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# FIRST- AND SECOND-ORDER CHARGED PARTICLE OPTICS*

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## DISCLAIMER

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## 1 SUMMARY AND INTRODUCTION

Since the invention of the alternating gradient principle there has been a rapid evolution of the mathematics and physics techniques applicable to charged particle optics. In this publication we derive a differential equation and a matrix algebra formalism valid to second-order to present the basic principles governing the design of charged particle beam transport systems. A notation first introduced by John Streib[1] is used to convey the essential principles dictating the design of such beam transport systems. For example the momentum dispersion, the momentum resolution, and all second-order aberrations are expressed as simple integrals of the first-order trajectories (matrix elements) and of the magnetic field parameters (multipole components) characterizing the system.

These integrals (listed in Tables I, II, and III) provide direct physical insight into the design of beam transport systems. From them one obtains an intuitive grasp of the mechanism of second-order aberrations. For example, the effects of magnetic symmetry on the minimization or elimination of the aberrations is immediately apparent. In fact it is demonstrated that all second-order aberrations will vanish under appropriate symmetry conditions.

It has also proved convenient to express the magnetic fields via a multipole expansion about a central trajectory. In this expansion, the constant term, proportional to the field strength at the central trajectory, is the dipole term. The term proportional to the first derivative of the field (with respect to transverse dimensions) is the quadrupole term and the second derivative is the sextupole term, etc.

At high energies, a considerable design simplification results if the dipole, quadrupole, and sextupole functions are physically separated such that cross product terms among them do not appear, and if the fringing field effects are small compared with the contributions of the multipole elements comprising the system.

At the risk of oversimplification, the basic function of the multipole elements may be identified in the following way:

The purpose of the dipole elements is to bend the central trajectory of the system and to generate the first-order momentum dispersion. The quadrupole elements provide the first-order imaging. In addition to their fundamental purpose, dipoles and quadrupoles will also introduce higher-order aberrations. If these aberrations are second order, they may be eliminated or at least modified by the introduction of sextupole elements at appropriate locations.

Dipoles introduce both second-order geometric and chromatic aberrations. Quadrupoles do not generate second-order geometric aberrations but they do have strong chromatic (energy dependent) aberrations.

In regions of zero momentum dispersion, a sextupole will couple with and modify only geometric aberrations. However, in a region where dispersion is present, sextupoles will also couple with and modify chromatic aberrations.

Quadrupole elements may be introduced in any one of three characteristic forms: (1) via an actual physical quadrupole consisting of four poles such that a first field derivative exists in the field expansion about the central trajectory; (2) via a rotated input or output face of a bending magnet; or (3) via a transverse field gradient in the dipole elements of the system. Clearly any one of these three fundamental mechanisms may be used as a means of achieving first-order imaging in a system. Dipole elements will tend to image in the radial bending plane independently of whether a transverse field derivative does or does not exist in the system, but imaging in the plane perpendicular to the plane of bend is not possible without the introduction of a first field derivative. Like the quadrupole element, a sextupole element may be generated in one of several ways; first by incorporating an actual sextupole, that is, a six-pole magnet, into the system. However, any mechanism that introduces a second-order derivative of the field with respect to transverse dimensions is, in effect, introducing a sextupole component.

We have included in the report a discussion of linear (first-order) optics as it relates to beam transport systems and to the design of circular machines and to the relationship between the two. Also included is a discussion of the basic optical building blocks that are most often used in the design of such systems. In addition we have provided some applications of second-order optics to the design of chromatic corrections in beam transport systems and circular machines.

It is our hope that the information supplied will provide readers with the necessary tools to design any beam transport system suited to their particular needs.

For the study of details beyond second order, computer ray tracing programs or higher-order formalisms such as the Lie algebra techniques developed by Alex Dragt and his students should be explored by the reader.

# 2 A GENERAL FIRST- AND SECOND-ORDER THEORY OF BEAM TRANSPORT OPTICS

The fundamental objective is to study the trajectories described by charged particles in a static magnetic field. To maintain the desired generality, only one major restriction is imposed on the field configuration: Relative to a plane that is designated as the magnetic midplane, the magnetic scalar potential $\phi$ is an odd function in the transverse coordinate $y$ (the direction perpendicular to the midplane), i.e. $\phi(x,y,s) = -\phi(x,-y,s)$. This restriction greatly simplifies the calculations, and from experience in designing beam transport systems it appears that for most applications there is little, if any, advantage to be gained from a more complicated field pattern. The trajectories are described by means of a Taylor expansion about a particular trajectory (which lies entirely within the magnetic midplane) designated henceforth as the central trajectory. Referring to Fig. 1, the coordinate $s$ is the arc length measured along the central trajectory; and $x, y$, and $s$ form a right-handed curvilinear coordinate system.

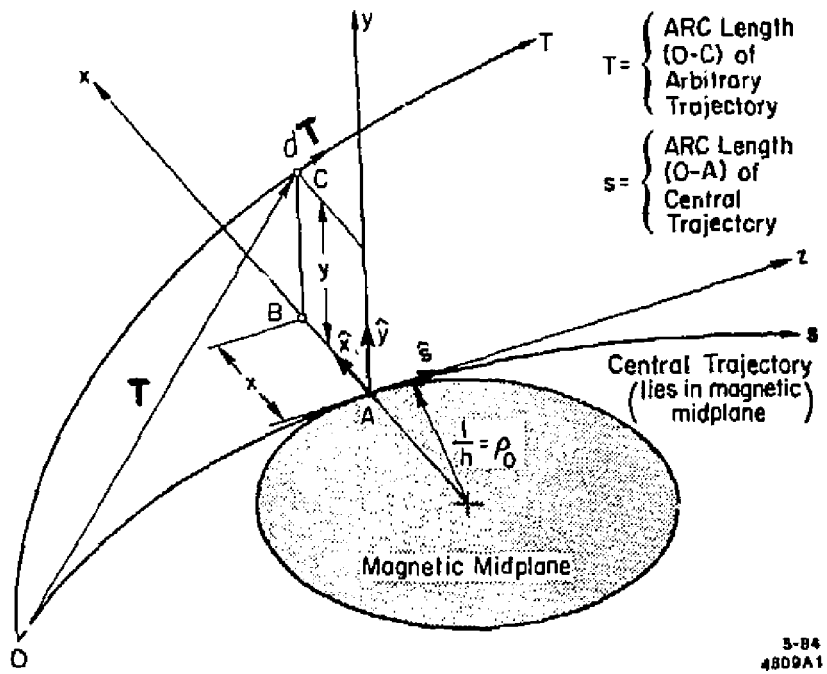


Fig. 1. Curvilinear coordinate system used in the derivation of the equations of motion.

