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TITLE: ON THE INFERENCE OF CRACK STATISTICS FROM OBSERVATIONS ON AN OUTCROPPING

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ABSTRACT

In rock mechanics it is often assumed that the number of cracks, faults or joints whose size exceeds c is given by the exponential $N_o e^{-c/\bar{c}}$. A mathematical argument making this distribution plausible, at least for a two-dimensional distribution of line segments in a plane, is given in the Appendix. It is difficult to examine the cracks in a three-dimensional body, however, and one is usually limited to observations on an outcropping, a cut, or a plane obtained by sectioning a sample. In this paper, we consider two problems. The direct problem is to find the distribution of line segments in a plane section when the three-dimensional distribution of cracks is homogeneous, isotropic, and exponential. This distribution can be expressed in closed form by means of the Hankel functions. It will be shown that the distribution in a plane section is qualitatively different from the three-dimensional distribution in having a peak for a finite value of segment length, i.e., there is a most probable (non-zero) segment length. It is also concluded that the mean segment size in the plane is $\pi/2$ times the mean crack diameter in three dimensions. This is consistent with the well-known observation that small cracks have a lower probability of being intercepted by a plane than larger cracks. The number density of line segments is finally expressed in terms of the Hankel function of order zero.

The indirect problem is to infer the three-dimensional distribution of cracks from the distribution on a section, which could be, for example, an outcropping. This problem is solved by deriving an integral equation relating the three-dimensional distribution of cracks and the distribution of line segments in a plane, and showing that it can be solved for an arbitrary distribution of segments on the outcropping. The special case of the Hankel distribution leads to the exponential distribution in three dimensions, verifying the solution method.

METHOD

We begin by considering a distribution of penny-shaped cracks (i.e., cracks of zero thickness whose edges are circles of radius c) isotropically and homogeneously distributed in space. The intersection of a typical crack with a plane, which we denote as the x-y plane, is shown in Fig. 1, and we denote by z the distance of the center of the crack from the x-y plane. The angle between the crack plane and the x-y plane is represented by θ. Then,

$$z = \sqrt{c^2 - \ell^2/4}\,\sin\theta. \qquad (1)$$

where ℓ is the length of the segment formed by the intersection.

Now consider the statistics of the distribution. The number of cracks whose radii lie in the range $(c, c+\Delta c)$, with normals in the range of solid angles $(\Omega, \Omega+\Delta\Omega)$, and having centers in the interval $(z, z+\Delta z)$ is written $n(c,\Omega,z)\Delta c\Delta\Omega\Delta z$. Then the number of intercepts of length greater than ℓ is

$$P(\ell) = \int_{\ell/2}^{\infty} dc \int_{\Omega} d\Omega \int_{-z}^{z} d\zeta\; n(c,\Omega,\zeta) \qquad (2)$$

where Ω is the range of solid angles in half the unit sphere and $d\Omega = d\theta d\phi \sin\theta$. It is convenient to make the assumption that the distributions of c, θ, and φ are independent. This is critical, both because it makes the integrals tractable, and because otherwise an extensive amount of research and data analysis would be required to develop a good correlation for rocks over a wide range of sizes. In any case, it seems unlikely that crack size or orientation would be significantly affected by altitude. It is more likely that crack size and orientation would be correlated in bedded materials, but this possibility will not be addressed here. The assumptions of statistical independence can be expressed mathematically in the form

$$n(c,\Omega,\zeta) = n_1(c)n_2(\Omega)n_3(\zeta) \quad . \tag{3}$$

It is common in rock mechanics to assume that the distribution of cracks is exponential

$$n_1(c) = (N_o/\bar{c})e^{-c/\bar{c}} \quad . \tag{4}$$

This is supported by the analysis and observations of Glynn, Veneziano, and Einstein,[1] Baecher and Lanney,[2] and Barton[3] and a theoretical argument leading to this form is made in the Appendix. The number density of cracks per unit volume being constant, we may write

$$A\int_o^\infty n_1(c)\, dc \int_{-z}^{z} d\zeta\, n_3(\zeta) = 2N_o z\, A \tag{5}$$

where A is the area of the control volume and 2z is its width. Then

$$n_3 = 1 \quad . \tag{6}$$

Isotropy of crack orientations can be expressed by putting n_2 constant. It follows that

$$\int_o^{2\pi} d\phi \int_o^{\pi/2} n_2\, d\theta\, \sin\theta = 1 \quad , \tag{7}$$

or

$$n_2 = 1/2\pi \quad . \tag{8}$$

In writing the integral over all (equally probable) angles, it is assumed that the crack normal makes an angle with the x-y plane that lies between zero and $\pi/2$. This can always be arranged by selecting the appropriate one of the two possible senses for the crack normal. Integrating over φ, we find that

$$P(\ell) = (2/\bar{c})\int_{\ell/2}^{\infty} dc \int_o^{\pi/2} d\theta \int_o^{z} d\zeta\, N_o e^{-c/\bar{c}} \sin\theta \quad . \tag{9}$$

