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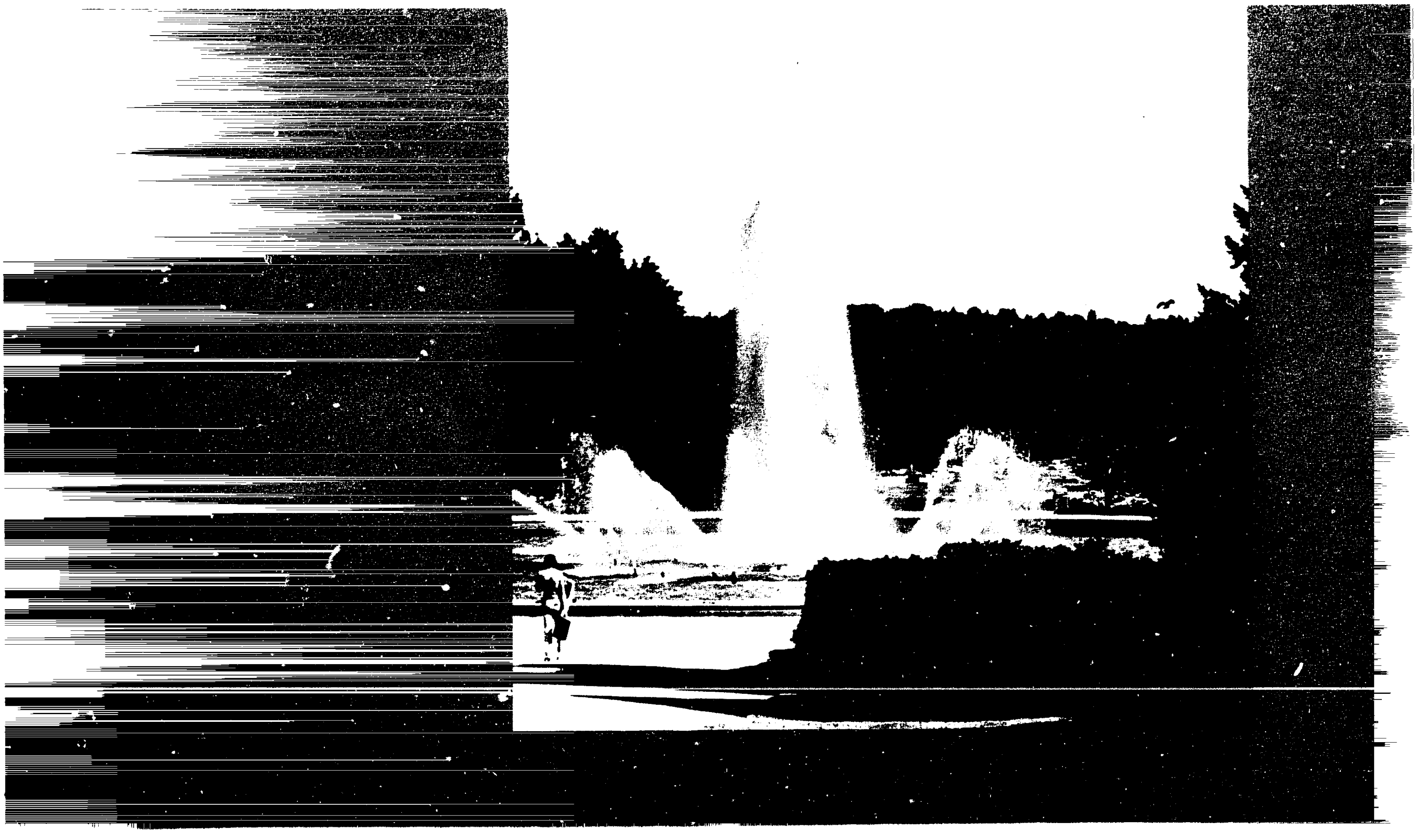

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INITIAL-VALUE PROBLEMS AND STABILITY IN SHEAR FLOWS

W.O. Criminale

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W. O. Criminale
Department of Applied Mathematics
University of Washington
Seattle, Washington 98195
U.S.A.

Introduction

Over a hundred years of history have been given to the general problem of perturbations of shear flows in fluid mechanics. In principle, the problem is well posed. Given a specific flow, an initial disturbance is introduced in a prescribed manner and the subsequent dynamics is sought subject to satisfying pertinent boundary conditions. In effect, it is the Navier-Stokes equations themselves that dictate this formulation since local immediate changes in any initial designation can be directly calculated by using these equations. Still, even today and with the use of high speed modern computers, significant results have been limited. Direct numerical calculations are not yet able to incorporate full three-dimensional effects and, strictly speaking, computing time is limited. At the other extreme, the linear mathematical problem is complex and does not lend itself well to solution. As a result, determination of the long-time stability of any flow has been taken as the hallmark and it is more than obtainable for almost all flows that have been investigated. Indeed, experiments–most notably those dealing with the boundary layer (Schubauer and Skramstad 1947) – have confirmed that the basic means of the approach is valid and does lead to correct physical output. From a theoretical basis, this means that the asymptotic fate of the flow subject to a small disturbance can be predicted by the well-known normal mode (travelling waves) form of solution to the analytical problem. In short, only one unstable eigenvalue is sufficient to make this assessment (Lin 1955; Betchov and Criminale 1967; Drazin and Reid 1981).

Computational fluid mechanical schemes actually rely on the output of the linear eigenvalue problem. For example, even though a real flow will be naturally unstable and a transition process will evolve, machine restrictions dictate that it is better to excite a flow with the most unstable eigenmode, thereby guaranting that the dynamics will rapidly occur. This strategy merely replaces one difficulty with another. True initial-value problems require detailed information far beyond the meager output that has been obtained by classical stability theory. Even with the linear problem, this fact has been well demonstrated by both Criminale and Kovasznay (1962) and Gaster (1968) where both of these works sought to do such a problem for the case of the laminar boundary layer. The initial transient period was impossible because very few of the eigenvalues (and the corresponding eigenfunctions) of the discrete spectrum were known. Extreme approximations had to be made in order to make any calculations and it was (and still is) clear that no arbitrary disturbance could be initially represented in this manner. Thus, although, the numerical treatment does use directly the Navier-Stokes equations, there is a considerable bias the certainly nothing is known about the transient dynamics. The analytical treatment omissions, on the other hand, are due completely to the complexity of the mathematics - non self-adjoint operators, singular perturbation needs – and not in the formulation. The early period has remained unknown throughout.

The importance in understanding the transient period has gained even more importance over and above the fact that the laminar flow problem continues unresolved. Turbulent flows are now known to depend upon the origin of the breakdown of the flow and arguments can be made to support the premise that the generation of large-scale oscillations that are found in turbulent shear flows (coherent structure as reported by experiments) can be synthesized using this form of modelling, that is, solving an initial-value problem. (cf. Jimenez 1981; Criminale 1987). Add to this the numerous facts of many laminar flows that are unexplained (such as the point that three-dimensionality occurs long before nonlinearity in laminar boundary layers, for example) resolution can only be made if the initial period of the dynamics is known explicity.