The results are valid for describing trajectories lying close to and making small angles with the central trajectory.

The basic steps in formulating the solution to the problem are as follows:

1) A general vector differential equation is derived describing the trajectory of a charged particle in an arbitrary static magnetic field which possesses midplane symmetry.

2) A Taylor series solution about the central trajectory is then assumed; this is substituted into the general differential equation, and terms to second order in the initial conditions are retained.

3) The first-order coefficients of the Taylor expansion (for monoenergetic rays) satisfy homogeneous second-order differential equations characteristic of simple harmonic oscillator theory; and the first-order dispersion and the second-order coefficients of the Taylor series satisfy second order differential equations having driving terms.

4) The first-order dispersion term and the second-order coefficients are then evaluated via a Green's function integral containing the driving function of the particular coefficient being evaluated and the characteristic solutions of the homogeneous equations.

In other words, the basic mathematical solution of beam transport optics is similar to the theory of forced vibrations or to the theory of the classical harmonic oscillator with driving terms.

It is useful to express the second-order results in terms of the first-order coefficients of the Taylor expansion. These first-order coefficients have a one to one correspondence with the following five characteristic first order trajectories (matrix elements) of the system.

1) The unit sinelike function $s_x(s)$ in the plane of bend defined by $s_x(0) = 0$ and $s'_x(0) = 1$. See Fig. 2.

2) The unit cosine-like function $c_x(s)$ in the plane of bend defined by $c_x(0) = 1$ and $c'_x(0) = 0$. See Fig. 3.

3) The dispersion function $d_x(s)$ in the plane of bend defined by $d_x(0) = 0$ and $d'_x(0) = 0$ and a momentum $p$ such that $(p - p_0)/p_0 = 1$. See Fig. 4.

4) The unit sinelike function $s_y(s)$ in the nonbend plane defined by $s_y(0) = 0$ and $s'_y(0) = 1$. See Fig. 5.

5) The unit cosinelike function $c_y(s)$ in the nonbend plane defined by $c_y(0) = 1$ and $c'_y(0) = 0$. See Fig. 6.

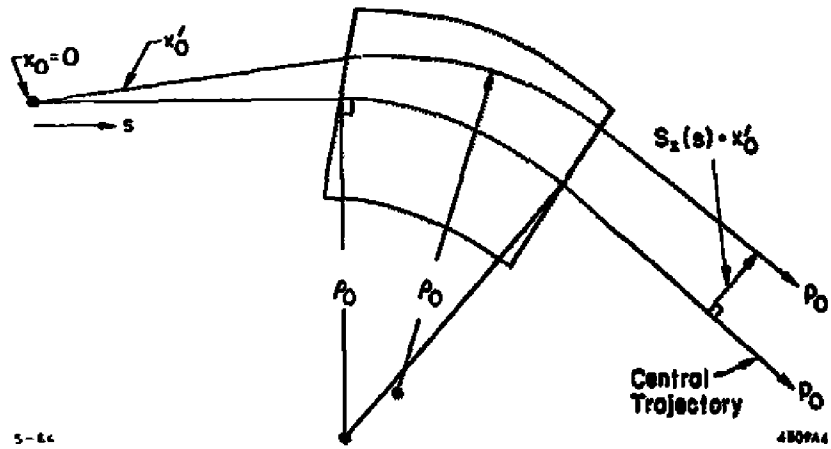


Fig. 2. Sinelike function $s_x(s)$ in the magnetic midplane.

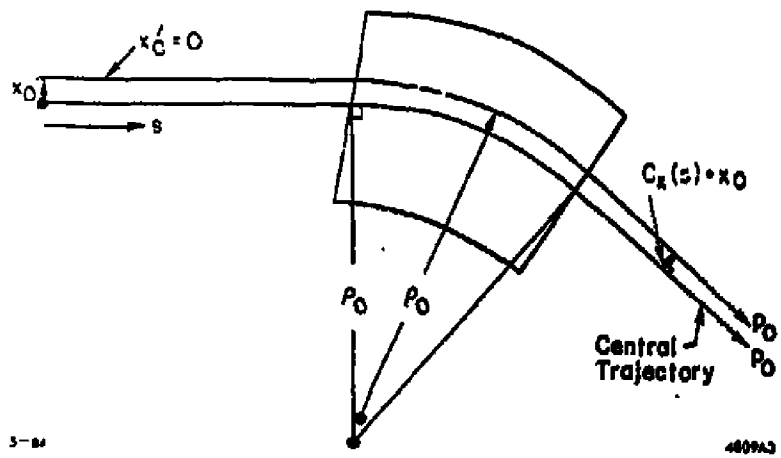


Fig. 3. Cosinelike function $c_x(s)$ in the magnetic midplane.

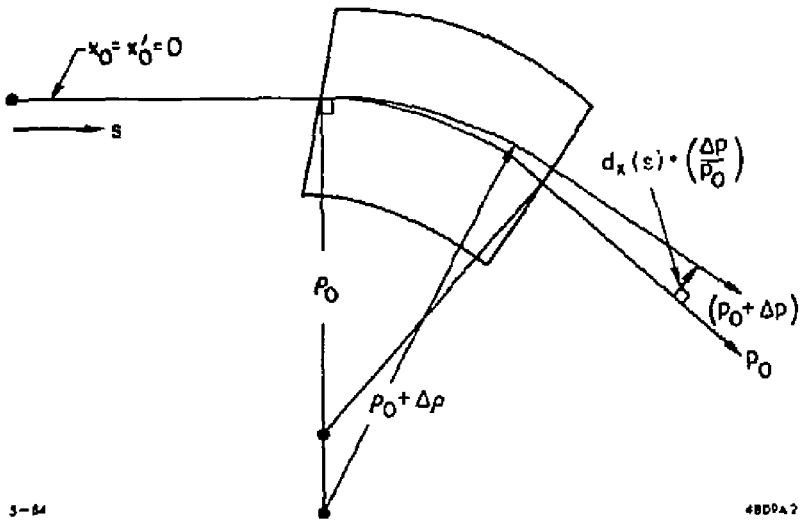


Fig. 4. Dispersion function $d_x(s)$ in the magnetic midplane.

