 6.7. *Let (X, T) and (Y, S) be topological dynamical systems and let φ witness that (Y, S) is a factor of (X, T).*

Then if φ is one-to-one, then φ is an isomorphism.

PROOF. Note that X and Y will be compact metric spaces and φ will be a homomorphism and is also continuous and onto. Since we assume φ is one-to-one, we see that it is also bijective, so φ^{-1} is defined. If φ^{-1} is continuous, that means for any open $U \subseteq X$, $(\varphi^{-1})^{-1}(U)$ must be open. So to show that φ is an isomorphism, we just need to show that φ is an open map.

Let F be a closed set in X. Since F is closed in X and X is compact, F is compact. Since φ is continuous, $\varphi(F)$ will be compact, and as Y is a compact metric space, compact subsets will be closed, so $\varphi(F)$ is compact. Therefore, φ is a closed map. Now let $U \in X$ be open. Then $X \backslash U$ is closed in X. Since φ is a bijection, $\varphi(X \backslash U) = \varphi(X) \backslash \varphi(U) = Y \backslash \varphi(U)$.

As φ is a closed map, and $X \backslash U$ is closed, $Y \backslash \varphi(U)$ is closed. But then $\varphi(U)$ is open, hence φ is an open map. □

So if we can show that a factor map must be one-to-one, then we can show that any such factor would have to be isomorphic to the original system.

We will start by looking at the situation where (X, T) has bounded spacer parameter.

6.3. Rank One Factors of Bounded Spacer Rank One Subshifts

For this section, we will assume that (X, T) is a rank one subshift with spacer parameter bounded by C_X and (Y, S) is an infinite rank one subshift which is a factor of (X, T). By Fact 6.4, we have that (Y, S) will also have bounded spacer parameter, and hence we will let C_Y be the bound on the spacer parameter for (Y, S).

We have the following useful fact about distinct points within a bounded rank one subshift which will combine nicely with Lemma 6.6,

FACT 6.8. *Let (X, T) be a rank one subshift with spacer parameter bounded by C_X. Let $x_0 \neq x_1 \in X$. Then for any n with $|v_n| > C_X$, there is some $k_0 \neq k_1 \in \mathbb{Z}$ with $|k_0 - k_1| < |v_n|$, so that $x_0 \in E_{v_n,k_0}$ and $x_1 \in E_{v_n,k_1}$.*

PROOF. Let $x_0 \neq x_1$. Then there is some index K (relabelling points if necessary) so that $x_0(K) = 0$ and $x_1(K) = 1$. Since every occurrence of 0 in x_0 is contained in some v_n we can find some index k_0 so that $x_0 \in E_{v_n,k_0}$ and $K - k_0 < |v_n|$ so the v_n contains the 0 at K.

Consider $x_1(K)$. If the 1 at K is contained in some v_n, then this v_n cannot start at k_0, but would overlap with index K, so there would be some k_1 with $|k_0 - k_1| < |v_n|$ with $x_1 \in E_{v_n,k_1}$.

If the 1 at K is not contained in some v_n, then it is a spacer adjacent to some v_n, hence it is contained in a string of at most C_X consecutive 1's bracketed by expected occurrences of v_n, i.e. there are l, l' with $x_1 \in E_{v_n,l}$ and $x \in E_{v_n,l'}$ so that $K - (l + |v_n|) < C_X$ and $l' - K < C_X$. Note that $l' = l + |v_n| + d$ for some $0 \leq d \leq C_X$. Suppose $|l' - k_0| \geq |v_n|$. By definition of l', $l' > K \geq k_0$, so $l' - k_0 > 0$ and so we have $l' - k_0 > |v_n|$. Note that $l' - K = l' - (K - k_0) - k_0 < C_X$. Since $|v_n| \geq K - k_0$, replacing $-(K - k_0)$ with $-|v_n|$ would only make things smaller, so $l' - |v_n| - k_0 < C_X$ and $l' - k_0 < C_X + |v_n|$. Since $l' = l + |v_n| + d$, we have $C_X + |v_n| > l' - k_0 = l + |v_n| + d - k_0$, so $l - k_0 < C_X - d \leq C_X$. Since $C_X < |v_n|$, we see that $l - k_0 < |v_n|$. Also, $l' - k_0 > |v_n|$, so $l + |v_n| + d - k_0 > |v_n|$, so $l - k_0 > -d > -|v_n|$. Therefore, $|l - k_0| < |v_n|$, so we can take $k_1 = l$.

Therefore, if $|l' - k_0| \geq |v_n|$, we can take $k_1 = l$, otherwise, we can take $k_1 = l'$. In either case, we can find a k_1 with $|k_0 - k_1| < |v_n|$ so that $x_1 \in E_{v_n,k_1}$. So in either case, there is some $k_0 \neq k_1 \in \mathbb{Z}$ with $|k_0 - k_1| < |v_n|$ so that $x_0 \in E_{v_n,k_0}$ and $x_1 \in E_{v_n,k_1}$. □

Note that since $|k_0 - k_1| < |v_{n'}|$, $E_{v_{n'},k_0}$ and $E_{v_{n'},k_1}$ are disjoint.

Combining this fact with Lemma 6.6, we get the following corollary.

Corollary 6.9. *Let (X,T) be a rank one subshift with bounded spacer and let (Y,S) be an infinite rank one subshift which is a factor of (X,T) as witnessed by sliding block code φ. Since (X,T) and (Y,S) are rank one subshifts, we will let v_n generate X and u_n generate Y.*

For any n, suppose that for some n', there is a string β and an $m \in \mathbb{Z}$ so that for any $x \in X$, if x has an occurrence of $v_{n'}$ starting at index k, then $\varphi(x)$ has an occurrence of β starting at index $k - m$. Also, suppose β starts and ends with expected occurrences of u_n and that for any $x \in X$, any expected occurrence of u_n in $\varphi(x)$ is contained in exactly one β which comes from an occurrence of $v_{n'}$ in x.

