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A NEW APPROACH TO CHIRAL FERMIONS ON THE LATTICE*

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We present a new method for implementing fermions with chiral couplings to gauge fields on the lattice.

1. GENERAL CONSIDERATIONS

We wish to describe a method for formulating, on the lattice, field theories that contain Dirac particles with chiral couplings to gauge fields. As is well-known, the most straight-forward lattice transcription of the continuum action for a Dirac particle leads to the doubling problem: for every particle of a given chirality in the continuum theory, there appear on the lattice, in d dimensions, 2^d particles, with equal numbers of particles of left- and right-handed chirality.[1] No-go theorems[1, 2] state that it is impossible to eliminate the doubling problem and still maintain an exact chiral gauge symmetry.

Rather than follow an approach that attempts to circumvent the no-go theorems let us, instead, explore the possibility of abandoning exact chiral symmetry. We introduce a Wilson term in the lattice Dirac action, thereby giving an infinite mass to the doublers in the continuum limit. Of course, the Wilson term, having the Dirac structure of a mass, breaks the chiral symmetry. Such a loss of chiral symmetry may be acceptable in a theory like QCD, which contains no γ_5's in the interactions, but it is potentially a disaster in theories, like the standard electroweak model, in which the chiral symmetry is one of the gauge symmetries. In particular, in such a theory, one needs the chiral Ward identities (*i.e.* chiral current conservation) to control the number of renormalization counterterms that can arise from superficially divergent Green's functions and to preserve relationships between counterterms. Furthermore, one needs chiral-current conservation in *renormalized* Green's functions in order to prove that the negative norm states—ghosts and would-be Goldstone bosons—decouple from the theory, thereby preserving unitarity.

2. THE NEW APPROACH

2.1. Perturbative Current Conservation

In weak-coupling perturbation theory, violations of the Ward identities corresponding to chiral-current conservation appear because the Wilson term in the action generates a propagator mass term and interaction vertices whose Dirac structures are those of scalars. The Wilson mass and vertex terms have the property that they vanish as one takes the lattice spacing a to zero with all momenta fixed. For example, the Wilson mass is $M(p) = \sum_\mu (2/a)\sin^2(\frac{1}{2}p_\mu a)$. Hence, a Wilson mass term or vertex term gives a vanishing contribution in the continuum limit to any convergent subgraph. However, the Ward identity that one obtains from a given graph by dotting a momentum into its associated vertex actually involves expressions whose degree of

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divergence D is one greater than that of the original graph. The reason for this is that the application of the Feynman identity and its generalizations,[3] leads to the cancellation of a propagator or the replacement of a dimensionless vertex with a Wilson mass term. Hence, we conclude that subgraphs with $D \geq -1$ potentially violate chiral current conservation even in the continuum limit, whereas subgraphs with $D < -1$ respect chiral current conservation as $a \to 0$.

2.2. Strategy

Our general strategy will be to restore chiral current conservation in the continuum limit, for renormalized Green's functions, by introducing suitable renormalization counterterms. Our approach differs from that of Borrelli *et al.*,[4] however, in that we do not add the many counterterms allowed by the lattice symmetry directly to the action. Rather, we implement a large class of counterterms by introducing two auxiliary Dirac species, one with Fermi statistics and one with Bose statistics. As we shall see, because of cancellations between the auxiliary species, the number of physical Dirac species is unchanged by this procedure, aside from a trivial (non-dynamical) doubling.

Of course, no counterterm can remove the triangle anomaly in the axial-vector current without producing a violation of vector-current conservation.[5] We deal with the anomaly by requiring that the theory in question contain a set of *physical* species satisfying the anomaly-cancellation condition $\mathrm{Tr}\,\lambda_a\{\lambda_b, \lambda_c\} = 0$. Here, the λ's are the coupling matrices associated with the chiral gauge interaction.

We illustrate our approach in the context of massless chiral QED—that is, QED but with left-handed couplings ($\propto \frac{1}{2}(1 - \gamma_5)$) of the photon to the electron. Of course, the weak-coupling perturbative analysis that we give here is strictly valid in the limit $a \to 0$ only in an asymptotically free (*i.e.*, non-Abelian) theory. However, in order to simplify the presentation, we discuss the Abelian theory here. The generalizations to the non-Abelian case and to theories involving both left- and right-handed gauge couplings and fermion-Higgs Yukawa couplings appear to be straightforward.