The integrals over ζ and θ are elementary, leading to

$$P(\ell) = \frac{\pi}{2}\frac{N_o}{\bar{c}}\int_{\ell/2}^{\infty} dc\; e^{-c/\bar{c}}\sqrt{c^2 - \ell^2/4} \quad . \tag{10}$$

By means of the change of variable

$$c = (\ell/2)\cosh\alpha \tag{11}$$

the integral can be transformed to the form of the Schlafli integral described by Watson[4] in Sec. 6.22,

$$K_\nu(\mu) = \int_o^\infty e^{-\mu\cosh\alpha}\cosh\nu\alpha\, d\alpha \quad . \tag{12}$$

Then

$$P(\ell) = \frac{\pi}{16}\frac{N_o}{\bar{c}}\ell^2\int_o^\infty d\alpha(\cosh 2\alpha - 1)e^{-\mu\cosh\alpha} \quad . \tag{13}$$

Using the recurrence formulas of Watson's Sec. 3.71, we find that

$$K_2 = K_o + (2/\mu)\, K_1 \tag{14}$$

and hence

$$P(\ell) = (\pi/2)N_o\,\bar{c}\,\mu K_1(\mu) \quad . \tag{15}$$

Since

$$\lim_{\mu\to o} \mu K_1(\mu) = 1 \tag{16}$$

according to the series given by Gradshteyn and Ryzhik[5] as 8.446, it follows that

$$P(o) = (\pi/2)N_o\,\bar{c} \quad . \tag{17}$$

This may be interpreted as the number of intersections per unit area with a fixed plane.

The mean size of the intercepts, $\bar{\ell}$, is the quantity such that

$$\int_0^\infty d\ell \, \frac{dP}{d\ell} \, (\ell - \bar{\ell}) = 0 \quad . \tag{18}$$

Now,

$$\int_0^\infty \ell d\ell \, \frac{dP}{d\ell} = \int_0^\infty P(\ell)d\ell = \pi N_o \bar{c}^2 \int_0^\infty \mu K_1(\mu)d\mu = \frac{\pi^2}{2} N_o \bar{c}^2 \tag{19}$$

where we make use of a special case of a result of Watson's in Sec. 13.21

$$\int_0^\infty K_1(\mu)\mu d\mu = \Gamma(1/2)\;\Gamma(3/2) = \pi/2 \quad . \tag{20}$$

Then

$$\bar{\ell} = \frac{(\pi^2/2)\; N_o \bar{c}^2}{(\pi/2)\; N_o \bar{c}} = \pi \bar{c} \tag{21}$$

so that the mean intercept length in a plane section is $\pi/2$ times larger than the mean diameter in three dimensions.

Now, using once again the recursion formulas of Watson's Sec. 3.71, the number density of intercepts is

$$n(\ell) = -\frac{dP}{d\ell} = \frac{\pi}{4} N_o \mu K_o(\mu) \quad , \tag{22}$$

and is shown graphically in Fig. 2. This has a maximum for $\mu = 0.60$, using Watson's tables of K_o, so that the most probable intercept length is

$$\tilde{\ell} = 1.20\;\bar{c} \tag{23}$$

or 60% of the mean crack diameter.

THE DETERMINATION OF THE GENERAL CRACK DISTRIBUTION FROM STATISTICS ON A SECTION

The indirect problem of crack statistics is to infer the distribution of crack size in three dimensions from observations on a plane. In the course of the preceding derivation, Eq. (10) was written with the assumption that the distribution of crack sizes is exponential. If, however, this assumption is not made, then the integral equation

$$P(\ell) = \frac{\pi}{2} \int_{\ell/2}^\infty dc \, \sqrt{c^2 - \ell^2/4} \; n_1(c) \tag{24}$$

for $n_1(c)$ is obtained. By making the change of variables

$$c^2 = x \quad , \; \ell^2/4 = y \tag{25}$$

it reduces to a special case of Abel's integral equation, whose solution is given, for example, by Whittaker and Watson.[6] Returning to the current variables, we obtain the solution to the indirect problem

$$n_1(c) = \frac{4}{\pi^2 c} \frac{d}{dc} \int_0^\infty c^3 \, \frac{P(\ell)\; d\ell/\ell}{(\ell^2/4 - c^2)^{3/2}} \quad . \tag{26}$$

Thus, the three-dimensional distribution of crack sizes can be obtained from the distribution on a section by quadrature.

Of special interest is the case when

$$P(\ell) = (\pi/4)N_o \; \ell \; K_1(\ell/2\bar{c}) \quad , \tag{27}$$

which is the solution obtained in the preceding section. Putting

$$\ell = 2c\sqrt{x} \tag{28}$$

the expression in (26) becomes

$$n_1(c) = \frac{N_o}{\pi c} \frac{d}{dc} \, c \int_0^\infty \frac{K_1(c\sqrt{x}/\bar{c})dx}{\sqrt{x}\,(x-1)^{3/2}} \quad . \tag{29}$$

The integral is given by Gradshteyn and Ryzhik[5] as 6.59.12 on p. 703, and after a series of elementary

$$n_1(c) = \frac{N_o}{\bar{c}} \, e^{-c/\bar{c}} \quad . \tag{30}$$

This is consistent with the assumption of (4), and verifies the validity of (26).