It has long been known that the early period can be dominated by other aspects of the problem. First, even if there are no growing eigen modes, the early period is easily controlled by the least damped modes. Second, most physical problems will possess a continuous eigen spectrum as well as the discrete set of modes. This aspect has been discussed by several authors (cf. Case 1960, 1961; Lin 1961) but actually little of the implications have been given. But, again, a complete knowledge of this information must be had in order to arbitrarily represent any initial disturbance.

Recent exact solutions found for shear flows have provided a means whereby a complete analysis of an initial-value problem is possible (Craik and Criminale 1986) and Criminale and Drazin (1990) have investigated various prototypical shear flows using the strategy. In a very general manner, exact solutions of the Navier-Stokes equations were presented for three-dimensional, time-dependent and nonparallel mean flows. The only restriction was that the spatial variation of the mean velocity be limited to constant shear. The result is that a set of basic solutions can be found for the linear perturbation problem that are (i) of closed from; (ii) contain both the discrete and continuous spectra allowing for arbitrary initial disturbances; (iii) the complications of critical layers or singular perturbation analysis are no longer required; (iv) even the near and far fields can be determined as well as the early and asymptotic temporal behaviors; (v) Lagrangian descriptions can also be persued using the solutions obtained, allowing for more insight into the mixing and vorticity physics. The solutions stem from earlier work of Kelvin (1887) and Orr (1907) and, unlike familiar normal mode solutions, these solutions are non separable in some of the independent variables. It will later be seen that his very property leads to pseudo nonlinear behavior denoted by a shortening of scales in certain directions and, if the amplitude grows as well, then an increase is coupled with the scale change. In essence, it will be shown that the asymptotic status can be a very moot point because, during the early transient period, the system can become completely nonlinear.

It is the plan of this presentation of outline three major topics of the novel analysis. (1) Use of exact solutions in solving initial-value problems; (2) detailed linear example that illustrates the relevant features of the early period and possible instability; (3) Lagrangian mechanics and the link to computational fluid dynamics.

General considerations

Although it is not a requirement, attention will be limited to flows of constant density and in the absence of body forces. Thus, the dependent variables are the three components of the velocity and the pressure. It will be assumed that all of these can be decomposed into a mean motion and a fluctuation, that is,

$$\underset{\sim}{u} = \underset{\sim}{U} + \underset{\sim}{u}' \qquad p/\rho_0 = P + p' \tag{1}$$

With ρ_0 a constant density. The perturbations are not necessarily small and therefore the governing Navier-Stokes equations are

$$\nabla \cdot \underset{\sim}{u}' = 0 \tag{2}$$

$$\frac{\partial \underset{\sim}{u}'}{\partial t} + \underset{\sim}{U} \cdot \nabla \underset{\sim}{u}' + \underset{\sim}{u}' \cdot \nabla \underset{\sim}{u}' = -\nabla p' + \nu \nabla^2 \underset{\sim}{u}' \tag{3}$$

where $\underset{\sim}{U}$ satisfies its own equation and ν is the kinematic viscosity.

In the special case for which the perturbation velocity is of the form

$$\underset{\sim}{u}'(\underset{\sim}{x}, t) = f(\underset{\sim}{x}, t)\underset{\sim}{\widehat{u}}(t) \tag{4}$$

it follows that

$$\underset{\sim}{\widehat{u}} \cdot \nabla f = \nabla \cdot \underset{\sim}{u}' = 0 \tag{5}$$

The consequence of this result is strongly evident because it implies that the nonlinear terms in the Navier-Stokes equations due to the fluctuations are identically zero for smooth functions f, $\underset{\sim}{\widehat{u}}$. Therefore. the perturbation problem can be solved by use of linear quations which are:

$$\nabla \cdot \underset{\sim}{u}' = 0 \tag{6}$$

$$\frac{\partial \underset{\sim}{u}'}{\partial t} + \underset{\sim}{U} \cdot \nabla' \underset{\sim}{u}' + \underset{\sim}{u}' \cdot \nabla \underset{\sim}{U} = -\nabla p' + \nu \nabla^2 \underset{\sim}{u}' \tag{7}$$

These equations are valid without approximation. Hence, if the velocity field is initially in the same direction everywhere so that $\underset{\sim}{u}'(\underset{\sim}{x}, 0) = f(\underset{\sim}{x}, 0)\underset{\sim}{\widehat{u}}(0)$ then a linearized problem can be solved to find the exact solution of the full governing equations. In terms of the new variables this means

$$\underset{\sim}{\widehat{u}} \cdot \nabla f = 0 \tag{8}$$

$$f\frac{d\widehat{\underset{\sim}{u}}}{dt} + \frac{\partial f}{\partial t}\widehat{\underset{\sim}{u}} + (\underset{\sim}{U}\cdot\nabla f)\widehat{\underset{\sim}{u}} + f\widehat{\underset{\sim}{u}}\cdot\nabla\underset{\sim}{U} = -\nabla p' + \nu(\nabla^2 f)\widehat{\underset{\sim}{u}} \tag{9}$$

For a stready basic flow it has been customary to solve the linearized problem by using the method of normal modes. This form of solution is actually one that is assumed separable in the dependent variables and represents travelling waves and can be written as

$$f(\underset{\sim}{x},t) = F(\xi),$$

$$\widehat{\underset{\sim}{u}} = \exp(\int \sigma dt) \tag{10}$$

with $$\xi = \underset{\sim}{\alpha}\cdot\underset{\sim}{x} + \int \omega dt.$$

For some wave number with vector $\underset{\sim}{\alpha}$, frequency ω, and relative growth (or decay) rate σ. For $\underset{\sim}{U}$ independent of t, all of these parameters become constants; more generally they are functions of t. Use of this form of solution leads to

$$\underset{\sim}{\alpha}\cdot\widehat{\underset{\sim}{u}} = 0 \tag{11}$$

$$\left\{\sigma F + \left(\frac{d\underset{\sim}{\alpha}}{dt}\cdot\underset{\sim}{x} + \omega + \underset{\sim}{\alpha}\cdot\underset{\sim}{U}\right)F'\right\}\widehat{\underset{\sim}{u}} + F\widehat{\underset{\sim}{u}}\cdot\nabla\underset{\sim}{U} = -\nabla p' + \nu\alpha^2 F''\widehat{\underset{\sim}{u}}. \tag{12}$$

This problem, along with the appropriate boundary conditions, may be solved in principle as an eigenvalue problem where ω, σ, $\underset{\sim}{\alpha}$ satisfy some eigenvalue relation. When $\underset{\sim}{U}$ is not a function of t, and the parameters are constants, then F is the exponential function and the eigenvalue relation will be the type traditional in the theory of hydrodynamic stability. Of course, in this case, it is expedient to use complex variables with $s = \sigma + i\omega$ etc.