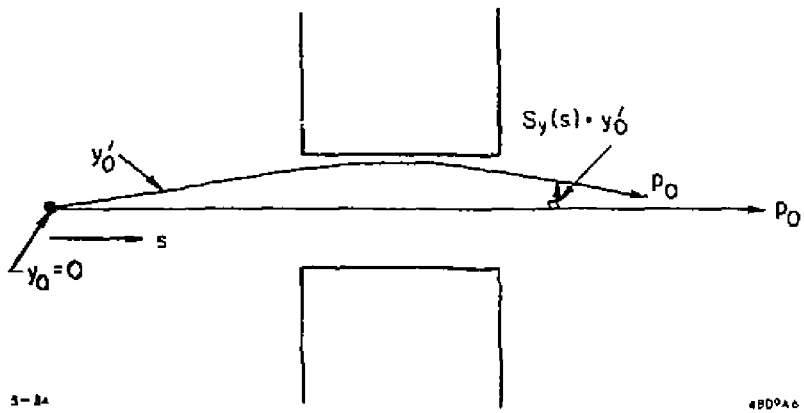


Fig. 5. Sinelike function $s_y(s)$ in the nonbend plane.

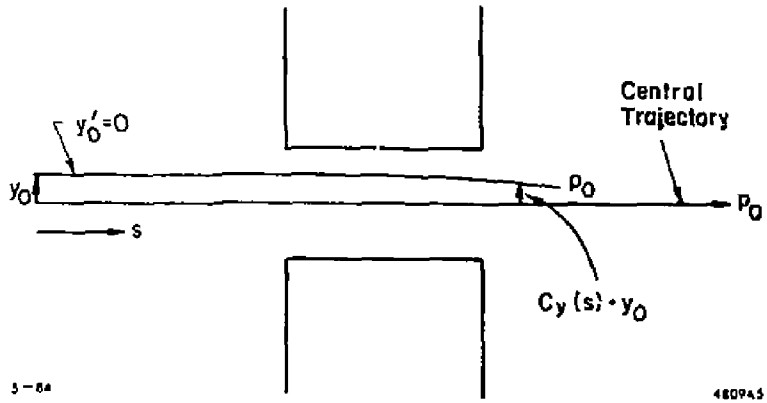


Fig. 6. Cosinelike function $c_y(s)$ in the nonbend plane.

In the first-order Taylor expansion for the transverse position of an arbitrary trajectory at position $s$ in terms of its initial conditions, the above five quantities are the coefficients appearing in the expansion for the transverse coordinates $x$ and $y$ as follows :

$$x(s) = c_x(s)x_0 + s_x(s)x'_0 + d_x(s)\delta$$

and

$$y(s) = c_y(s)y_0 + s_y(s)y'_0$$

where $x_0$ and $y_0$ are the initial transverse coordinates and $x'_0$ and $y'_0$ are the initial slopes of the arbitrary ray with respect to the central trajectory. $\delta = \Delta p/p_0 = (p - p_0)/p_0$ is the fractional momentum deviation of the ray from that of the central trajectory.

## 2.1 The Vector Differential Equation of Motion

We begin with the usual vector relativistic equation of motion for a charged particle in a static magnetic field, equating the time rate of change of momentum to the Lorentz force:

$$\dot{\mathbf{p}} = e(\mathbf{V} \times \mathbf{B})$$

and immediately transform this equation to one in which time has been eliminated and we are left with only spatial coordinates. The curvilinear coordinate system used is shown in Fig. 1. Note that the variable $s$ is the arc distance measured along the central trajectory. With a little algebra, the equation of motion is readily transformed to the following vector forms:

Let $e$ be the charge of the particle, $\mathbf{V}$ its velocity, $V$ its speed, $p$ its momentum magnitude, $\mathbf{T}$ its position vector and $T$ its distance to the origin. The unit tangent vector of the trajectory is $d\mathbf{T}/dT$. Thus the velocity and momentum of the particle are, respectively, $(d\mathbf{T}/dT)V$ and $(d\mathbf{T}/dT)p$. The vector equation of motion then becomes:

$$V\frac{d}{dT}\left(\frac{d\mathbf{T}}{dT}p\right) = eV\left(\frac{d\mathbf{T}}{dT}\times\mathbf{B}\right)$$

or

$$p\frac{d^2\mathbf{T}}{dT^2} + \frac{d\mathbf{T}}{dT}\left(\frac{dp}{dT}\right) = e\left(\frac{d\mathbf{T}}{dT}\times\mathbf{B}\right)$$

where $\mathbf{B}$ is the magnetic induction. Then, since the derivative of a unit vector is perpendicular to the unit vector, $d^2\mathbf{T}/dT^2$ is perpendicular to $d\mathbf{T}/dT$. It follows that $dp/dT = 0$; that is, $p$ is a constant of the motion as expected from the fact that the magnetic force is always perpendicular to the velocity in a static magnetic field. The final result is

$$\frac{d^2\mathbf{T}}{dT^2} = \frac{e}{p}\left(\frac{d\mathbf{T}}{dT}\times\mathbf{B}\right). \tag{2.1}$$

## 2.2 The Coordinate System

The general right-handed curvilinear coordinate system $(x, y, s)$ used is illustrated in Fig. 1. A point $O$ on the central trajectory is designated as the origin. The direction of motion of particles on the central trajectory is designated as the positive direction of the coordinate $s$. A point $A$ on the central trajectory is specified by the arc length $s$ measured along that curve from the origin $O$ to point $A$. The two sides of the magnetic symmetry plane are designated the positive and negative sides by the sign of the coordinate $y$. To specify an arbitrary point $B$ which lies in the symmetry plane, construct a line segment from that point to the central trajectory (which also lies in the symmetry plane) intersecting the latter perpendicularly at $A$; the point $A$ provides one coordinate $s$; the second coordinate $x$ is the length of the line segment $BA$, combined with a sign, (+) or (−) according as an observer, on the positive side of the symmetry plane and facing in the positive direction of the central trajectory, finds the point on the left or right side. In other words, $x$, $y$, and $s$ form a right-handed curvilinear coordinate system. To specify a point $C$ which lies off the symmetry plane, we construct a line segment from the point to the plane, intersecting the latter perpendicularly at $B$; then $B$ provides the two coordinates $s$ and $x$; the third coordinate $y$ is the length of the line segment $CB$.

We now define three mutually perpendicular unit vectors $(\hat{x}, \hat{y}, \hat{s})$. $\hat{s}$ is tangent to the central trajectory and directed in the positive $s$ direction at the

point $A$ corresponding to the coordinate s; $\hat{x}$ is perpendicular to the principal trajectory at the same point, parallel to the symmetry plane, and directed in the positive $x$ direction. $\hat{y}$ is perpendicular to the symmetry plane, and directed away from that plane on its positive side. The unit vectors $(\hat{x}, \hat{y}, \hat{s})$ constitute a right-handed system and satisfy the relations

$$\begin{aligned} \hat{x} &= \hat{y} \times \hat{s} \ , \\ \hat{y} &= \hat{s} \times \hat{x} \ , \\ \hat{s} &= \hat{x} \times \hat{y} \ . \end{aligned} \tag{2.2}$$

The coordinate s is the primary independent variable, and we shall use the prime to indicate the operation $d/ds$. The unit vectors depend only on the coordinate s, and, from differential vector calculus, we may write

$$\begin{aligned} \hat{x}' &= h\hat{s} \ , \\ \hat{y}' &= 0 \ , \\ \hat{s}' &= -h\hat{x} \ , \end{aligned} \tag{2.3}$$

where $h(s) = 1/\rho_0$ is the curvature of the central trajectory at point $A$ defined as positive, as shown in Fig. 1.