Then, φ is one-to-one.

Proof. By Lemma 6.6, we see that the above hypotheses give us that φ is one-to-one on points x_0, x_1 so that we can find $k_0 \neq k_1 \in \mathbb{Z}$ with $|k_0 - k_1| < |v_{n'}|$ where $x_0 \in E_{v_{n'},k_0}$ and $x_1 \in E_{v_{n'},k_1}$.

But by Fact 6.8, when $x_0 \neq x_1 \in X$, we can always find a large enough n' and k_0, k_1 satisfying these properties. Therefore, φ is one-to-one on all of X. □

The rest of this section will be devoted to showing that we can use the presence of the sliding block code to define such a φ.

Lemma 6.10. *Let (X,T) be a rank one subshift with bounded spacer parameter and let (Y,S) be an infinite rank one subshift which is a factor of (X,T) as witnessed by sliding block code φ. Since (X,T) and (Y,S) are rank one subshifts, we will let v_n generate X and u_n generate Y.*

For any large enough n, for some n', there is a string β and an $m \in \mathbb{Z}$ so that for any $x \in X$, if x has an occurrence of $v_{n'}$ starting at index k, then $\varphi(x)$ has an occurrence of β starting at index $k-m$. Also, β will start and end with expected occurrences of u_n and that for any $x \in X$, any expected occurrence of u_n in $\varphi(x)$ in contained in exactly one β which comes from an occurrence of $v_{n'}$ in x.

Proof. Let N be the block length and m' be the offset for the sliding block code. Recall that X has bound C_X on the spacer parameter. We let n be large enough that $|u_n| \geq N + C_X + 1$.

By Corollary 2.25, there is some t so that if $y \in Y$ contains an occurrence of u_n, we can determine whether that occurrence of u_n is expected by expanding the context around u_n to encompass a substring of y of length $t\,|u_n|$ and checking whether that string is part of a finite set of strings. We will let n' be large enough that $|v_{n'}| - N \geq t\,|u_n| + 2C_X$ and $|v_{n'}| \geq 3\,|u_n|$. Note that this n' is large enough so that $|v_{n'}| > N$.

We claim that for any $k \in \mathbb{Z}$ and for any $x \in X$, if $x \in E_{v_{n'},k}$, then we can determine all expected occurrences of u_n in $\varphi(x)[k-m', k-m'+|v_{n'}|-N-1]$. By Proposition 6.3, we have that the occurrence of $v_{n'}$ in x starting at index k determines $\varphi(x)[k-m', k-m'+|v_{n'}|-N-1]$. We will let $\beta' = \varphi(x)[k-m', k-m'+|v_{n'}|-N-1]$. So we have that any occurrence of $v_{n'}$ in x starting at index k induces an occurrence of β' in $\varphi(x)$ starting at index $k-m'$.

Note that since $|v_{n'}| - N \geq t\,|u_n|$, we have $k - m' + |v_{n'}| - N - 1 \geq k - m' + t\,|u_n| - 1$, so β' has length at least $t\,|u_n|$. By Corollary 2.25, we have that since β' is a string of length at least $t\,|u_n|$, any occurrence of u_n completely within β' can be determined as expected by looking at the rest of β'. Therefore, there are a finite number of l_i, so whenever β' occurs within $\varphi(x)$ starting at an index k, we get that $\varphi(x) \in E_{u_n,k+l_i}$.

Furthermore, because of the definition of the sliding block code, we see that the string β' will occur in $\varphi(x)$ whenever any $v_{n'}$ occurs in x, hence each occurrence of $v_{n'}$ generates the same expected u_n structure. Specifically, for any $k \in \mathbb{Z}$, we have that if $x \in E_{v_{n'},k}$, then $\varphi(x) \in E_{u_n,k-m'+l_i}$ for each of the finitely many l_i.

Next, we claim that every expected occurrence of u_n in $\varphi(x)$ must overlap with an occurrence of β' generated by an expected occurrence of $v_{n'}$ in x. Note that since (X,T) has spacer parameter bounded by C_X, we have that the distance between starting points of consecutive occurrences of $v_{n'}$ can be no more than $|v_{n'}| + C_X$. So if $x \in E_{v_{n'},k}$, then there is some $0 \leq d \leq C_X$ so $x \in E_{v_{n'},k+|v_{n'}|+d}$. But $x \in E_{v_{n'},k}$ means $\varphi(x)[k-m', k-m'+|v_{n'}|-N-1]$ is β' and $x \in E_{v_{n'},k+|v_{n'}|+d}$ means $\varphi(x)[k-m'+|v_{n'}|+d, k-m'+2|v_{n'}|+d-N-1]$ is β'. Therefore, we see that $\varphi(x)[k-m'+|v_{n'}|-N, k-m'+|v_{n'}|+d-1]$ (which has length $N+d$) is left undetermined between the occurrences of β' from consecutive occurrences of $v_{n'}$. But this is true for arbitrary indices k, so we see that $\varphi(x)$ only contains undetermined strings of length $N+d \leq N+C_X$. But $|u_n| \geq N+C_X+1 > N+C_X$, so there cannot be an occurrence of u_n completely within the undetermined portion. Therefore, every occurrence of u_n must at least overlap such an occurrence of β'.