This formulation can be seen at once to allow a superposition of normal modes ***provided*** $\underset{\sim}{u}'$ has the form taken in (4). Then, for example,

$$\underset{\sim}{u}'(\underset{\sim}{x},t) = \widehat{\underset{\sim}{u}}(t)\sum_n F_n(\xi_n);$$

$$\widehat{\underset{\sim}{u}} \propto \exp(\sigma_n t) \tag{13}$$

where $\xi_n = \underset{\sim}{\alpha}_n\cdot\underset{\sim}{x} + \omega_n$ and $\omega_n, \sigma_n, \underset{\sim}{\alpha}_n$ satisfy the same eigenvalue relation for all n for a steady $\underset{\sim}{U}$. Indeed, to solve an initial-value problem for $\underset{\sim}{u}'(\underset{\sim}{x},0)$ in the same direction everywhere, there will be a general need for a complete set of eigenfunctions $\{F_n\}$ in order to expand specified distributions. It should also be noted that all $\underset{\sim}{\alpha}_n$ lie in the plane perpendicular to $\underset{\sim}{\widehat{u}}$.

The Linear Problem

a. Background

Normal mode solutions are not the only solutions to the linear poblem. This point can be noted if it is recognized that the eigenvalue problem is one that produces a discrete spectrum. The continuous spectrum, if it exists, remains to be determined.

Craik and Criminale (1986) obtained that most general solutions for the linear problem (12) when the main flow was of the form $U_i = \sigma_{ij}(t)x_j + U_i^{(0)}(t)$. It appears that this approach can be used for other profiles as well, for example, plane Poiseuille flow (Drazin and Criminale 1991), but it is sufficient to discuss the basic linear shear flow to illustrate the general needs. As was noted, the steady problem can have normal mode solution and all parameters are constants. The more implicit form is

$$u_i' = \widehat{u}_i(t)\exp[i\alpha_j(t)x_j + i\delta(t)] \tag{14}$$

when the mean flow has linear shear. At time $t = 0$, (14) becomes a Fourier expansion. If the mean flow is infinite in extent then the solutions are bounded. In fact, these are the only solutions for such a flow that meets the required boundary conditions.

Problems that are prescribed with one or more solid boundaries in the flow require alteration to meet boundary conditions. A sum of (14) is still valid for initial values but cannot be adapted to fit physical constraints at one or more boundaries for all time. In order to made this task tractable, unsteady irrotional disturbances must be introduced together with the solutions to the set of equations that have been developed. In short, Laplace's equation must now be employed as well as the vortical perturbations described by the modes; such additional solutions do not exist when the boundaries are removed. This fact was overlooked by both Kelvin (1887) and Orr (1907), but was recognized later by Marcus and Press (1977) when investigating plane bounded Couette flow in these terms for two-dimensional distrubances.

The modal approach is strongly similar to the classical travelling wave (normal modes) assumption used in stability theory, but differences should be noted. For example, the fact that the wave numbers are time-dependent implies that these solutions are nonseparable, at least in this coordinate system. Normal modes for the linearly varying basic shear do constitute a separable solution if the perturbation vorticity is taken as the dependent variable. For basic shear flows with curvature in the velocity profile, the linear equations are not separable, even in terms of the vorticity. This topic is discussed somewhat by Marcus and Press with the central suggestion being that a change of corrdinates can be made that does render the problem again separable. More specifically, the shift is from an Eulerian to a moving frame of reference and, as well be shown, this step has advantages with plane bounded Couette flow as considered by Marcus and Press being but one reducible case of a more general transformation.

A stronger difference between the two approaches is that the travelling wave assumption leads to a boundary-value problem with a differential equation that is not of standard form, i.e. a non self-adjoint differential equation. Considered with viscosity, this route also requires treatment of a singular perturbation problem. Even with the computer, success is restricted to determining a stability boundary in terms of the most dangerous eigenvalue. Eigenfunctions are few and the important continuous eigen spectrum is essentially nonexistent. Most, if not all, of these complications are completely resolved with a generalized model approach and initial-value problems can be solved completely.

As suggested, Marcus and Press outlined a routine for making the necessary calculations for plane bounded Couette flow. With the addition of the unsteady irrotational velocity field, the method of images was employed to meet boundary conditions. Unfortunately, no significant output was generated but rather conclusions were left in the form of a determinant that was not evaluated. The proposed alternative allows for expounding such details in the investigation with the result being the understanding of a vast amount of physics of the early period of dynamical evolution.

The salient points to be noted a *priori* are:

(i) The system is linear;

(ii) the fundamental nature of the problem – as shown by (14) – is different because the Fourier amplitudes are functions of time rather than spatial variable as is common with normal mdoes;

(iii) since the strict requirement of boundary conditions for the disturbances in the fully infinite shear flow is boundedness at infinity such solutions can be incorporated into the more general problem without alteration; and

(iv) in less exacting terms, it would greatly facilitate the results if an infinite sum could be replaced with a closed form solution.

The first two points leads to the suggestion that (a) a transformation of the distrubance equations can be found that results in a set of linear partial differential equations whose coefficients are functions of time only and, subsequently, (b) Fourier transforms can then be used to derive a set of ordinary coupled differential equations for the Fourier amplitudes. This step insures the boundedness at infinity as well. The culmination of all steps does provide closed form solutions.

b. Coordinate transformation

It is sufficient to find a transforamtion of the independent variables (x_i, t) to a moving set (ξ_i, T) in order that the new set of linear partial differential equations depends only on T or are constants. Applying this chain rule to equations (6) to (7) gives

$$\frac{\partial \xi_k}{\partial x_i}\frac{\partial u_i}{\partial \xi_i} + \frac{\partial T}{\partial x_i}\frac{\partial u_i}{\partial T} = 0 \tag{15}$$

$$\begin{aligned} &\left(\frac{\partial \xi_k}{\partial t}\frac{\partial u_i}{\partial \xi_k} + \frac{\partial T}{\partial t}\frac{\partial u_i}{\partial T}\right) + U_j\left(\frac{\partial \xi_k}{\partial x_j}\frac{\partial u_i}{\partial \xi_k} + \frac{\partial T}{\partial x_j}\frac{\partial u_i}{\partial T}\right) \\ &\quad + \sigma_{ij}u_j = -\left(\frac{\partial \xi_k}{\partial x_i}\frac{\partial p}{\partial \xi_k} + \frac{\partial T}{\partial x_i}\frac{\partial p}{\partial T}\right) \\ &\quad + \frac{\partial \xi_k}{\partial x_j}\frac{\partial}{\partial \xi_k}\left(\frac{\partial \xi_s}{\partial x_j}\frac{\partial u_i}{\partial \xi_s} + \frac{\partial T}{\partial x_j}\frac{\partial u_i}{\partial T}\right) + \nu\frac{\partial T}{\partial x_j}\frac{\partial}{\partial T}\left(\frac{\partial \xi_s}{\partial x_j}\frac{\partial u_i}{\partial \xi_s} + \frac{\partial T}{\partial x_j}\frac{\partial u_i}{\partial T}\right) \end{aligned} \tag{16}$$

where $U_i = \sigma_{ij}x_j + U_i^0$ has been used for the basic gradients respectively. Inspection of (15) to (16) shows that the transformation is achieved if

$$\frac{\partial T}{\partial t} + U_j\frac{\partial T}{\partial x_j} = F(T) \tag{17}$$

$$\frac{\partial \xi_k}{\partial t} + U_j\frac{\partial \xi_k}{\partial x_j} = Q_k(T) \tag{18}$$

$$\frac{\partial \xi_k}{\partial x_i} = a_{ki}(T) \tag{19}$$

$$\frac{\partial T}{\partial x_i} = S_i(T) \tag{20}$$

where F, Q_k, a_{ki} and S_i are all arbitrary functions of T.