2.3. Electron Loops with No Radiative Corrections

First we examine the case of an electron loop with no radiative corrections (*i.e.*, no photon propagators). We consider explicitly only electron-photon vertices, although our method may be of use in loops involving external currents as well. The one-electron-loop graphs that potentially violate chiral-current conservation ($D \geq -1$) are those with 2–5 external photons. We call these graphs (including all permutations of external photons and seagull graphs with the same number of external photons) the "vacuum polarization", "triangle", "rectangle", and "pentagon", respectively. For these graphs, let us decompose each left-handed vertex into a vector vertex (corresponding to the 1 in the left-handed projector) and an axial-vector vertex (corresponding to the γ_5), so that each graph is written as a sum of graphs involving vector and axial-vector vertices.

Consider first those graphs in the sum in which the electron loop contains an odd number of γ_5's. Current conservation at the vector vertices holds exactly, since a (gauged) Wilson mass term in the action does not break the vector gauge symmetry. The vacuum polarization graph is identically zero because the trace of an odd number of γ_5's is proportional to the completely anti-symmetric tensor $\epsilon_{\mu\nu\rho\sigma}$, and there are not enough independent four-vectors to saturate the indices. In the triangle graph, current conservation is violated at the axial-vector vertices. However, this violation is precisely the triangle anomaly, and it cancels, provided that the physical complement of Dirac species satisfies the anomaly-cancellation condition. The rectangle and pentagon graphs also potentially violate current conservation at the axial-vector vertices. In an Abelian theory, these violations cancel because of Bose symmetry; in a non-Abelian theory, the violations have the group structure of the triangle anomaly and cancel when one sums over a physical complement of Dirac species.

Now consider the graphs in which the electron loop contains an even number of γ_5's. Suppose we could anti-commute the γ_5's through propagators and vertices to bring pairs of γ_5's together and then use $\gamma_5^2 = 1$ to eliminate them. The resulting expressions would be vector-like, and current conservation would be satisfied exactly. The problem with this manipulation is that the Wilson mass and vertex terms, having the Dirac structures of scalars, commute, rather than anti-commute with γ_5. However, the Wilson mass terms in rationalized-propagator numerators and the Wilson vertex terms are irrelevant unless the electron loop momentum is of order $\pi/a \gg$ external momenta. Thus, the net effect of anti-commuting, rather than commuting, the γ_5's with the Wilson mass terms and vertices, is to add counterterms to the graphs. Similar counterterms arise in dimensional regularization.

A perturbative rule for anti-commuting γ_5's does no good in a lattice simulation unless it can be implemented at the level of the action. To that end, we consider the following lattice action for the Dirac particle in chiral QED.

$$S_e = S_K + S_{KI} + S_W + S_{WI}, \qquad (1a)$$

where

$$S_K = a^4 \sum_{x,\mu} \overline{\psi}(x)\gamma_\mu \frac{1}{2a}[\psi(x+a_\mu) - \psi(x-a_\mu)] \quad (1b)$$

is the "naive" kinetic term,

$$S_{KI} = a^4 \sum_{x,\mu} \overline{\psi}(x)\gamma_\mu \frac{1}{2}[1-\gamma_5 \begin{pmatrix} 1 \\ \tau_1 \\ \tau_1 \end{pmatrix}] \times \frac{1}{2a}[(U_\mu(x)-1)\psi(x+a_\mu) - (U_\mu^\dagger(x-a_\mu)-1)\psi(x-a_\mu)] \quad (1c)$$

is the contribution to the action that arises from gauging the kinetic term,

$$S_W = a^4 \sum_{x,\mu} \overline{\psi}(x) \begin{pmatrix} 1 \\ 1 \\ \tau_2 \end{pmatrix} \times \frac{1}{2a}[\psi(x+a_\mu) + \psi(x-a_\mu) - 2\psi(x)] \quad (1d)$$

is the Wilson term, and

$$S_{WI} = a^4 \sum_{x,\mu} \overline{\psi}(x) \begin{pmatrix} 1 \\ 1 \\ \tau_2 \end{pmatrix} \frac{1}{2}[1-\gamma_5 \begin{pmatrix} 1 \\ \tau_1 \\ \tau_1 \end{pmatrix}] \times \frac{1}{2a}[(U_\mu(x)-1)\psi(x+a_\mu) + (U_\mu^\dagger(x-a_\mu)-1)\psi(x-a_\mu)] \quad (1e)$$