The equation of motion may now be rewritten in terms of the curvilinear coordinates defined above. To facilitate this, it is convenient to express $d\mathbf{T}/dT$ and $d^2\mathbf{T}/dT^2$ in the following forms:

$$\frac{d\mathbf{T}}{dT} = \frac{(d\mathbf{T}/ds)}{(dT/ds)} = \frac{\mathbf{T}'}{T'} \ ,$$

$$\frac{d^2\mathbf{T}}{dT^2} = \frac{1}{T'} \frac{d}{ds}\left(\frac{\mathbf{T}'}{T'}\right) \ ,$$

or

$$T'^2 \frac{d^2\mathbf{T}}{dT^2} = \mathbf{T}'' - \frac{1}{2} \frac{\mathbf{T}'}{T'^2} \frac{d}{ds} (T'^2) \ .$$

The equation of motion now takes the form

$$\mathbf{T}'' - \frac{1}{2} \frac{\mathbf{T}'}{T'^2} \frac{d}{ds} (T'^2) = \frac{e}{p} T'(\mathbf{T}' \times \mathbf{B}) \ . \tag{2.4}$$

In this coordinate system, the differential line element is given by

$$d\mathbf{T} = \hat{x}dx + \hat{y}dy + (1 + hx)\hat{s}ds$$

and

$$(dT)^2 = d\mathbf{T} \cdot d\mathbf{T} = dx^2 + dy^2 + (1 + hx)^2 ds^2 \ .$$

Differentiating these equations with respect to s, it follows that

$$T'^2 = x'^2 + y'^2 + (1+hx)^2 ,$$

$$\frac{1}{2}\frac{d}{ds}(T'^2) = x'x'' + y'y'' + (1+hx)(hx' + h'x) ,$$

$$\mathbf{T}' = \hat{x}x' + \hat{y}y' + (1+hx)\hat{s} ,$$

$$\mathbf{T}'' = \hat{x}x'' + \hat{x}'x' + \hat{y}y'' + \hat{y}'y' + (1+hx)\hat{s}' + (hx' + h'x)\hat{s} .$$

Using the differential vector relations of Eq. (2.3), the expression for $\mathbf{T}''$ reduces to

$$\mathbf{T}'' = \hat{x}(x'' - h(1+hx)) + \hat{y}y'' + \hat{s}(2hx' + h'x) .$$

The vector equation may now be separated into its component parts with the result

$$\hat{x}\{(x'' - h(1+hx)) - \frac{x'}{T'^2}(x'x'' + y'y'' + (1+hx)(hx' + h'x))\}$$

$$+ \hat{y}\{y'' - \frac{y'}{T'^2}(x'x'' + y'y'' + (1+hx)(hx' + h'x))\}$$

$$+ \hat{s}\{(2hx' + h'x) - \frac{(1+hx)}{T'^2}(x'x'' + y'y'' + (1+hx)(hx' + h'x))\}$$

$$= \frac{e}{p}T'(\mathbf{T}' \times \mathbf{B})$$

$$= \frac{e}{p}T'\{\hat{x}(y'B_s - (1+hx)B_y) + \hat{y}((1+hx)B_x - x'B_s)$$

$$+ \hat{s}(x'B_y - y'B_x)\} . \qquad (2.5)$$

Note that in this form, no approximations have been made; the equation of motion (2.5) is still valid to all orders in the variables $x$ and $y$ and their derivatives.

If now we retain only terms through second order in $x$ and $y$ and their derivatives and note that $T'^2 = 1 + 2hx + \cdots$, then the $x$ and $y$ components of the equation of motion become

$$x'' - h(1+hx) - x'(hx' + h'x) = \frac{e}{p}T'(y'B_s - (1+hx)B_y) ,$$

$$y'' - y'(hx' + h'x) = \frac{e}{p}T'((1+hx)B_x - x'B_s) . \qquad (2.6)$$

The equation of motion of the central orbit is readily obtained by setting $x$ and $y$ and their derivatives equal to zero. We thus obtain

$$h = \frac{e}{p_0} B_y(0,0,s) \qquad \text{or} \qquad B\rho_0 = \frac{p_0}{e} . \tag{2.7}$$

This result will be useful for simplifying the final equations of motion. $p_0$ is the momentum of a particle on the central trajectory. Note that this equation establishes the sign convention between $h$,$e$ and $B_y$.

## 2.3 Expanded Form of a Magnetic Field Having Median Plane Symmetry

We now evolve the field components of a static magnetic field possessing median or midplane symmetry. See Fig. 7. We define midplane symmetry as follows. Relative to the plane containing the central trajectory, the magnetic potential $\phi$ is an odd function in $y$ : i.e. $\phi(x,y,s) = -\phi(x,-y,s)$. Stated in terms of the magnetic field components $B_x$, $B_y$, and $B_s$ this is equivalent to

$$B_x(x,y,s) = -B_x(x,-y,s) ,$$

$$B_y(x,y,s) = B_y(x,-y,s) ,$$

$$B_s(x,y,s) = -B_s(x,-y,s) .$$

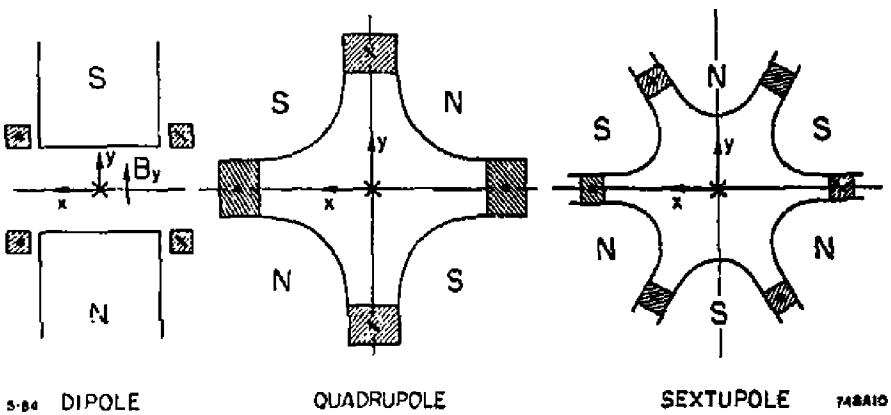


Fig. 7. Illustration of the magnetic midplane for dipole, quadrupole, and sextupole elements.