Further determination is best made by recognizing that the new variables should be independent and the realization that it is most useful to have $T = t$. Thus, $S_i = 0$ in (20) and $F = 1$ in (17) to satisfy these demands. And, immediately from (19)

$$\xi_k = a_{ki}(t)x_i. \tag{21}$$

Certainly, by requiring $\xi_k = x_k$ at time $t = T = 0$, then

$$a_{ki}(0) = \delta_{ki}. \tag{22}$$

Substitution of U_i, (21), and (22) into the final equation (18) gives the set of differential equations for the transformation matrix or

$$\frac{da_{ki}}{dt}x_i + [\sigma_{is}x_s + U_i^0]a_{ki} = Q_k(t) \tag{23}$$

which can only be satisfied if

$$\frac{da_{ki}}{dt} + \sigma_{is}a_{ki} = 0 \tag{24}$$

and

$$Q_k(t) = a_{ki}U_i^0. \tag{25}$$

The relations (24), (25) are closely aligned to the discussions already given by Craik and Criminale.

Defining $A = \{a_{ij}(t)\}$, $S = \{\sigma_{ij}(t)\}$, and $U_i^0 = \{U_1^0(t), U_2^0(t), U_3^0(t)\}$, this new problem results from the following: A linear transform, given by

$$\underset{\sim}{\xi} = A\underset{\sim}{x}, \tag{26}$$

can be found that will result in changing the set of linear partial differential equations ((6) to (7)) to a set whose coefficients are functions of time ($t = T$) only. The matrix A is time dependent and must satisfy the ordinary differential equation

$$\frac{dA}{dt} + AS = 0 \tag{27}$$

with the initial condition

$$A(0) = I. \tag{28}$$

The new set of equations then become

$$a_{ki}\frac{\partial u_i}{\partial \xi_k} = 0 \tag{29}$$

$$\frac{\partial a_i}{\partial T} + \sigma_{ij}u_j = -a_{ki}\frac{\partial p}{\partial \xi_k} + \nu a_{\ell j}a_{kj}\frac{\partial^2 u_i}{\partial \xi_a \partial \xi_\ell} \tag{30}$$

for the disturbance variables.

c. Solution by Fourier transforms

The set of equations (29) to (30) is now in a form that can admit separable solutions, regardless of the dependent variable. In view of the fact that the conditions at infinity are boundedness, it is more expedient to employ Fourier transforms for each of the ξ_i spatial variables and thus, if the transforms are defined, the far field conditions are met. Accordingly, define

$$\widehat{u}_i(k_s;T) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_i(\xi_s;T)e^{i\kappa_q\xi_q}\,d\xi_1 d\xi_2 d\xi_3, \tag{31}$$

and for $\widehat{p}$ as well.

Performing the transformations of the equations results in the new set

$$\kappa_k a_{ki}\widehat{u}_i = 0 \tag{32}$$

$$\frac{d\widehat{u}_s}{dT} + \sigma_{sj}\widehat{u}_j = +i\kappa_k a_{ks}\widehat{p} - \nu\kappa_\ell a_{\ell j}\kappa_k a_{kj}\widehat{u}_s \tag{33}$$

Unlike the case where the wave numbers were time-dependent, this system of equations is one of conventional form. In addition, closed form solutions can be anticipated (rather than an infinite sum) subject to initial input. In short, the dynamics can be ascertained including the important questions dealing with discrete and continuous spectra as long as the initial disturbances have Fourier transforms that are defined.

The set of ordinary differential equations for the Fourier amplitudes is tantamount to the one presented by Craik and Criminale except that the time dependent coefficients reflect the coordinate transformations rather than the wave numbers. Other similarities have already been noted for (24) and (25) that determine the coordinate changes proper. It is important to recognize that solutions to this sytem of equations will yield the most general time dependence and, if only the infinite extent shear flow problem is considered, the solutions will be exact.

A general strategy can be developed for solving the perturbation equations. First, by virtue of the incompressibility condition, the pressure can be written as

$$\widehat{p} = -\frac{i}{\kappa^2} 2\kappa_\ell a_{\ell q} \sigma_{qr} \widehat{u}_r \tag{34}$$

with

$$\kappa^2 = \kappa_\ell \kappa_m a_{\ell i} a_{mi} = (\kappa_i a_{i1})^2 + (\kappa_i a_{i2})^2 + (\kappa_i a_{i3})^2. \tag{35}$$

As a result of (34), the resulting equations become

$$\frac{d\widehat{u}_s}{dT} + \sigma_{sj}\widehat{u}_j = \frac{2}{\kappa^2}\kappa_m a_{ms} \kappa_\ell a_{\ell q} \widehat{\sigma}_{qr} \widehat{u}_r - \nu \kappa^2 \widehat{u}_s \tag{36}$$

Proceeding beyond this point is best left to specific problem cases.

Specific example: The mixing layer

Under the formulation given, four pototypical shear flows were examed by Criminale and Drazin (1990). Basically, all of these cases were parallel mean flows where $\underset{\sim}{U} = (\sigma y, 0, 0)$ in the x-direction only. Without solid boundaries, the perturbation problem for the mixing layer can be done inviscidly. Using the moving coordinate transformation it can be seen that, for such a parallel flow,

$$T = \sigma t,$$

$$\xi = x - \sigma y t,$$

$$\eta = y, \tag{37}$$

and $\zeta = z.$

This problem immediately reduces to

$$\frac{\partial}{\partial T}\Delta \overline{v} = 0 \tag{38}$$

where

$$\overline{v} = \iint_{-\infty}^{+\infty} v(\xi,\eta,\zeta,T)e^{+i\alpha\xi+i\gamma\zeta}d\xi d\zeta, \tag{39}$$

and

$$\Delta\overline{v} = \frac{\partial^2\overline{v}}{\partial\eta^2} + i2\alpha T\frac{\partial\overline{v}}{\partial\eta} - (\widetilde{\alpha}^2 + \alpha^2 T^2)\overline{v} \tag{40}$$

with $\widetilde{\alpha}^2 = \alpha^2 + \gamma^2$

for the vertical disturbance velocity component by eliminating the pressure and using incompressibility. Only two-dimensional Fourier transforms have been used here in view of the particular flow model under consideration. In short, since a piece-wise linear mean velocity is used - as shown in Figure 1 - matching conditions at the interfaces will be needed as well as boundedness as $\eta \to \pm\infty$.