Combining the two previous paragraphs, we see that the expected occurrences of $v_{n'}$ within x completely determine all expected occurrences of u_n within $\varphi(x)$, because every expected occurrence of u_n in $\varphi(x)$ must overlap an occurrence of β' generated by some expected occurrence of $v_{n'}$ in x. (For the rest of the proof, when we refer to occurrences of β' which come from some $v_{n'}$, we are referring to only to those coming from expected occurrences of $v_{n'}$ and may omit the expected.) Also, we saw that the expected occurrence structure of the u_n within β' is fixed, i.e. there is a finite set of l_i, so that if β' starts at index k in $\varphi(x)$, then $\varphi(x) \in E_{u_n,k+l_i}$. This comes from the fact that every occurrence of u_n contained entirely in β' can be determined as expected by looking at the rest of the β'. We will show that we can modify β' into a β which starts and ends with expected occurrences of u_n and that $x \in E_{v_{n'},k}$ implies $\varphi(x)$ has an occurrence of β starting at index $k-m$. We will only show how to modify the end of β' to get β, but modifying the beginning of β' follows the same process. We have three possibilities for the end of β': β' ends with an expected occurrence of u_n which is contained in β' or β' has extra characters after the last expected occurrence of u_n which is completely contained in β', and the extra characters can contain a 0 or not. So the three possibilities are that $\beta' = \alpha^\frown u_n$, $\beta' = \alpha^\frown u_n{}^\frown 1^a$ for some $a > 1$, or $\beta' = \alpha^\frown u_n{}^\frown 1^{e_0}01^{e_1}0...01^{e_s}$, where each $e_i \in \mathbb{N}$ and $s \geq 1$.

If the end of β' is an expected occurrence of u_n which is contained entirely in β', then β' is as necessary, so we can leave β' alone and it will end with the necessary expected u_n, will still generate the necessary overlap with each expected u_n, and will be determined by $v_{n'}$.

If the end of β' contains extra characters which are not contained in the last expected occurrence of u_n and the extra characters do not contain a 0, then we are in the case $\beta' = \alpha^\frown u_n{}^\frown 1^a$. In this case, we remove the 1^a, and modify β' to $\alpha^\frown u_n$. By definition, this ends with an expected u_n which is contained entirely in $\alpha^\frown u_n$. Furthermore, since β' was entirely determined from the sliding block code, and $\alpha^\frown u_n$ is a substring of β', $\alpha^\frown u_n$ will also be completely determined by $v_{n'}$. Finally, since u_n must start and end with 0 and the string 1^a we removed from β' does not contain any 0's, we have that $\alpha^\frown u_n$ will still overlap every expected occurrence of u_n that β' did, and so all occurrences of $\alpha^\frown u_n$ will still overlap with each expected u_n in $\varphi(x)$.

Now suppose that the end of β' contains extra characters which are not contained in the last expected occurrence of u_n and the extra characters do contain a 0. So we are in the case where $\beta' = \alpha^\frown u_n{}^\frown 1^{e_0}01^{e_1}0...01^{e_s}$. Since the u_n denoted in $\beta' = \alpha^\frown u_n{}^\frown 1^{e_0}01^{e_1}0...01^{e_s}$ is expected, we see that the 0 immediately after 1^{e_0} will be the start of an expected occurrence of u_n as well.

Note that if p is the number of occurrences of 0 in each u_n, we have that $s < p$, since otherwise, we would have that the first p many 0's would form another expected occurrence of u_n which is contained entirely in β' and we assumed that the u_n in $\alpha^\frown u_n$ was the last expected occurrence of u_n occurring completely within β'.

We claim that we can extend $\beta' = \alpha^\frown u_n{}^\frown 1^{e_0}01^{e_1}0...01^{e_s}$ to $\alpha^\frown u_n{}^\frown 1^{e_0}{}^\frown u_n$ and this will satisfy the necessary properties. By definition, it ends with the necessary u_n. Furthermore, we saw that there was only one possible occurrence of u_n overlapping the $1^{e_0}01^{e_1}0...01^{e_s}$, and we extended $\alpha^\frown u_n{}^\frown 1^{e_0\frown} u_n$ so it contained the entirety of that occurrence, so the system of $\alpha^\frown u_n{}^\frown 1^{e_0\frown} u_n$ will contain all the occurrences of u_n that overlap some β'. So it remains to show that each $v_{n'}$ induces $\alpha^\frown u_n{}^\frown 1^{e_0\frown} u_n$.

Let $x \in E_{v_{n'},k}$. Then $\varphi(x)$ has an occurrence of β' starting at $k - m'$. But $\beta' = \alpha^\frown u_n{}^\frown 1^{e_0} 0 1^{e_1} 0 ... 0 1^{e_s}$ where the u_n is expected. Since each 0 in $\varphi(x)$ is contained in exactly one expected occurrence of u_n, the 0 immediately after 1^{e_0} will be contained in some expected occurrence of u_n. Also, since expected occurrences of u_n cannot overlap and u_n must start with a 0, we have that the u_n containing that 0 must start immediately after 1^{e_0}. Therefore, there is an expected occurrence of u_n starting at that 0, so there is an occurrence of $\alpha^\frown u_n{}^\frown 1^{e_0\frown} u_n$ starting at index $k - m'$. Furthermore, since $s < p$, $0 1^{e_1} ... 0 1^{e_s}$ is a proper initial segment of u_n and so β' is a substring of $\alpha^\frown u_n{}^\frown 1^{e_0\frown} u_n$, so we have that $\alpha^\frown u_n{}^\frown 1^{e_0\frown} u_n$ is induced at the proper location by $v_{n'}$.

To get β from β', we apply the similar process to the front of β' as well. So β starts and ends with u_n, is induced by $v_{n'}$, and the system of occurrences of β which are induced by the $v_{n'}$ overlaps with all the expected occurrences of u_n. Note that we may have needed to include the rest of the expected u_n at the front or remove the extra string of 1^a from the front, but this modifies the starting point $k - m'$ to $k - m$ in a well-defined way. So if we removed the string 1^a from β', then β starts at index $k - m$ where $m = m' - a$ and if we added the rest of the expected u_n, then $m = m' + |u_n| - s - \sum_{i=1}^{s} e_i$, as we added the portion of the u_n which was not the s many 0's or the 1^{e_i} where $1 \leq i \leq s$.