It follows immediately that on the midplane $B_x = B_s = 0$ and only $B_y$ remains nonzero; in other words, on the midplane B is always normal to the plane. As such, any trajectory initially lying in the midplane will remain in the midplane throughout the system.

The expanded form of a magnetic field with median plane symmetry has been worked out by many people; one of the most convenient and comprehensible references is provided by Teng.[16]

For a magnetic field in vacuum, the field may be expressed in terms of a scalar potential $\phi$ by $\mathbf{B} = \nabla\phi$.[11] The scalar potential will be expanded in the curvilinear coordinates about the central trajectory lying in the median plane $y = 0$. The curvilinear coordinates have been defined in Fig. 1, where $x$ is the outward normal distance in the median plane away from the central trajectory, $y$ is the perpendicular distance from the median plane, s is the distance along the central trajectory, and $h = h(s)$ is the curvature of the central trajectory. As previously stated, these coordinates $(x, y, s)$ form a right-handed orthogonal curvilinear coordinate system.

As has been stated, the existence of the median plane requires that $\phi$ be an odd function of $y$, i.e. $\phi(x, y, s) = -\phi(x, -y, s)$. The most general expanded form of $\phi$ may be expressed as follows:

$$\begin{aligned}\phi(x,y,s) &= (A_{10} + A_{11}x + A_{12}(x^2/2!) + A_{13}(x^3/3!) + \cdots)y \\ &\quad + (A_{30} + A_{31}x + A_{32}(x^2/2!) + \cdots)y^3/3! + \cdots \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{2m+1,n}\frac{x^n}{n!}\frac{y^{2m+1}}{(2m+1)!} \qquad (2.8)\end{aligned}$$

where the coefficients $A_{2m+1,n}$ are functions of $s$.

The differential line element $dT$ of the coordinate system is

$$dT^2 = dx^2 + dy^2 + (1 + hx)^2(ds)^2 \qquad (2.9)$$

and the Laplace equation has the form

$$\nabla^2\phi = \frac{1}{(1+hx)}\frac{\partial}{\partial x}\left((1+hx)\frac{\partial\phi}{\partial x}\right) + \frac{\partial^2\phi}{\partial y^2} + \frac{1}{(1+hx)}\frac{\partial}{\partial s}\left(\frac{1}{(1+hx)}\frac{\partial\phi}{\partial s}\right) = 0\,. \qquad (2.10)$$

Substitution of Eq. (2.8) into Eq. (2.10) gives the following recursion formula

[11] For convenience, we omit the minus sign since we are restricting the problem to static magnetic fields.

for the coefficients:

$$
\begin{aligned}
-A_{2m+3,n} = {} & A''_{2m+1,n} + nhA''_{2m+1,n-1} - nh'A'_{2m+1,n-1} + A_{2m+1,n+2} \\
& + (3n+1)hA_{2m+1,n+1} + n(3n-1)h^2A_{2m+1,n} \\
& + n(n-1)^2h^3A_{2m+1,n-1} \\
& + 3nhA_{2m+3,n-1} + 3n(n-1)h^2A_{2m+3,n-2} \\
& + n(n-1)(n-2)h^3A_{2m+3,n-3}
\end{aligned}
\tag{2.11}
$$

where prime means $d/ds$, and where it is understood that all coefficients $A$ with one or more negative subscripts are zero. This recursion formula expresses all the coefficients in terms of the midplane field $B_y(x,0,s)$ via the coefficients $A_{1,n}$:

$$
A_{1,n} = \left(\frac{\partial^n B_y}{\partial x^n}\right)_{\substack{x=0 \\ y=0}} = \text{functions of } s \ . \tag{2.12}
$$

Since $\phi$ is an odd function of y, on the median plane we have $B_x = B_s = 0$. The normal (in $x$ direction) derivatives of $B_y$ on the reference curve defines $B_y$ over the entire median plane, hence the magnetic field $\mathbf{B}$ over the whole space. The components of the field are expressed in terms of $\phi$ explicitly by $\mathbf{B} = \nabla\phi$ or

$$
\begin{aligned}
B_x &= \frac{\partial\phi}{\partial x} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{2m+1,n+1}\frac{x^n}{n!}\frac{y^{2m+1}}{(2m+1)!} \ , \\
B_y &= \frac{\partial\phi}{\partial y} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{2m+1,n}\frac{x^n}{n!}\frac{y^{2m}}{(2m)!} \ , \\
B_s &= \frac{1}{(1+hx)}\frac{\partial\phi}{\partial s} = \frac{1}{(1+hx)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A'_{2m+1,n}\frac{x^n}{n!}\frac{y^{2m+1}}{(2m+1)!} \ ,
\end{aligned}
\tag{2.13}
$$

where $B_s$ is not expressed in pure expansion form. This form can be obtained in a straightforward way by expanding $1/(1+hx)$ in a power series of $hx$ and multiplying out the two series; however, there does not seem to be any advantage gained over the form given in Eq. (2.13).

The coefficients up to sixth-degree terms in $x$ and $y$ are given explicitly below as derived from Eq. (2.11).

$$A_{30} = -A''_{10} - A_{13} - hA_{11} ,$$

$$A_{31} = -A''_{11} + 2hA''_{10} + h'A'_{10} - A_{13} - hA_{12} + h^2A_{11} ,$$

$$A_{32} = -A''_{12} + 4hA''_{11} + 2h'A'_{11} - 6h^2A''_{10} - 6hh'A'_{10} - A_{14}$$
$$- hA_{13} + 2h^2A_{12} - 2h^3A_{11} ,$$

$$A_{33} = -A''_{13} + 6hA''_{12} + 3h'A'_{12} - 18h^2A''_{11} - 18hh'A'_{11}$$
$$+ 24h^3A''_{10} + 36h^2h'A'_{10} - A_{15} - hA_{14} + 3h^2A_{13}$$
$$- 6h^3A_{12} + 6h^4A_{11} ,$$

$$A_{30} = A'''_{10} + 2A''_{12} - 2hA''_{11} + h''A_{11} + 4h^2A''_{10} + 5hh'A'_{10}$$
$$+ A_{14} + 2hA_{13} - h^2A_{12} + h^3A_{11} ,$$

$$A_{31} = A'''_{11} - 4hA'''_{10} - 6h'A''_{10} - 4h''A'_{10} - h'''A_{10} + 2A''_{13}$$
$$- 6hA''_{12} - 2h'A'_{12} + h''A_{12} + 10h^2A''_{11} + 7hh'A'_{11} - 4hh''A_{11}$$
$$- 3h'^2A_{11} - 16h^3A''_{10} - 29h^2h'A'_{10} + A_{15} + 2hA_{14}$$
$$- 3h^2A_{13} + 3h^3A_{12} - 3h^4A_{11} . \qquad (2.14)$$

In the special case when the field has cylindrical symmetry about $\hat{y}$, we can choose a circle with radius $\rho_0 = 1/h$ constant for the reference curve. The coefficients $A_{2m+1,n}$ in Eq. (2.8) and the curvature $h$ of the reference curve are then all independent of $s$. Equations (2.14) are greatly simplified by putting all terms with primed quantities equal to zero.