Solving (38) in each of the regions is the central issue to the problem but it does not complete the needs. The other velocity components and pressure still must be found. Introducing the transformation of the planar velocity components $\overline{u}$, $\overline{w}$ (Fourier transforms of u, w as by use of (39)) as

$$\begin{aligned} \widetilde{\alpha}\widetilde{u} &= \alpha\overline{u} + \gamma\overline{w} \\ \widetilde{\alpha}\widetilde{w} &= -\gamma\overline{u} + \alpha\overline{w}, \end{aligned} \tag{41}$$

then the equation

$$\frac{\partial\widetilde{w}}{\partial T} = \sin\varphi\overline{v} \tag{42}$$

where $\sin\varphi = \gamma/\widetilde{\alpha}$ can be found. Systematically, (38) is solved for $\overline{v}$, (42) for $\widetilde{w}$. Incompressibility shows that

$$\widetilde{\alpha}\widetilde{u} = -(\frac{\partial\overline{v}}{\partial\eta} + i\alpha T\overline{v}). \tag{43}$$

Thus (43) can be combined with the solutions and the inversion of (41) provides $\overline{u}, \overline{w}$. The pressure follows in a similar manner.

This problem has been solved fully by Bun (1991) where $[\Delta\overline{v}]_{T=0}$ was considered as a combination of an oblique wave in the xz plane and a pulse (Dirac delta function) in y in the inner shear zone. Such an initial input easily allows for the inverse of (39) to give $v(\xi,\eta,\zeta,\tau)$ or $v(x,y,z,t)$. The exterior non-shear regions are straight forward as well and have zero initial values.

The mechanics of the problem construction allows for the complete closed form solution for the dynamics *and* spatial variation. Define $\overline{v}_i$ to be the solution obtained for $\overline{v}$ in the inner shear zero. Let $\overline{v}_0$ be the equivalent solution in the outer region. The conditions $\overline{v}_i = \overline{v}_0$ and $\overline{p}_i = \overline{p}_0$ must be met at the corners. Since

$$-\widetilde{\alpha}^2\overline{p} = i\left\{\frac{\partial^2\overline{v}}{\partial\eta\partial T} + i\alpha T\frac{\partial\overline{v}}{\partial T} + i2\alpha\overline{v}\right\} \tag{43}$$

and irrotational solutions must be added, that is, those solutions that satisfy $\Delta\overline{v} = 0$, the calculations involve $\overline{v}_0 = \overline{v}_{oI}$; $\overline{v}_i = \overline{v}_{iR} + \overline{v}_{iI}$ where I and R refer to the irrotational and rotational parts of the velocity. The irrotational solutions are all proportional to $F(\alpha,\gamma;T)e^{\pm\widetilde{\alpha}\eta - i\alpha\eta T}$ and consequently the matching problem is tantamount to solving the linear system

$$\mathring{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{f} \tag{44}$$

where $\underset{\sim}{x}$ are the coefficients of the irrotational components and $\underset{\sim}{f}$ is due to the initial vorticity input.

Clearly, (44) has two solutions. The first are those of the homogeneous problem and the eigenvalues of A are the normal modes. The second are the forced solutions and, at least for this problem, give rise to algebraic behavior in T which is due to the continuous spectrum. Because the problem can be taken as symmetric and there are only two corners present in the modelled flow, there are either two discrete or one eigenfrequency. Schematically, Figure 1 shows the variation. Although not shown in the figure, the inviscid problem has a damped mode for every growing one for $0 \le \widetilde{\alpha} \le \widetilde{\alpha}_s$ where $\widetilde{\alpha}_s$ is the cutoff value.

The initial-value problem is complete then when the following synthesis is made: Outer region, $\overline{v} = \overline{v}_0$ and $[\overline{v}_0]_{T=0} = 0$. Inner shear zone, $\overline{v} = \overline{v}_i = \overline{v}_{iR} + \overline{v}_{iI}$ and $[\overline{v}_{iI}]_{T=0} = 0$. This combination translates to $\underset{\sim}{x}(0) = 0$ for (44). In effect, no normal modes are used at $T = 0$ but, of course, as time goes on, these modes will play a role and, asymptotically, any exponential growth from any normal mode will dominate. The transient period, on the other hand, is a different matter implying that there will be a change from one dominence to the other as time increases.

Figures 2 and 3 illustrate several aspects of the dynamics. In Figure 2, the combination of the most amplified normal made is used; in Figure 3, the neutual normal mode is combined with the continuous spectrum. Effects of three-dimensionality are pronounced. Even though there is an eventual exponential growth, the early period is controlled by the algebraic variation. And, as Criminale and Drazin have reported, the proper three-dimensionality can lead to complete nonlinearlity before a normal mode will be influential. The neutual normal mode makes the consequences even stronger.

Lagrangian mechanics

A large benefit comes from the means of solution employed here, namely the veloctiy components can be written in explicit closed form as functions of x, y, z, t. This means it is possible to deal directly with the Lagrangian representation where x, y, z are the coordinates of a fluid particle and

$$\begin{aligned} \frac{dx}{dt} &= U + u' &&= f(x, y, z, t), \\ \frac{dy}{dt} &= \quad v' &&= g(x, y, z, t), \\ \frac{dz}{dt} &= \quad w' &&= h(x, y, z, t). \end{aligned} \tag{41}$$

The solutions obtained here were used in such a manner and the coupled nonlinear equations were integrated in time. Figures 4 and 5 show one set of possibilities for the initial most amplified normal mode and the neutual mode respectively. The classic roll up process is clearly more robust under algebraic dynamics. In addition, other features known from experiments for this problem, such as the cross-ribing, are directly due to the three-dimensionality. For the first time, the complete early period dynamics is known and, if desired, can be controlled.

Conclusions

The bases for use of exact solutions in shear flows has been presented in terms of initial-value problems. By use of piece-wise continuous mean velocity profiles, shear flows can be modelled and the perturbation problem can be solved arbitrarily and in closed form. Both the Eulerian and the Langranian problems can be analyzed. The early transient period does indeed set the scene for future development of a flow and three-dimensionality is strongly influential.

Although not reported here, investigations with viscous effects has also been done (cf. Easthope and Criminale 1991) where the boundary layer has considered. Here, it was found that the near field dynamics is akin to a quadrapole field and the wave packet has double maxima and develops a streaky pattern, increasing in the downstream direction in addition to the strong three-dimensionality that is characteristic throughout all such flows.