So from above, we have found a string β which starts and ends with expected occurrences of u_n, so that for any $k \in \mathbb{Z}$, if $x \in E_{v_{n'},k}$, then there is an occurrence of β starting at index $k - m$ in $\varphi(x)$. Also, for any expected occurrence of u_n in $\varphi(x)$, there is some expected occurrence of $v_{n'}$ in x which induces an occurrence of β containing that expected occurrence of u_n in $\varphi(x)$. Since we showed that β' was long enough so that every occurrence of u_n which is entirely contained in β can be determined to be expected without looking at any other characters, we cannot have an occurrence of β in $\varphi(x)$ which is disjoint from those induced by an expected occurrence of $v_{n'}$ in x. Such a β would necessarily contain expected u_n, all of which must be contained in occurrences of β which are induced by expected $v_{n'}$.

Note that it is possible that some occurrence of β in $\varphi(x)$ comes from a fortuitous combination of spacers between those occurrences of β which are induced by $v_{n'}$. However, the previous paragraph shows that the "expected" occurrences of β which come from occurrences of $v_{n'}$ are sufficient to cover all of the u_n and are well-defined. Therefore, we will from now on refer to occurrences of β in $\varphi(x)$ as the occurrences which come from occurrences of $v_{n'}$, and we will not include "unexpected" occurrences when we refer to occurrences of β.

It remains to show that each u_n in $\varphi(x)$ is contained in a unique β. Since we know there is at least one β containing each expected occurrence of u_n in $\varphi(x)$, it is enough to show that two distinct occurrences of β in $\varphi(x)$ cannot overlap. Suppose that there are overlapping occurrences of β. We note that each β occurring in $\varphi(x)$ comes from an occurrence of $v_{n'}$ in x, as all of the u_n in $\varphi(x)$ are contained in occurrences of β which come from the $v_{n'}$ in x.

Note that occurrences of β' are non-overlapping, since for any k, if $x[k, k+|v_{n'}|-1] = v_{n'}$, then β' is the string $\varphi(x)[k-m', k-m'+|v_{n'}|-N-1]$ and since occurrences of $v_{n'}$ must be at least $|v_{n'}|$ apart, so if $x \in E_{v_{n'},k}$, then the earliest another $v_{n'}$ in x could start is $k+|v_{n'}|$. For occurrences of β' coming from those consecutive $v_{n'}$ to overlap, we would need $k-m'+|v_{n'}|-N-1 \geq k-m'+|v_{n'}|$ which is impossible. Also, note that $|\beta'| \leq |v_{n'}|$.

Note that if we left β' alone or removed a string of ones from β' to get β, then β is a substring of β' and we get that there is no overlap. So we only need to consider the case where we extended β' to get β. In the case of the end of β', this was when $\beta' = \alpha^\frown u_n{}^\frown 1^{e_0}01^{e_1}0...01^{e_s}$ and we extended it to $\alpha^\frown u_n{}^\frown 1^{e_0}{}^\frown u_n$ by taking the entire expected occurrence of u_n containing the 0 immediately after 1^{e_0}. This would result in β having at most one additional occurrence of u_n to each side of β'. So $|\beta| \leq |\beta'| + 2|u_n| \leq |v_{n'}| + 2|u_n|$.

We claim that two overlapping β must be induced by consecutive occurrences of $v_{n'}$. If one occurrence of $v_{n'}$ starts at index k, then without loss of generality, looking to the right, the closest the non-consecutive occurrence of $v_{n'}$ could occur would be starting at $k+2|v_{n'}|$. This would induce occurrences of β at $\varphi(x)[k-m, k-m+|\beta|-1]$ and $\varphi(x)[k+2|v_{n'}|-m, k+2|v_{n'}|-m+|\beta|-1]$. So for there to be overlap, we would need

$k-m+|\beta| > k+2|v_{n'}|-m$, so $|\beta| > 2|v_{n'}|$. But $|\beta| \leq |v_{n'}|+2|u_n|$, and since $|v_{n'}| \geq 3|u_n|$, $|v_{n'}|+2|u_n| < 2|v_{n'}|$. But this is a contradiction, since we get $2|v_{n'}| < |\beta| < 2|v_{n'}|$.

So if two distinct occurrences of β overlap in $\varphi(x)$, then they are induced by consecutive occurrences of $v_{n'}$. Also, since β starts and ends with expected occurrences of u_n, any overlap would involve overlapping some expected occurrences of u_n. Since Y is infinite, we have that expected occurrences of u_n are disjoint, so if they overlap, they must be the same occurrence. Therefore, we see that such overlapping occurrences of β must share expected occurrences of u_n.

Let d be the spacer between occurrences of $v_{n'}$ inducing this overlap and let the first such $v_{n'}$ start at index k. Then $x \in E_{v_{n'},k} \cap E_{v_{n'},k+d+|v_{n'}|}$ and so the β occur as $\varphi(x)[k-m, k-m+|\beta|-1]$ and $\varphi(x)[k+d+|v_{n'}|-m, k+d+|v_{n'}|-m+|\beta|-1]$. Therefore, we see that $k+d+|v_{n'}|-m < k-m+|\beta|-1$ and in particular, the interval $[k+d+|v_{n'}|-m, k-m+|\beta|-1]$ is a substring of β which starts and ends with expected occurrences of u_n. Since the β started at index $k-m$, we see that $\beta[d+|v_{n'}|, |\beta|-1]$ is a string starting and ending with expected occurrences of u_n. Hence $|\beta|-d-|v_{n'}| \geq |u_n|$, so $|\beta|-|v_{n'}| \geq |u_n|+d$.