APPENDIX

ON THE PERSISTENCE OF POISSON STATISTICS

The Griffith theory of cracks predicts that the largest cracks are the most unstable, and hence one might expect rock masses to be sectioned by extension of the largest crack as soon as its critical stress

$$\sigma = \alpha\sqrt{\gamma E/c}$$

is reached. Here α denotes a constant that depends on the geometry and type of stress, γ denotes the surface energy; E, Young's modulus; and c, the crack radius or half-length. In fact, this behavior is not usually observed, but one sees a distribution of cracks that is often idealized by representing it by an exponential distribution. The number of cracks whose size exceeds c is then written as

$$N = N_0 e^{-c/\bar{c}} \quad .$$

where N denotes the number of cracks per unit volume and $\bar{c}$ denotes the mean size. The distribution is difficult to verify because it would be necessary to sample a large number of sections, and the observed distribution is difficult to quantify. In fact, the distribution in planes is not the same as in three dimensions, as shown in the body of the paper. It appears, therefore, useful to make a mathematical argument in favor of the exponential distribution. The argument is similar to one made by Rice[7] in connection with electronic shot noise.

We begin by considering a segment of length s confined to a plane, illustrated in Fig. A1. The frequency with which the line intersects other segments is denoted by ν (which for an exponential distribution in the plane is $2\pi/\bar{c}$, with $\bar{c}$ the mean crack size). We divide the segment into n sections of length s. Then the probability that a given section intersects another segment is $\nu s/n$. The probability that it does not intersect another segment is $1-\nu s/n$, and the probability that no segment intersects another segment is $(1-\nu s/n)$. For large n this can be written

$$p = 1-e^{-\nu s} \quad .$$

Hence, if the total number of segments in a unit area is N_0, the expected number of intersections is $N_0(1-e^{-\nu s})$. Assuming that intersection terminates growth of a segment, the number of segments whose size is less than s is $N_0(1-e^{\nu s})$. Hence the number of segments exceeding s in length is $N_0 e^{-\nu s}$. A detailed argument in three dimensions would be much more intricate because of the geometric and topological complications. A simple, but approximate, argument can be made by considering the frequency with which a random line intersects cracks in three dimensions, which we again denote by ν. As before the number of segments whose length exceeds s is $N_0 e^{-\nu s}$. The difficulty with this argument is that in three dimensions the crack edges are closed, and intersections may occur either tangentially or by edge contacts. These intersections may or may not inhibit growth, and if inhibited, growth is not necessarily terminated. It is conjectured that the same distribution holds in spite of these complications, but the problem certainly needs further attention.

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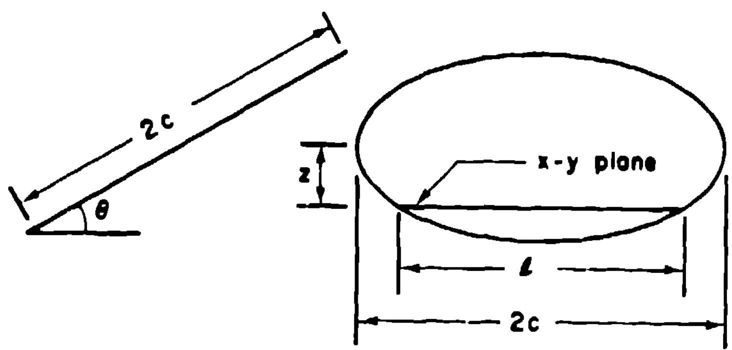

Fig. 1. Intersection of a typical penny-shaped crack with the x-y plane.

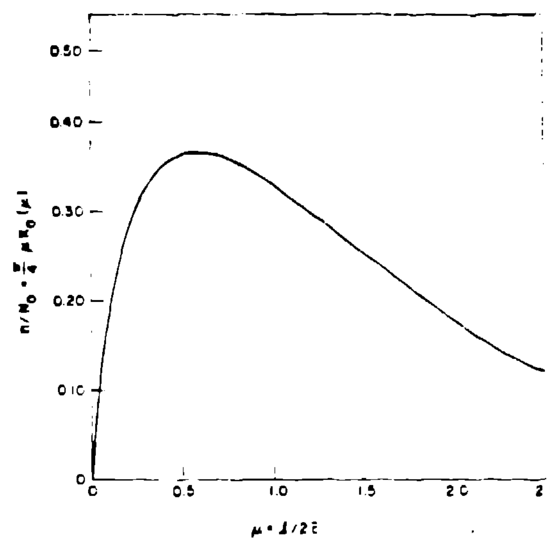

Fig. 2. The density of line segments formed by intersection of an exponential distribution of penny-shaped cracks with a plane, expressed in dimensionless form.

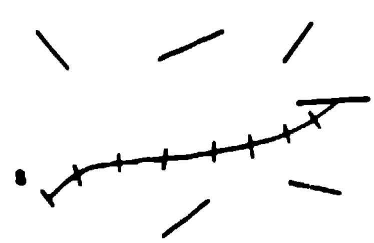

Fig. A-1. Intersection of a crack of length s with one of many other cracks in the plane, showing its division into n segments.