## 2.4 Field Expansion to Second Order Only

If the field expansion is terminated with second-order terms, the results may be considerably simplified. For this case, the scalar potential $\phi$ and the field $\mathbf{B} = \nabla\phi$ become

$$\phi(x,y,z) = \left(A_{10} + A_{11}x + \frac{1}{2!}A_{12}x^2 + \cdots\right)y + (A_{30} + \cdots)\frac{y^3}{3!} + \cdots ,$$

$$A_{i,n} = \left.\frac{\partial^n B_y}{\partial x^n}\right|_{\substack{x=0\\ z=0}} = \text{functions of } s \text{ only} ,$$

and

$$A_{30} = -(A''_{10} + hA_{11} + A_{12})$$

where prime means the total derivative with respect to s. Then $\mathbf{B} = \nabla\phi$ gives

$$B_x(x,y,s) = \frac{\partial\phi}{\partial x} = A_{11}y + A_{12}xy + \cdots ,$$

$$B_y(x,y,s) = \frac{\partial\phi}{\partial y} = A_{10} + A_{11}x + \frac{1}{2!}A_{12}x^2 + \frac{1}{2!}A_{30}y^2 + \cdots ,$$

$$B_s(x,y,s) = \frac{1}{(1+hx)}\frac{\partial\phi}{\partial s} = \frac{1}{(1+hx)}(A'_{10}y + A'_{11}xy + \cdots) . \qquad (2.15)$$

By inspection it is evident that $B_x$, $B_y$ and $B_s$, are all expressed in terms of $A_{10}$, $A_{11}$, and $A_{12}$ and their derivatives with respect to s. Consider then $B_y$ on the midplane only:

$$B_y(x,0,s) = A_{10} + A_{11}x + \frac{1}{2!}A_{12}x^2 + \cdots$$

dipole quadrupole sextupole etc.

$$= B_y\Big|_{\substack{x=0\\y=0}} + \frac{\partial B_y}{\partial x}\Big|_{\substack{x=0\\y=0}} x + \frac{1}{2!}\frac{\partial^2 B_y}{\partial x^2}\Big|_{\substack{x=0\\y=0}} x^2 + \cdots . \qquad (2.16)$$

The successive derivatives identify the terms as being dipole, quadrupole, sextupole, octupole, etc., in the expansion of the field. To eliminate the necessity of continually writing these derivatives, it is useful to express the midplane field in terms of dimensionless quantities $n(s)$, $\beta(s)$, etc., or

$$B_y(x,0,s) = B_y(0,0,s)(1 - nhx + \beta h^2x^2 + \gamma h^3x^3 + \cdots) \qquad (2.17)$$

where, as before, $h(s) = 1/\rho_0$, and $n$, $\beta$ and $\gamma$ are functions of s. Direct comparison of Eqs. (2.16) and (2.17) yields

$$n = -\left[\frac{1}{hB_y}\left(\frac{\partial B_y}{\partial x}\right)\right]_{\substack{x=0\\y=0}} \quad \text{and} \quad \beta = \left[\frac{1}{2!h^2B_y}\left(\frac{\partial^2 B_y}{\partial x^2}\right)\right]_{\substack{x=0\\y=0}} . \qquad (2.18)$$

We now make use of Eq. (2.7), the equation of motion of the central trajectory:

$$B_y(0,0,s) = h\left(\frac{p_0}{e}\right) .$$

Combining Eqs. (2.7) and (2.18), the coefficients of the field expansions become

$$A_{10} = B_y(0,0,s) = h\left(\frac{p_0}{e}\right) ,$$

$$A_{11} = \frac{\partial B_y}{\partial x}\Big|_{\substack{x=0\\y=0}} = -nh^2\left(\frac{p_0}{e}\right),$$

$$\frac{1}{2!}A_{12} = \frac{1}{2!}\frac{\partial^2 B_y}{\partial x^2}\Big|_{\substack{x=0\\y=0}} = \beta h^3\left(\frac{p_0}{e}\right),$$

$$A_{30} = -\left(h'' - nh^3 + 2\beta h^3\right)\left(\frac{p_0}{e}\right),$$

$$A_{10} = h'\left(\frac{p_0}{e}\right),$$

$$A_{11} = -\left(2nhh' + n'h^2\right)\left(\frac{p_0}{e}\right). \tag{2.19}$$

To second order the expansions for the magnetic field components may now be expressed in the form

$$B_x(x,y,s) = \left(\frac{p_0}{e}\right)\left(-nh^2y + 2\beta h^3xy + \cdots\right),$$

$$B_y(x,y,s) = \left(\frac{p_0}{e}\right)\left(h - nh^2x + \beta h^3x^2 - \frac{1}{2}(h'' - nh^3 + 2\beta h^3)y^2 + \cdots\right),$$

$$B_s(x,y,s) = \left(\frac{p_0}{e}\right)\left(h'y - (n'h^2 + 2nhh' + hh')xy + \cdots\right), \tag{2.20}$$

where $p_0$ is the momentum of the central trajectory.

## 2.5 Expansion of the Magnetic Field as a Function of Multipole Components

The magnetic field on the midplane may also be expressed as follows:

$$B_y(x,0,s) = B\rho\sum_{n=0}^{\infty} K_n(s)x^n \tag{2.21}$$

where $B\rho = B/h = p_0/e$ is the magnetic rigidity of a particle of momentum $p_0$ and charge $e$ along the central trajectory. From Eq. (2.21) it follows that

$$K_n(s) = \left(\frac{1}{B\rho}\right)\left(\frac{1}{n!}\right)\left(\frac{\partial^n B_y}{\partial x^n}\right)_{x=y=0} \tag{2.22}$$

and

$$S_n = \int_0^L K_n(s)ds \tag{2.23}$$

where $S_n$ is the integrated strength of an nth order multipole component of length $L$.

### 2.5.1 *Multipole Strengths for Pure Multipole Fields*

Consider the scalar potential of an nth-order $(2(n+1)$pole) pure multipole element:

$$\phi = \frac{B_0 r^{n+1}}{(n+1)a^n} \sin(n+1)\theta \tag{2.24}$$

where

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta\,,$$

$B_0$ is the field at the pole and $a$ is the radial distance to the pole from the central trajectory.