Since Y is infinite and $\varphi : X \to Y$ maps onto Y, X is infinite, hence must have distinct spacers occur infinitely often in x. Consider two $v_{n'}$'s in x separated by a spacer $d' \neq d$. Then for some $k' \in \mathbb{Z}$, $x \in E_{v_{n'},k'} \cap E_{v_{n'},k'+d'+|v_{n'}|}$. By our above discussion, we see that occurrences of β will happen as $\varphi(x)[k'-m, k'-m+|\beta|-1]$ and $\varphi(x)[k'+d'+|v_{n'}|-m, k'+d'+|v_{n'}|-m+|\beta|-1]$.

Suppose these overlap. Then $k'-m+|\beta|-1 > k'+d'+|v_{n'}|-m$ and by considering the starting point, we see that the overlap would be $\beta[d'+|v_{n'}|, |\beta|-1]$. Since β starts and ends with expected occurrences of u_n, this overlap would have to align such occurrences. So $\beta[d'+|v_{n'}|, |\beta|-1]$ is a string starting and ending with expected occurrences of u_n. But $\beta[d+|v_{n'}|, |\beta|-1]$ is also a string starting and ending with expected occurrences of u_n. Since these are necessarily distinct strings, the difference must contain at least one expected occurrence of u_n.

Without loss of generality, since $d \neq d'$, let $d > d'$. Then $d'+|v_{n'}| < d+|v_{n'}|$ and so we have $\beta[d' + |v_{n'}|, d + |v_{n'}| - 1]$ contains at least one expected occurrence of u_n, hence $d + |v_{n'}| - (d' + |v_{n'}|) \geq |u_n|$, so $d - d' \geq |u_n|$. But C_X was the bound on spacer for X, and d, d' are each spacers from X so $0 \leq d < d' \leq C_X$, so $|u_n| \leq d - d' \leq C_X$. But $|u_n| \geq N + C_X + 1 > C_X$. But this gives $|u_n| \leq d - d' \leq C_X < |u_n|$, which is a contradiction. For $d < d'$, simply interchange d and d' in the previous argument.

Therefore, $\varphi(x)[k' - m, k' - m + |\beta| - 1]$ and $\varphi(x)[k' + d' + |v_{n'}| - m, k' + d' + |v_{n'}| - m + |\beta| - 1]$ cannot overlap. So $k' + d' + |v_{n'}| - m \geq k' - m + |\beta|$ and so $d' \geq |\beta| - |v_{n'}|$. But we showed that $|\beta| - |v_{n'}| \geq |u_n| + d$, so $d' \geq |u_n| + d$, therefore $d' - d \geq |u_n|$. But similarly to above, $0 \leq d, d' \leq C_X$ since d, d' each come from spacers in x, so $d' - d \leq C_X$. But $|u_n| > C_X$ and hence we get $|u_n| \leq d' - d \leq C_X < |u_n|$ which is a contradiction. Therefore, we cannot have that two distinct occurrences of β overlap. □

We can combine Lemma 6.10 and Corollary 6.9 to see that the map φ is one-to-one, and then use Fact 6.7 to get the following theorem.

THEOREM 6.11. *Let (X, T) be a rank one subshift with bounded spacer parameter. Let (Y, S) be a rank one subshift which is a factor of (X, T) and suppose Y is infinite. Then (X, T) is isomorphic to (Y, S).*

6.4. Rank One Factors of Unbounded Rank One Subshifts

Now we look to show a similar theorem for rank one subshifts with unbounded spacer parameter. We will use a similar technique of creating a word β which is induced by occurrences of some large $v_{n'}$ and use Lemma 6.6 to help show that the factor map is one-to-one. However, when we proved this for bounded rank one subshifts, we needed to use the bound on the spacer in several places. For those subshifts with unbounded spacer parameter, we do not have such a bound and hence will need to use different techniques.

We use a similar setup. We let (X, T) be a rank one subshift with unbounded spacer parameter and (Y, S) be an infinite rank one subshift which is a factor of (X, T) witnessed by φ. Note that by Fact 6.5, we have that (Y, S) will be also be an unbounded rank one subshift.

Again using Proposition 6.2, we have that the factor map φ will be given by a sliding block code. Letting N be the block length and m' be the offset, by Proposition 6.3, there is some $N \in \mathbb{N}$ and $m' \in \mathbb{Z}$, so that for any n' so that $|v_{n'}| > N$, the image of any $v_{n'}$ starting at k will be a fixed string of length $|v_{n'}| - N$ starting at index $k - m'$.

Since the spacer parameter is unbounded in X, we will see arbitrarily long sequences of 1's, and so we note the following useful property of the sliding block code.

LEMMA 6.12. *Let (X,T) be an unbounded rank one subshift and (Y,S) is a rank one subshift which is a factor of (X,T) witnessed by φ. Let $N \in \mathbb{N}$ be the block length for the sliding block code and m' be the offset.*

For any $x \in X$ and $k \in \mathbb{Z}$, if $x[k, k+N-1] = 1^N$, then $\varphi(x)(k-m') = 1$. In particular, if $M > N$ and x contains a sequence of M consecutive 1's starting at index k, then $\varphi(x)$ contains a sequence of $M-N$ consecutive 1's starting at $k-m'$.