Finally, two other uses of this approach should be cited. First, there is the obvious need to link this form of input to computational fluid dynamic shemes. Second, it turns out that it is possible to find basic fundamental solutions if only the infinite shear flow problem is considered. In other words, just as the source-sink is fundamental to solutions of Laplace's equation, these solutions can be used for constructing other boundary-value, initial-value problems in the case of flow with constant shear. Such analysis has been made by Criminale and Smith (1991).

Aknowledgement

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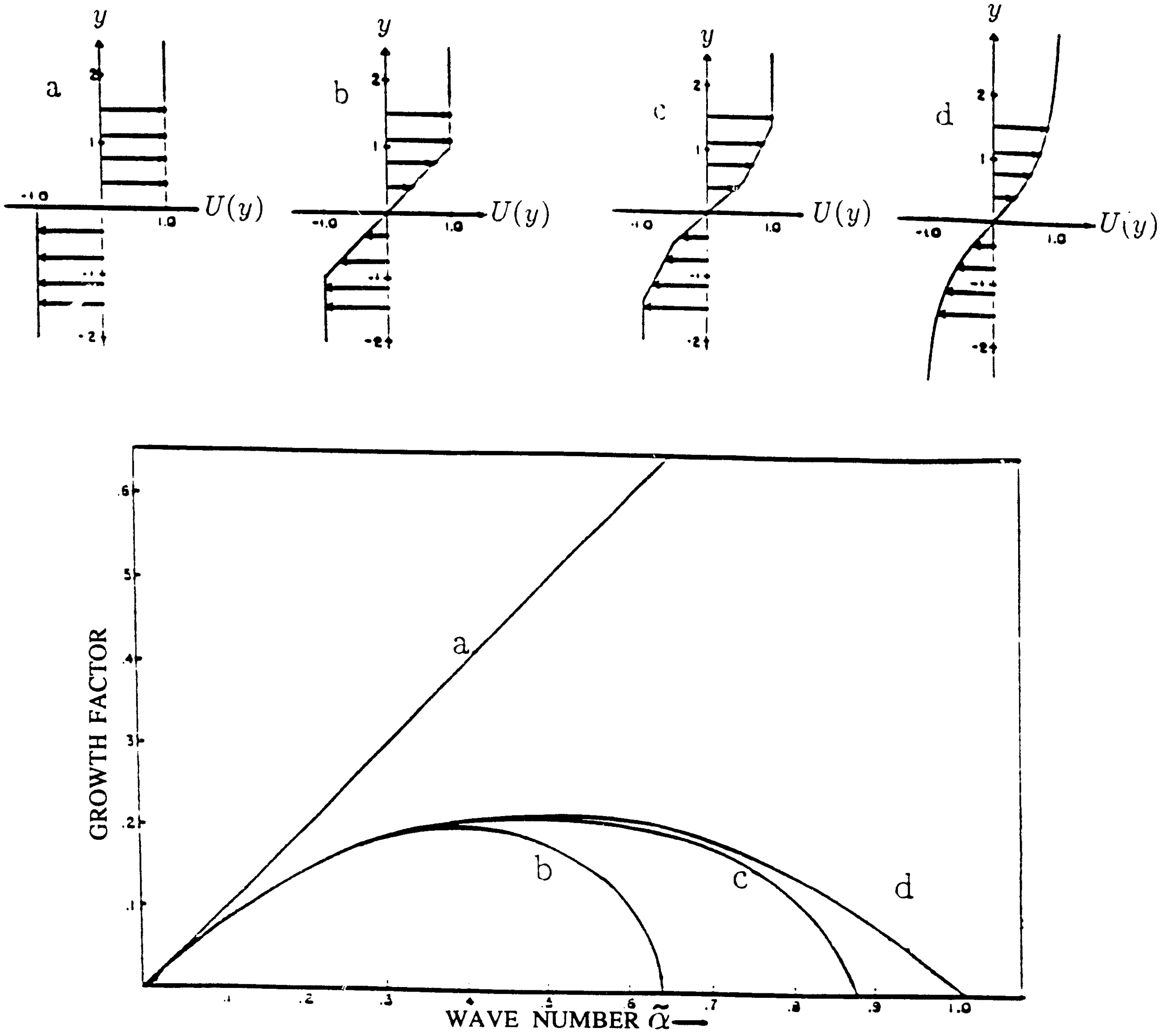