Expanding $\phi$ as a function of $x$ and $y$ and differentiating, we have

$$B_y = \frac{\partial\phi}{\partial y} = \frac{B_0}{a^n}(x^n + \ldots)$$

from which

$$K_n = \left(\frac{B_0}{a^n}\right)\left(\frac{1}{B\rho}\right) \tag{2.25}$$

and

$$S_n = \left(\frac{B_0}{a^n}\right)\left(\frac{L}{B\rho}\right) \tag{2.26}$$

where $L$ is the length of the multipole element.[12]

For a dipole $(n = 0)$, the dipole strength is

$$S_0 = \frac{B_0 L}{B\rho} = \alpha$$

where $\alpha$ is the angle of bend of the central trajectory.

---

[12] Note that in most European publications the monomial $K_n(s)x^n$ in Eq. (2.21) would be replaced by $-K_n(s)x^n/n!$, which would result in a change of sign and the introduction of the factor $n!$ in the definition of $K_n$ in Eq. (2.25) and $S_n$ in Eq. (2.26).

For a quadrupole ($n = 1$),

$$S_1 = \left(\frac{B_0}{a}\right)\left(\frac{L}{B\rho}\right) ;$$

for a sextupole ($n = 2$),

$$S_2 = \left(\frac{B_0}{a^2}\right)\left(\frac{L}{B\rho}\right) ;$$

and so on for higher-order multipoles.

### 2.5.2 *Multipole Strengths for a Non-Uniform Field Expansion*

Consider the midplane field expansion of a non-uniform field:

$$\begin{aligned} B_y(x,0,s) &= B_y(0,0,s)(1 - nhx + \beta(hx)^2 + \gamma(hx)^3 + \ldots) \\ &= B\rho(h - nh^2x + \beta h^3x^2 + \gamma h^4x^3 + \cdots) \\ &= B\rho\sum_{n=0}^{\infty} K_n(s)x^n . \end{aligned} \tag{2.27}$$

The multipole strength factors are

$$K_0 = h , \qquad K_1 = -nh^2 , \qquad K_2 = \beta h^3 , \quad \ldots .$$

The integrated strengths $S_n$ are

$$S_0 = hL = \alpha , \qquad S_1 = -nh^2L , \qquad S_2 = \beta h^3 L , \quad \ldots .$$

### 2.5.3 *Multipole Strengths for a Contoured Entrance or Exit Boundary of a Dipole*

A third method of introducing multipole components is via a curved entrance or exit boundary of a dipole magnet. To calculate the multipole strengths in this case, we integrate Eq. (2.21), holding $x$ constant. The shape of the exit (or entrance) pole face is introduced by letting the limit of integration $L(x)$ vary with $x$. Thus we have the following relation:

$$\int_0^{L(x)} B_y(x,0,s)ds = B\rho\sum x^n \int_0^{L(x)} K_n(s)ds = B\rho\sum S_n x^n . \tag{2.28}$$

We assume $B_y$ to be a constant inside the effective field boundary and zero outside (the finite extent of the fringe field is ignored). In this sharp-cutoff approximation, the field boundary $L(x)$ is given by

$$L(x) = \frac{1}{B_y}\int_0^{L(x)} B_y(x,0,z)dz = \frac{S_1}{h}x + \frac{S_2}{h}x^2 + \cdots \qquad (2.29)$$

where $h = 1/\rho$.

The slope of the boundary at $x = 0$ is $S_1/h$. If we denote the boundary angle by $\beta$, the slope is also $-\tan\beta$. The minus sign defines the positive orientation of the angle. Thus we have

$$S_1 = -h\tan\beta\ .$$

A positive $\beta$ implies radial ($x$-plane) defocusing and transverse ($y$-plane) focusing.

The boundary defining a sextupole component is parabolic. It is convenient (from a construction point of view) to relate the sextupole strength to the radius of curvature $R$ of the parabola at $x = 0$:

$$\frac{1}{R} = \frac{Z''}{(1+Z'^2)^{3/2}} = \frac{2S_2}{h\sec^3\beta}$$

or

$$S_2 = \frac{h\sec^3\beta}{2R}\ .$$

From Eq. (2.28) we conclude that a positive multipole component of the field increases the value $\int Bds$ as $x$ increases. Thus a positive sextupole is represented by a concave surface at the boundary. Figure 8 shows the sign conventions used in the TRANSPORT program for $\beta$ and $R$.

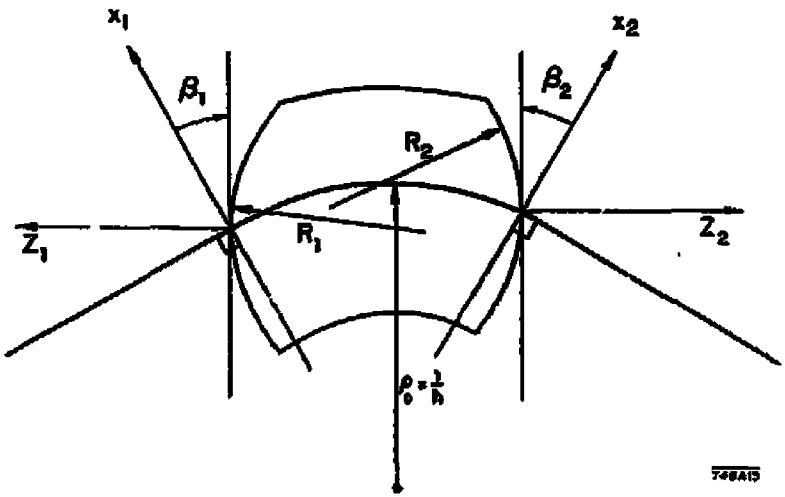


Fig. 8. Field boundaries for bending magnets. The quantities illustrated in the figure have a positive sign when using the convention of the TRANSPORT program.

## 2.6 The Equations of Motion in Their Final Form to Second Order

Having derived the expressions (2.20), we are now in a position to substitute them into the general second-order equations of motion (2.6). We find for $x$

$$x''-h(1+hx)-x'(hx'+h'x)$$

$$=\left(\frac{p_0}{p}\right)T'\left\{(1+hx)[-h+nh^2x-\beta h^3x^2+\frac{1}{2}(h''-nh^3+2\beta h^3)y^2]\right.$$

$$\left.+h'yy'+\cdots\right\}$$

and for $y$

$$y''-y'(hx'+h'x)$$

$$=\left(\frac{p_0}{p}\right)T'(-x'h'y-(1+hx)(nh^2y-2\beta h^3xy)+\cdots)\,.$$

Note that we have eliminated the charge of the particle $e$ in the equations of motion. This has resulted from the use of Eq. (2.7), which is the equation of motion of the central trajectory.