PROOF. Suppose toward a contradiction, that $x[k, k+N-1] = 1^N$ gives that $\varphi(x)(k-m') = 0$. Since φ is a sliding block code, this applies for any $k \in \mathbb{Z}$. Also, since X has unbounded spacer parameter, we can find arbitrarily long spacers in each $x \in X$. So for any $M \in \mathbb{N}$ with $M > N$, we can find some $k \in \mathbb{Z}$ so that $x[k, k+M-1] = 1^M$. But by considering substrings of $x[k, k+M-1]$ which have length N, we see that these substrings are all 1^N, hence we have $\varphi(x)[k-m', k-m'+M-N-1] = 0^{M-N}$. So we can find arbitrarily long strings of consecutive 0's in each $\varphi(x)$ and since (Y,S) is a shift space, we have $0^{\mathbb{Z}} \in Y$. But (Y,S) is rank one and so Y can contain only one fixed point, which is $1^{\mathbb{Z}}$. But $0^{\mathbb{Z}}$ is a fixed point which is not $1^{\mathbb{Z}}$, which is impossible. Therefore, we must have $x[k, k+N-1] = 1^N$ gives that $\varphi(x)(k-m') = 1$ and so it follows that if $x[k, k+M-1] = 1^M$, then $\varphi(x)[k-m', k-m'+M-N-1] = 1^{M-N}$. □

We will use this instead of the bound on the spacer to show that we can get the necessary β to use Lemma 6.6.

Now we show the unbounded version of Lemma 6.10

LEMMA 6.13. *Let (X,T) be an unbounded rank one subshift and let (Y,S) be an infinite rank one subshift which is a factor of (X,T) as witnessed by sliding block code φ. Since (X,T)*

and (Y, S) are rank one subshifts, we will let v_n generate X and u_n generate Y.

For any large enough n, for some n', there is a string β and an $m \in \mathbb{Z}$ so that for any $x \in X$, if x has an occurrence of $v_{n'}$ starting at index k, then $\varphi(x)$ has an occurrence of β starting at index $k - m$. Also, β will start and end with expected occurrences of u_n and that for any $x \in X$, any expected occurrence of u_n in $\varphi(x)$ in contained in exactly one β which comes from an expected occurrence of $v_{n'}$ in x.

Proof. Let N be the block length and m' be the offset for the sliding block code. Since (X, T) is an unbounded rank one subshift and (Y, S) is a rank one subshift which is a factor of (X, T), (Y, S) is unbounded as well.

We let n be large enough that $|u_n| > 3N$.

By Corollary 2.25, there is some t so that if $y \in Y$ contains an occurrence of u_n, we can determine whether that occurrence of u_n is expected by expanding the context around u_n to encompass a substring of y of length $t\,|u_n|$ and checking whether that string is part of a finite set of strings. We will let n' be large enough that $|v_{n'}| - N \geq t\,|u_n|$ and $|v_{n'}| \geq 3\,|u_n|$. Note that this n' is large enough so that $|v_{n'}| > N$.

Note that if $x \in E_{v_{n'},k}$, then φ induces a particular string $\varphi(x)[k - m', k - m' + |v_{n'}| - N - 1]$, which we will call β'. Following the same argument as the proof of Lemma 6.10, the expected occurrences of u_n within β' are determined by the string β', hence there are finitely many l_i, so that if $x \in E_{v_{n'},k}$, then $\varphi(x) \in E_{u_n,k-m'+l_i}$ for each of the finitely many l_i.

Next, we claim that every occurrence of u_n must overlap with some occurrence of β' coming from an expected occurrence of $v_{n'}$.

Suppose otherwise. Then there is some $x \in X$ and some $k' \in \mathbb{Z}$ so that $\varphi(x)[k' - m', k' - m' + |u_n| - 1] = u_n$, but there is no expected occurrence of $v_{n'}$ so that the occurrence of β' overlaps with this u_n. Note that the occurrences of β' that come from expected occurrences of $v_{n'}$ come from the sliding block code applied to strings of length N which are contained in the $v_{n'}$. Since x in built from $v_{n'}$ and spacers, an occurrence of u_n which does not overlap with β' coming from expected occurrences of $v_{n'}$ would come from necessarily come from the sliding block code applied to strings of length N in x which overlap expected occurrences of

$v_{n'}$ by fewer than N characters or are entirely contained in spacers. So we see that such u_n must come from the sliding block code applied to a string of the form $\gamma^\frown 1^{M\frown}\gamma'$ in x, where γ is the final $N-1$ characters of $v_{n'}$ and γ' is the initial $N-1$ characters in $v_{n'}$.

First, we will rule out the possibility that u_n can come from such a string where $M \leq N$. If $M \leq N$, then $\left|\gamma^\frown 1^{M\frown}\gamma'\right| < 3N$, since $M \leq N$ and $|\gamma| = |\gamma'| = N-1$. But $|u_n| > 3N$ and we cannot have a sliding block code applied to a shorter string induce a longer string.

Therefore, we must have that u_n is induced by a string with $M > N$. Let $\gamma^\frown 1^{M\frown}\gamma'$ induce an expected occurrence of u_n. Note that since $M > N$, by Lemma 6.12, the string 1^M induces a string 1^{M-N} in $\varphi(x)$, and in particular, we cannot have an occurrence of 0 which is induced by the blocks in 1^M. Since $|\gamma| = |\gamma'| = N-1$ and $|u_n| > 3N$, we have that the u_n cannot be induced entirely by blocks overlapping γ alone or blocks overlapping γ' entirely. Since u_n starts and ends with 0, we see that we must have the first 0 in u_n generated by a block overlapping γ and the last 0 in u_n generated by a block overlapping γ'.

Since (X, T) has unbounded spacer parameter, we can find a spacer with length larger than $|u_n| + N$. Let K be such a spacer length and consider an occurrence of $v_{n'}{}^\frown 1^{K\frown} v_{n'}$ within x where each $v_{n'}$ is expected. Note that this contains a string of the form $\gamma^\frown 1^{K\frown}\gamma'$ which will induce a string which is disjoint from the occurrences of β' induced by the $v_{n'}$. Note that since $M > N$, each block of $\gamma^\frown 1^{K\frown}\gamma'$ which overlaps γ will be of the form $\delta 1_0^N$, where $N_0 < N < M$ and δ is a final segment of γ, and so the blocks of $\gamma^\frown 1^{K\frown}\gamma'$ which overlap γ are exactly the same as the blocks of $\gamma^\frown 1^{M\frown}\gamma'$ which overlap γ. Similarly, since the blocks which overlap γ' will be of the form $1^{N_0}\delta'$ where δ' is an initial segment of γ' and $N_0 < N$, we get the same correspondence between blocks of $\gamma^\frown 1^{K\frown}\gamma'$ and $\gamma^\frown 1^{K\frown}\gamma'$ which overlap γ'. Since φ is a sliding block code, we see that this $\gamma^\frown 1^{K\frown}\gamma'$ will induce the same first $N-1$ characters as $\gamma^\frown 1^{M\frown}\gamma'$, a string of 1's with length $K-N$, and finally the same final $N-1$ characters as $\gamma^\frown 1^{M\frown}\gamma'$.