Figure 1: where $\tilde{\alpha}^2 = \alpha^2 + \gamma^2$

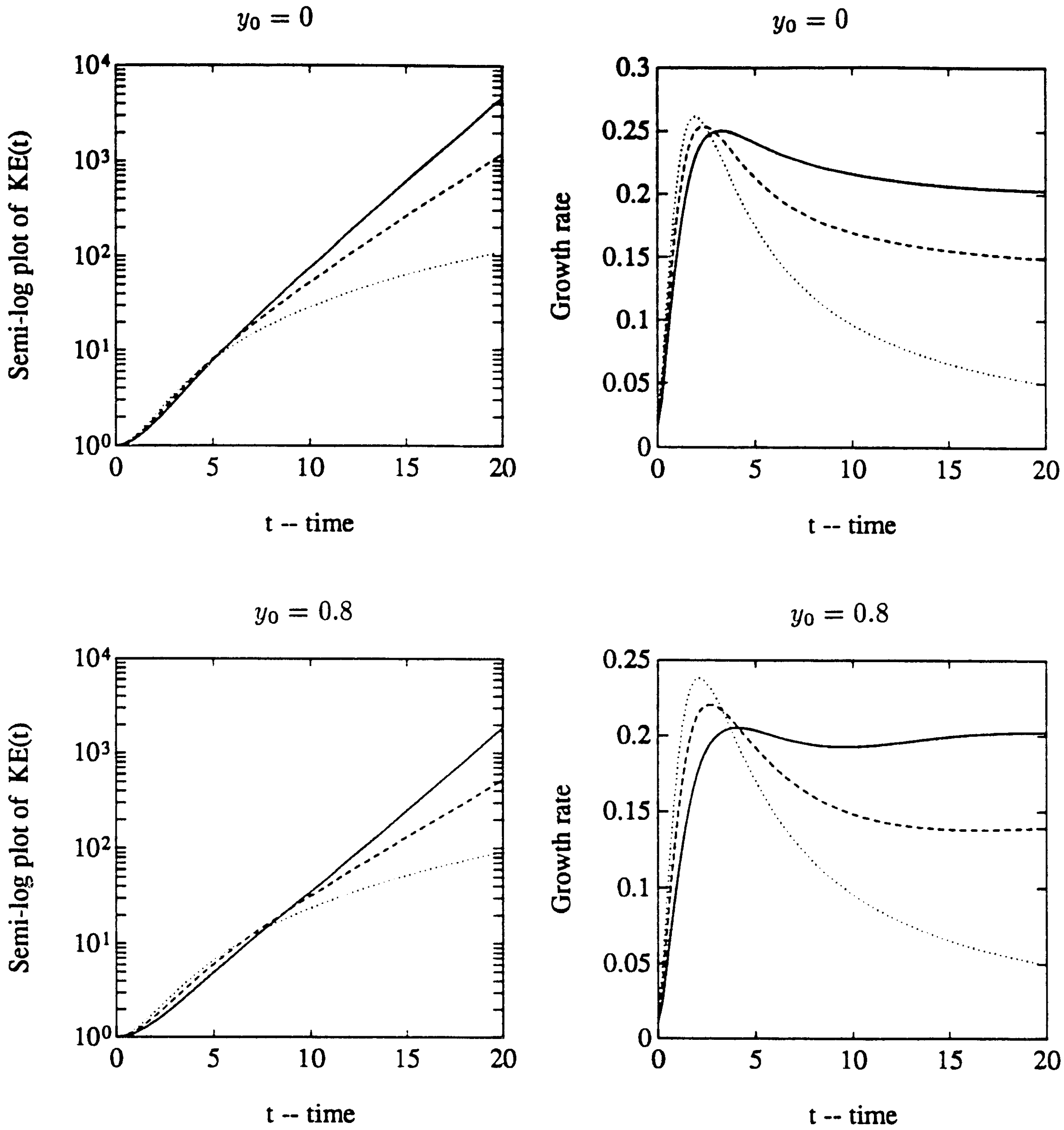

Figure 2: KE for the Green's function: $\tilde{\alpha}_0 = 0.4$, $W_0 \equiv 0$ and fixed y_0.

——— $\phi = 0$

- - - $\phi = \pi/4$

...... $\phi = \pi/2$

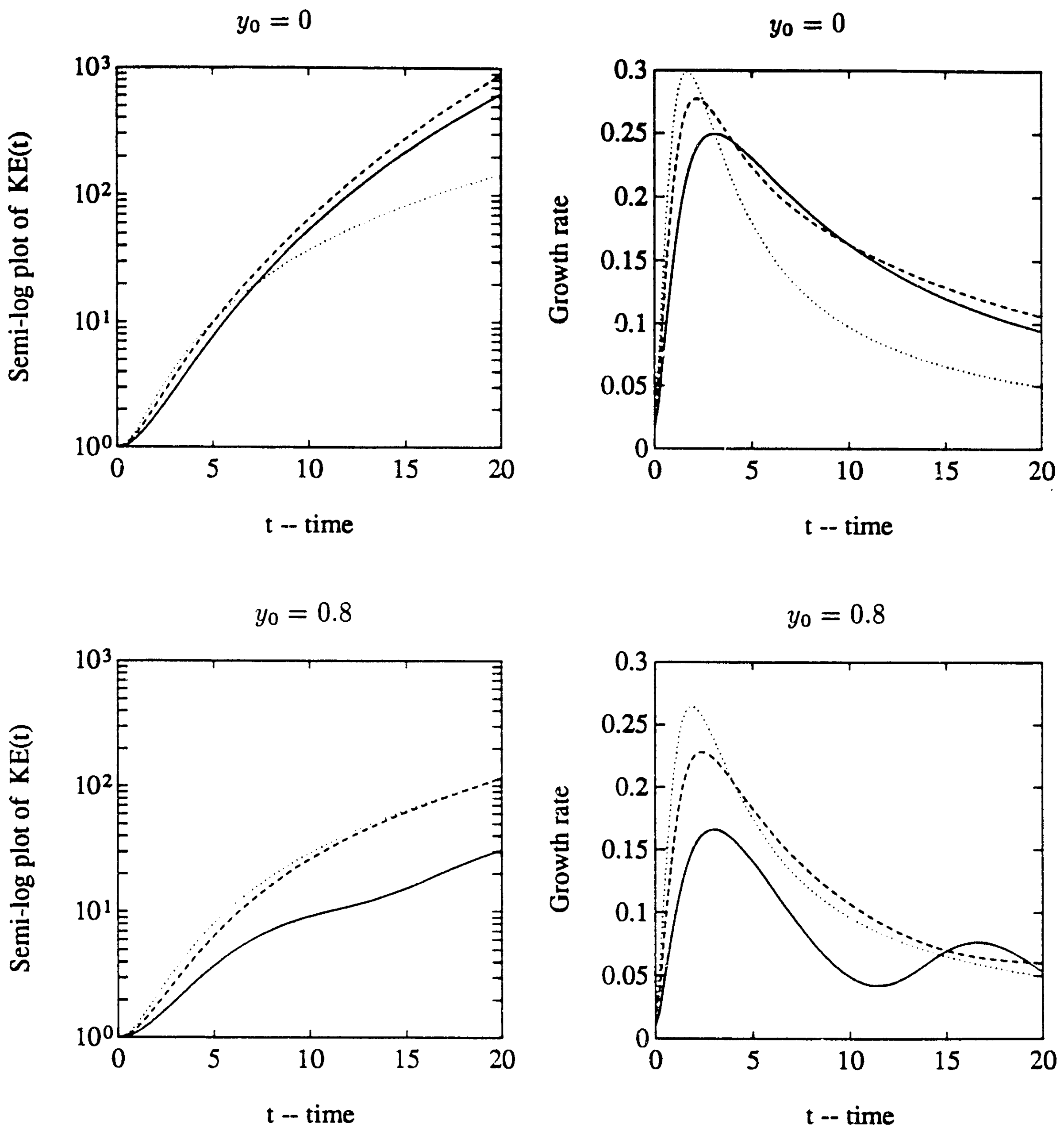

Figure 3: KE for the Green's function: $\tilde{\alpha}_0 = \tilde{\alpha}_s$, $W_0 \equiv 0$ and fixed y_0.