is the contribution to the action that arises from gauging the Wilson term. The Dirac matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and $\gamma_5^2 = 1$. The U_μ's are the usual gauge-field link variables. The τ_i's are matrices in a two-dimensional auxiliary "flavor" space, with $\{\tau_1, \tau_2\} = 0$ and $\tau_i^2 = 1$, and **1** is a unit matrix in that space. The fields $\overline{\psi}$ and ψ are Dirac spinors and are doublets in the τ space. The action (1) is actually a short hand for the actions for three separate species. In the column vectors that appear in (1), the upper entry corresponds to "electron" type 1, which is an ordinary Wilson-Dirac particle satisfying Fermi statistics. This is the original electron in the theory. The middle entry corresponds to "electron" type 2, which is an auxiliary Dirac particle satisfying Bose statistics. The bottom entry corresponds to "electron" type 3, which is an auxiliary Dirac particle satisfying Fermi statistics. The type 2 electron differs from the type 1 electron in that there is a factor τ_1 associated with every vertex γ_5 and there is no minus sign for a closed loop. The type 3 electron differs from the type 1 electron in that there is a factor τ_1 associated with every vertex γ_5 and a factor τ_2 associated with every Wilson mass term or Wilson vertex term. The action (1) is to be employed only in computing electron loop contributions (Dirac determinants). For external electron lines, one uses the ordinary Wilson-Dirac action.

Now let us see how the auxiliary fermions implement the counterterm strategy that we have outlined. For electron loops with an odd number of γ_5's, the type 2 and type 3 contributions vanish because the τ trace contains an odd number of τ_1's. Then we are left with the type 1 contribution, which is that of the original electron (multiplied by a factor of 2 from the τ trace). The violations of chiral-current conservation cancel because the physical species satisfy the anomaly-cancellation condition. For electron loops with an even number of γ_5's, we can use $\tau_1^2 = 1$ to eliminate the τ_1 matrices from the type 2 contribution. Then the type 1 and type 2 contributions cancel because of the relative minus sign from their opposite statistics. In the type 3 contribution, the $\gamma_5\tau_1$ factors anti-commute with all vertices and propagator factors—either because the γ_5 anti-commutes with another γ matrix in a non-Wilson propagator or vertex term or because the τ_1 anti-commutes with a τ_2 in a Wilson mass or vertex term. Then we can use $(\gamma_5\tau_1)^2 = 1$ to eliminate the γ_5's and τ_1's from the type 3 contribution. Now, the Dirac trace vanishes unless it contains an even number of Wilson mass terms, and, hence, an even number of τ_2's. Thus, we can use $\tau_2^2 = 1$ to eliminate the τ_2's from the type 3 contribution. At this point, the type 3 contribution has reduced to the contribution from a Wilson-Dirac particle with vector-like gauge interactions (aside from a factor of 2 from the τ trace). For it, current conservation is exact.

The type 2 and type 3 contributions would cancel each other precisely if one could neglect the Wilson vertex terms and the numerator Wilson mass terms. Such Wilson terms can contribute only for electron loop momenta of $O(\pi/a)$. Thus, the net effect, in the continuum limit, of the type 2 and type 3 contributions is to add local counterterms to the electron loops. Power counting shows that the counterterms are nonvanishing in the continuum limit only for loops with $D \geq 0$. Since the counterterms arise from loop momenta of $O(\pi/a)$, they cannot contribute to the imaginary parts of loops with fixed (finite) external momenta. Hence, the type 2 and type 3 contributions have no effect on Minkowski-space unitarity in the limit $a \to 0$.

In the method that we have presented, the τ trace introduces a factor of two in the contribution of each electron loop. This is a trivial, non-dynamical doubling, which merely replicates the contribution of the original electron without introducing species of opposite chirality. We expect, then, that the doubling can be removed in a simulation by computing the square roots of the determinants (or inverse determinant for type 2) of the Dirac operators. Alternatively, one could avoid taking the square root of the type 1 determinant by introducing the type 1 electron as a singlet in the τ space.

Finally, we note that there is some freedom with regard to the gauging of the Wilson term. For the type 3 electron we *must* gauge the Wilson term; that is, we must retain interactions of the type S_{WI} in (1e) in order the preserve current conservation for the vector-like interactions. However, for the type 1 and type 2 electrons, we require only that their gaugings be the same, so that their contributions cancel exactly in loops containing an even number of γ_5's. That is, we are free to drop the interactions in S_{WI} for the type 1 and type 2 electrons. This would lead to violations of vector-current conservation in loops containing an odd number of γ_5's. However, these violations have the same group structure as those that appear in the axial-vector-current anomaly, and their cancellation is guaranteed by the anomaly-cancellation condition.

2.4. Radiative Corrections

Now let us consider radiative corrections to external electron lines and electron loops. We assume that the photon field has been fixed to a renormalizable gauge. Then the radiative corrections with $D \geq -1$ are the corrections to the electron proper self-energy, the corrections to the electron–one-photon proper vertex, and the corrections to the electron–two-photon proper vertex.