But we saw that $\gamma^\frown 1^{M\frown}\gamma'$ induced an expected u_n and induced the first 0 in this u_n with a block which overlapped γ and induced the last 0 in this u_n with a block which overlapped γ'. Since we saw that blocks that overlap γ or γ' are the same in both $\gamma^\frown 1^{K\frown}\gamma'$ and $\gamma^\frown 1^{M\frown}\gamma'$, we have that they will both induce occurrences of 0 which are not contained in any expected u_n which overlaps with occurrences of β'. But consider the 0's induced by $\gamma^\frown 1^{K\frown}\gamma'$. The first 0 which is not part of an expected u_n which overlaps β' will be contained in the $N-1$ characters induced by blocks which overlap γ. Since it cannot be part of a u_n which overlaps β', we have that it must be the start of an expected u_n. But then, after at most $N-2$ additional characters, it is followed by a string of 1's of length $K-N$. But $K > |u_n| + N$, so $K - N > |u_n|$. But this is impossible since u_n must end with 0, so we have that there is a 0 which cannot be contained in an expected u_n. But we assumed (Y, S) is rank one and hence, every occurrence of 0 must be contained in some expected u_n. This is a contradiction, so we cannot have that $\gamma^\frown 1^{M\frown}\gamma'$ induces an expected occurrence of u_n. This shows that every expected occurrence of u_n overlaps with some occurrence of β' which comes from an expected occurrence of $v_{n'}$.

Next, we follow the same process as in the proof of Lemma 6.10 and modify β' into β so that β starts and ends with expected occurrences of u_n, every expected occurrence of u_n in $\varphi(x)$ is contained in some β induced by an expected $v_{n'}$ in X, and for any $x \in E_{v_{n'},k}$, $\varphi(x)$ has an occurrence of β starting at index $k-m$. Recall that we did this by either removing an excess string of 1's from the beginning and/or end of β' or by looking at any 0's in β' which occurred outside of the expected occurrences of u_n which are entirely contained in β', and including the entire expected occurrence of u_n containing these 0's. In that proof, we showed that such an expected occurrence of u_n occurred in the same place within β', as β' was long enough to determine the expectedness of all occurrences of u_n within β', and so β was well-defined and entirely determined by the occurrence of $v_{n'}$ which generated β'. Furthermore, $|\beta| \leq |\beta'| + 2\,|u_n|$, as we added at most one expected occurrence of u_n to each end of β'.

So by the same process as in the proof of Lemma 6.10, we have found a string β which starts and ends with expected occurrences of u_n, so that for any $k \in \mathbb{Z}$, if $x \in E_{v_{n'},k}$, then there is an occurrence of β starting at index $k-m$ in $\varphi(x)$. Also, for any expected occurrence of u_n in $\varphi(x)$, there is some expected occurrence of $v_{n'}$ in x which induces an occurrence of β containing that expected occurrence of u_n in $\varphi(x)$.

To avoid unwieldly wording, when we refer to occurrences of β, we only reference those occurrences of β which are induced by expected occurrences of $v_{n'}$ and we do not refer to "unexpected" occurrences that happen to occur as combinations coming from fortuitous placement of occurrences of β's which are induced by expected $v_{n'}$ or that come from unexpected $v_{n'}$.

We will show that two different occurrences of β cannot overlap. Note that β starts and ends with an expected occurrence of u_n, and since any expected occurrences of u_n which overlap must coincide, we have that if two occurrences of β overlap, they must share some expected occurrence of u_n.

Recall that by definition of β', occurrences of β' which come from expected $v_{n'}$ cannot overlap. Also, we showed that there cannot be an occurrence of 0 in β' where the expected occurrence of u_n containing that 0 extends to another β'. But since occurrences of β' do not overlap, and the only way we could make β longer than β' is by extending β' to contain the entirety of some expected u_n which overlaps it. So we see that if two occurrences of β overlap, then they overlap on an expected occurrence of u_n which necessarily overlaps with two occurrences of β'. However, we showed that this cannot occur. Therefore, occurrences of β cannot overlap, and so each expected u_n is contained in exactly one occurrence of β which comes from an occurrence of $v_{n'}$.

Therefore, β witnesses that the lemma holds. □

Therefore, we can combine this with Lemma 6.6 and get that φ is one-to-one on many points. Therefore, if we can show φ is one-to-one on points which do not fit with the hypothesis for Lemma 6.6, then we get the following theorem.

THEOREM 6.14. *Let (X, T) be a rank one subshift with unbounded spacer parameter. Let (Y, S) be a rank one subshift which is a factor of (X, T) and suppose Y is infinite. Then (X, T) is isomorphic to (Y, S).*

Proof. Let $\varphi : X \to Y$ be the factor map. By Fact 6.7, we see that it is enough to show that φ is one-to-one.

Let $x_0 \neq x_1 \in X$.

By Proposition 6.3, φ is a sliding block code and we can invoke Lemma 6.13 for some n'. Note that for such an n', we have $|v_{n'}| > N$.