—— $\phi = 0$

- - - $\phi = \pi/4$

...... $\phi = \pi/2$

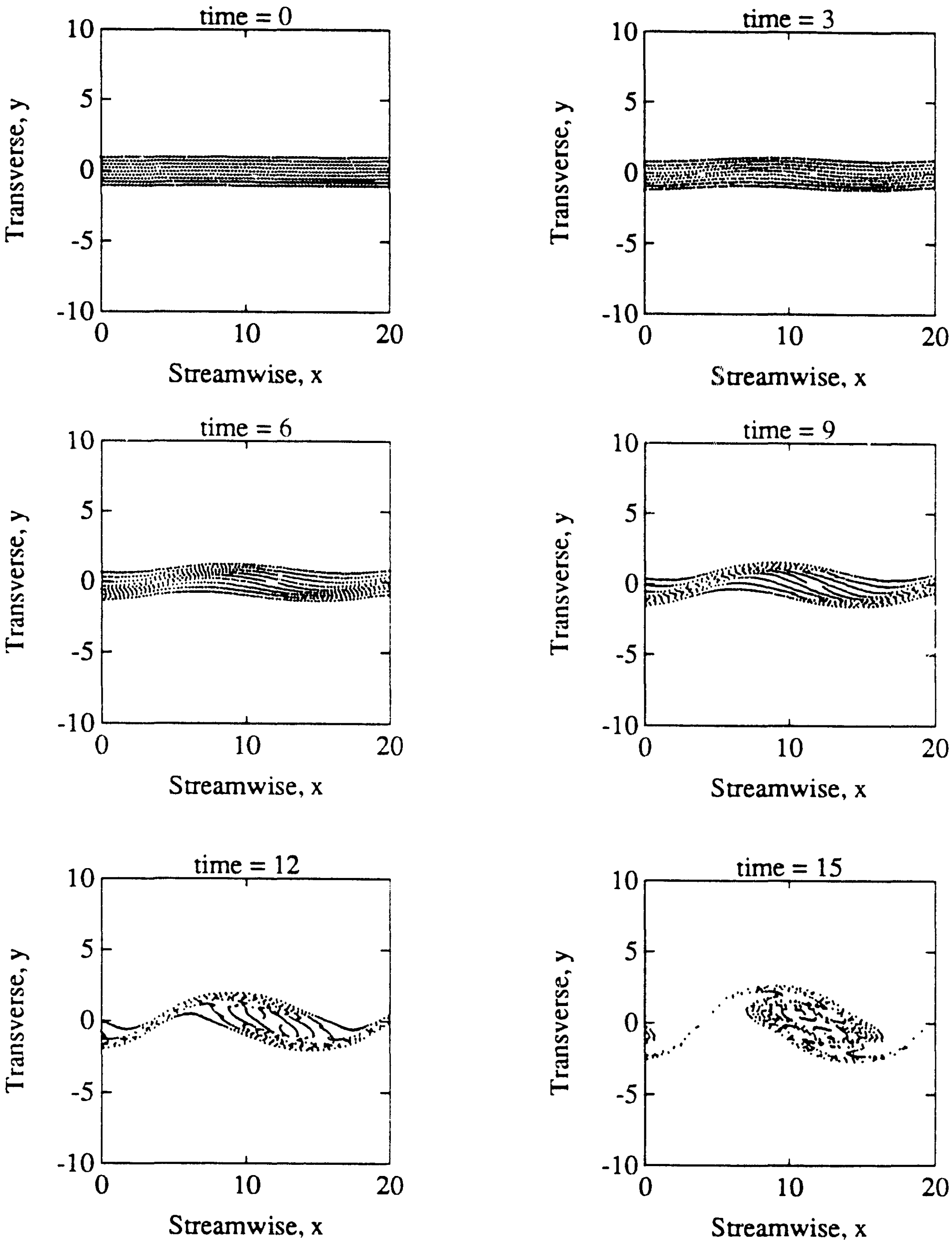

Figure 4: material particles, $\varphi = 0$, $W_0 = 0$, KE(0) = 0.01 and $\widetilde{\alpha}_0 = 0.4$

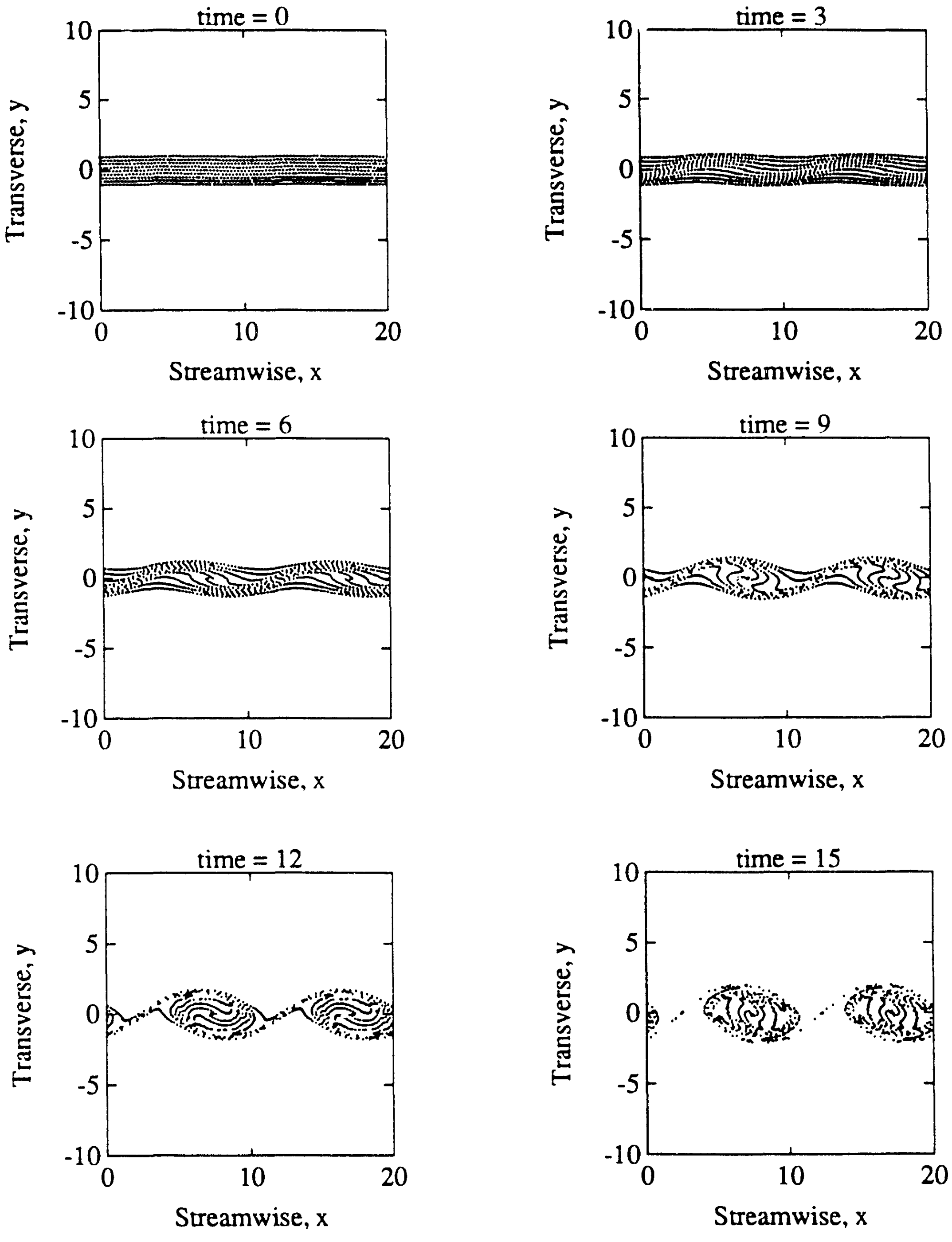

Figure 5: material particles, $\varphi = 0$, $W_0 = 0$, $KE(0) = 0.01$ and $\widetilde{\alpha}_0 = \widetilde{\alpha}_s \simeq 0.64$

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