For a type 3 electron, all of the interaction vertices are effectively vector-like, even in the radiative corrections, so there are no violations of current conservation. However, owing to the presence of the Wilson vertex terms, the self-energy correction does generate a mass for the type 3 electron, so it is necessary to tune a mass counterterm in order to keep the type 3 electron massless. For the type 1 and type 2 electrons, we can avoid the generation of a mass by dropping S_{WI}, the gauge interactions associated with the Wilson term.[6]

For the type 1 and type 2 electrons, the Ward identity relating the proper vertex and proper self-energy contains a term that violates current conservation:

$$\sum_{\mu} d_\mu(l)\Gamma_\mu(p,l) = \Sigma(p+l) - \Sigma(p) + \tilde{\Gamma}(p,l). \quad (2)$$

Here l is the photon momentum; $d_\mu(l) = \frac{2}{a}\sin(\frac{1}{2}l_\mu a)$; p is the external electron momentum; Γ is the proper vertex; Σ is the proper self-energy; and $\tilde{\Gamma}$ is a proper vertex obtained by replacing the electron-photon vertex in Γ by $\frac{1}{2}[(1-\gamma_5)M(p'+l) - (1+\gamma_5)M(p')]$, where p' is the electron momentum flowing into the elementary vertex. $\tilde{\Gamma}$ violates current conservation, and, because it arises from the Wilson mass term, it contributes in $O(a^0)$ only for loop momenta of order π/a. That is, to leading order in a, $\tilde{\Gamma}$ behaves like a counterterm: it is a polynomial in the external momenta, namely, $\sum_\mu l_\mu \gamma_\mu \frac{1}{2}(1-\gamma_5)$. Hence, $\tilde{\Gamma}$ can be removed by a renormalization of the electron-photon vertex. The counterterm is

$$(\tilde{Z}_1 - 1)a^4 \sum_{x,\mu}\Big\{\overline{\psi}(x)\gamma_\mu \frac{1}{2}[1-\gamma_5\begin{pmatrix}1\\ \tau_1\\ 0\end{pmatrix}]\frac{ig}{2} \times [A_\mu(x)\psi(x+a_\mu) + A_\mu(x-a_\mu)\psi(x-a_\mu)]\Big\}, \quad (3)$$

where A_μ is the gauge field. We note that such a counterterm also appears in dimensional regularization.[7] Because contributions to the renormalization factor $\tilde{Z}_1$ are dominated by loop momenta of order π/a, they are calculable in lattice perturbation theory for g in the scaling region. It is easy to see that $\tilde{Z}_1$ is actually a finite renormalization. It can be shown that violations of current conservation associated with contributions to the proper electron–two-photon vertex for type 1 and type 2 electrons are suppressed by at least one power of a.

Thus far, we have argued that, at fixed electron momentum, violations of current conservation associated with radiative corrections to type 1 and type 2 electrons can be removed by finite renormalization of the electron–one-photon proper vertex. We have ignored violations of current conservation that are suppressed by powers of a. These are potentially important when the electron-loop momentum itself is of order π/a. However, it can be shown, by proving a variant of the Adler-Bardeen anomaly-no-renormalization theorem,[8] that such violations of current conservation do not, in fact, arise. Specifically, we have proven that, if the renormalized radiative corrections do not violate current conservation in the limit $a \to 0$ for fixed electron momentum, and there are no violations of current conservation in type 1 and type 2 loops without radiative corrections (physical species cancellation of the anomaly), then there are no violations of current conservation in the limit $a \to 0$, to all orders in perturbation theory.

3. SIMULATIONS

Let us summarize the novel requirements that arise in carrying out a simulation using the lattice chiral-fermion method we have presented. First, one must simulate a theory in which the physical fermion species satisfy the anomaly cancellation condition. One must fix to a renormalizable gauge. For external legs, one can use the ordinary Wilson-Dirac fermion. However, in loops, the three Dirac particles of (1) must be used. For the type 1 and type 3 particles, one computes the square root of the determinant of the modified Dirac operator, while for the type 2 particle, owing to its Bose statistics, one computes the inverse of the square root of the determinant. One has the option of introducing the type 1 particle as a singlet in the auxiliary flavor space, which would eliminate the need to compute the square root of its determinant. The Wilson term is gauged for the type 3 particle, but not for type 1 and type 2. (S_{WI} in (1e) is dropped for type 1 and type 2.) A mass counterterm (hopping parameter) must be tuned for the type 3 particle in order to obtain a vanishing renormalized mass. Finally, a finite vertex renormalization counterterm (3) must be tuned, or its coefficient computed from lattice perturbation theory, in order to insure that the chiral current is conserved. For purposes of a simulation, it may be most convenient to write the counterterm (3) in terms of the link variables U rather than the gauge fields A. The expressions in terms of U and A need be equal only to order a^0. Any differences of $O(a)$ merely change the value of $\tilde{Z}_1$ at higher orders in weak-coupling perturbation theory.

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