Either there exists $k_0 \neq k_1 \in \mathbb{Z}$ with $|k_0 - k_1|$ so that $x_0 \in E_{v_{n'},k_0}$ and $x_1 \in E_{v_{n'},k_1}$ or there are no such k_0 and k_1.

If there is such a k_0 and k_1, we can combine Lemma 6.13 and Lemma 6.6 and we see that $\varphi(x_0) \neq \varphi(x_1)$.

So suppose there are no such k_0 and k_1. Then for any k with $x_0 \in E_{v_{n'},k}$, for any k' with $k - |v_{n'}| + 1 \leq k' \leq k + |v_{n'}| - 1$, we have $x_1 \notin E_{v_{n'},k'}$. Since X is rank one, each point in X is generated by expected occurrences of $v_{n'}$ and spacers which are strings of 1's, so if $x_0 \in E_{v_{n'},k}$, then we see that we must have $x_1[k, k + |v_{n'}| - 1] = 1^{|v_{n'}|}$.

Without loss of generality, let $x_0 \in E_{v_{n'},k}$ for some $k \in \mathbb{Z}$. (Note that there is exactly one point $1^{\mathbb{Z}}$ in X for which this fails, so if x_0 happens to be $1^{\mathbb{Z}}$, switch x_1 and x_0). By the proof of 6.13, since $x_0 \in E_{v_{n'},k}$, $\varphi(x_0)[k - m', k - m' + |v_{n'}| - N - 1] = \beta'$ contains a 0. But since $x_1[k, k + |v_{n'}| - 1] = 1^{|v_{n'}|}$ and $|v_{n'}| > N$, we can invoke Lemma 6.12 and we see that $\varphi(x_1)[k-m', k-m'+|v_{n'}|-N-1] = 1^{|v_{n'}|-N}$. But then we see $\varphi(x_0)[k-m', k-m'+|v_{n'}|-N-1]$ contains a 0, but $\varphi(x_1)[k - m', k - m' + |v_{n'}| - N - 1]$ does not. Therefore, $\varphi(x_0) \neq \varphi(x_1)$.

Therefore, φ is one-to-one and we get the theorem. □

We can combine Theorem 6.14 and Theorem 6.11 to get Theorem 1.4 from the introduction.

Theorem 1.4. *Let (X, T) and (Y, S) be rank one subshifts. Suppose (Y, S) is a factor of (X, T). Then either Y is finite or (Y, S) is isomorphic to (X, T).*

CHAPTER 7

OPEN PROBLEMS

In this section, we will discuss some open problems and areas for further research.

Recall that we completely categorised mixing properties for bounded rank one subshifts. In particular, we saw that a bounded rank one subshift is weakly mixing iff for all large enough n, 1 is the gcd of the set $\{|v_n| + a_{m,i} : m \geq n, 1 \leq i \leq q_m\}$. The proof for this required the bound on the spacer parameter. We do not have a complete classification of weakly mixing for unbounded rank one subshifts. However, the counterexamples listed violate this condition. So one question is whether it is possible to prove a similar dichotomy.

QUESTION 7.1. *Let (X, T) be a rank one subshift with unbounded spacer parameter. Is it the case that (X, T) is weakly mixing iff for all large enough n, 1 is the gcd of the set $\{|v_n| + a_{m,i} : m \geq n, 1 \leq i \leq q_m\}$?*

As a step along the way to this question, it may be helpful to add additional assumptions on how fast the spacer parameters can grow, which might help make the combinatorial calculations initially more feasible.

Also, we do not yet have a complete classification for the mixing property for unbounded rank one subshifts. For weakly mixing, we were able to prove that having a spacer parameter that makes up a large enough measure subset of $\mathbb{N}$ guarantees weakly mixing. We can ask a similar question for the mixing property.

QUESTION 7.2. *Let (X, T) be a rank one subshift with unbounded spacer parameter.*

If the spacer parameter makes up a measure 1 subset of $\mathbb{N}$, is the space mixing?

We also saw that when the spacer parameter forms a finite set, the resulting transformation is never mixing. We can ask if this generalises to a small enough infinite set.

QUESTION 7.3. *Let (X, T) be a rank one subshift with unbounded spacer parameter.*

If the spacer parameter makes up a measure 0 subset of $\mathbb{N}$, is the space necessary not mixing?

Looking at the factor problem, Del Junco [8] proved that for measure theoretic rank one transformations, a factor of a rank one transformation is rank one. We showed that for a rank one subshift, any rank one factor will be either finite or isomorphic, so a similar result to that of Del Junco would severely restrict the factor structure of rank one subshifts. So we can ask the following initial question.

QUESTION 7.4. *Let* (X, T) *be a rank one subshift and* (Y, S) *be a factor of* (X, T)*. Suppose* (Y, S) *is also a subshift.*

Is it possible that (Y, S) *is not isomorphic to a rank one subshift?*

Along the lines of this question, we would likely need a result that helps characterise when an arbitrary subshift is isomorphic to a rank one subshift. So we can ask the following question.

QUESTION 7.5. *Given a subshift, what (non-constructive) properties are necessary for this subshift to satisfy so that it is isomorphic to a rank one subshift?*

Can we find a necessary and sufficient condition?

Other possibilities for continuing this line of research might include trying to apply the techniques to other classes of topological dynamical systems. The first starting point would be trying to generalise rank one subshifts to a different group action and applying the same techniques. For example, trying to develop a two dimensional analogue, where $X \subseteq 2^{\mathbb{Z}\times\mathbb{Z}}$ and we allow both a vertical and horizontal shift. Further research could also involve generalising rank one subshifts to arbitrary countable groups.

Another possibility would be looking to apply the techniques to other shift spaces. For example, we could attempt to apply the techniques to subshifts that are generated by a more complicated schema, such as that in rank two subshifts. Another possibility would be looking at subshifts generated as an extension of rank one subshifts.

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