Inserting a second-order expansion for

$$T' = \sqrt{x'^2 + y'^2 + (1 + hx)^2}$$

and letting

$$\frac{p_0}{p} = \frac{p_0}{p_0(1+\delta)} = 1 - \delta + \delta^2 + \cdots \tag{2.30}$$

we finally express the differential equations for $x$ and $y$ to second order as follows:

$$\begin{aligned} x'' + (1-n)h^2 x = {} & h\delta + (2n - 1 - \beta)h^3x^2 + h'xx' + \frac{1}{2}hx'^2 \\ & + (2-n)h^2x\delta + \frac{1}{2}(h'' - nh^3 + 2\beta h^3)y^2 \\ & + h'yy' - \frac{1}{2}hy'^2 - h\delta^2 + \text{higher-order terms}, \\ y'' + nh^2 y = {} & 2(\beta - n)h^3xy + h'xy' - h'x'y + hx'y' + nh^2y\delta \\ & + \text{higher-order terms}. \end{aligned} \tag{2.31}$$

From Eqs. (2.31) the familiar equations of motion for the first-order terms may be extracted:

$$x'' + (1-n)h^2x = h\delta \qquad \text{and} \qquad y'' + nh^2y = 0. \tag{2.32}$$

Substituting $K_1 = -nh^2$ into Eqs. (2.31), the equations of motion for a pure quadrupole field result by taking the limit $h \to 0$, $h' \to 0$, and $h'' \to 0$. They are

$$\begin{aligned} x'' + K_1 x &= K_1 x\delta, \\ y'' - K_1 y &= -K_1 y\delta, \end{aligned}$$

where

$$K_1 = \left(\frac{B_0}{a}\right)\left(\frac{a}{\rho_0}\right) = \left(\frac{B_0}{a}\right)\left(\frac{1}{B\rho_0}\right). \tag{2.33}$$

Similarly, to find the equations of motion for a pure sextupole field, we substitute $K_2 = \beta h^3$ into Eqs. (2.31) and take the limit $h \to 0$, $h' \to 0$, and $h'' \to 0$. The

equations are

$$x'' + K_2(x^2 - y^2) = 0 ,$$

$$y'' - 2K_2xy = 0 ,$$

where

$$K_2 = \left(\frac{B_0}{a^2}\right)\left(\frac{e}{p_0}\right) = \left(\frac{B_0}{a^2}\right)\left(\frac{1}{B\rho_0}\right) . \tag{2.34}$$

## 2.7 The Description of the Trajectories and the Coefficients of the Taylor Expansion

The deviation of an arbitrary trajectory from the central trajectory is described by expressing $x$ and $y$ as functions of $s$. The expressions will also contain $x_0$, $y_0$, $x'_0$, $y'_0$, and $\delta$, where the subscript 0 indicates that the quantity is evaluated at $s = 0$; these five initial values will have the value 0 for the central trajectory itself. The procedure for expressing $x$ and $y$ as a fivefold Taylor expansion will be considered in a general way using these initial values, and detailed formulas are given for the calculations of the coefficients through quadratic terms. The expansions are written

$$x = \sum (x|x_0^\kappa y_0^\lambda x'^\mu y'^\nu \delta^\chi) x_0^\kappa y_0^\lambda x'^\mu y'^\nu \delta^\chi ,$$

$$y = \sum (y|x_0^\kappa y_0^\lambda x'^\mu y'^\nu \delta^\chi) x_0^\kappa y_0^\lambda x'^\mu y'^\nu \delta^\chi . \tag{2.35}$$

Here, the parentheses are symbols for the Taylor coefficients; the first part of the symbol identifies the coordinate represented by the expansion, and the second indicates the term in question. These coefficients are functions of $s$ to be determined. The $\Sigma$ indicates summation over zero and all positive integer values of the exponents $\kappa$, $\lambda$, $\mu$, $\nu$, $\chi$, however, the detailed calculations will involve only the terms up to the second power. The constant term is zero, and the first-order terms that would indicate a coupling between the coordinates $x$ and $y$ are also zero; this results from the midplane symmetry. Thus we have

$$(x|1) = (y|1) = 0 ,$$

$$(x|y_0) = (y|x_0) = 0 ,$$

$$(x|y'_0) = (y|x'_0) = 0 . \tag{2.36}$$

Here, the first line is a consequence of choosing $x_0 = y_0 = 0$, while the second and third lines follow directly from considerations of symmetry, or, more formally, from the formulas at the end of this section.

As mentioned in the introduction, it is convenient to introduce the following abbreviations for the first-order Taylor coefficients:

$$(x|x_0) = c_x(s)\ , \qquad (x|x_b) = s_x(s)\ , \qquad (x|\delta) = d_x(s)\ ,$$

$$(y|y_0) = c_y(s)\ , \qquad (y|y_b) = s_y(s)\ . \tag{2.37}$$

Retaining terms to second order and using Eqs. (2.35) and (2.37), the Taylor expansions of Eqs. (2.35) reduce to the following terms:

$$\begin{aligned} x = \quad & \overbrace{(x|x_0)}^{c_x} x_0 && + \overbrace{(x|x_b)}^{s_x} x_b && + \overbrace{(x|\delta)}^{d_x} \delta \\ & + (x|x_0^2) x_0^2 && + (x|x_0 x_b) x_0 x_b && + (x|x_0\delta) x_0\delta \\ & + (x|x_b^2) x_b^2 && + (x|x_b\delta) x_b\delta && + (x|\delta^2)\delta^2 \\ & + (x|y_0^2) y_0^2 && + (x|y_0 y_b) y_0 y_b && + (x|y_b^2) y_b^2 \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} y = \quad & \overbrace{(y|y_0)}^{c_y} y_0 && + \overbrace{(y|y_b)}^{s_y} y_b \\ & + (y|x_0 y_0) x_0 y_0 && + (y|x_0 y_b) x_0 y_b && + (y|x_b y_0) x_b y_0 \\ & + (y|x_b y_b) x_b y_b && + (y|y_0\delta) y_0\delta && + (y|y_b\delta) y_b\delta \quad . \end{aligned}$$

Substituting these expansions into Eqs. (2.31), we derive a differential equation for each of the first and second order coefficients contained in the Taylor expansions for $x$ and $y$. When this is done a systematic pattern evolves, namely

$$\begin{aligned} & c_x'' + k_x^2 c_x = 0\ , && c_y'' + k_y^2 c_y = 0\ , \\ & s_x'' + k_x^2 s_x = 0\ , && s_y'' + k_y^2 s_y = 0\ , \\ & q_x'' + k_x^2 q_x = f_x\ , && q_y'' + k_y^2 q_y = f_y\ , \end{aligned} \tag{2.39}$$

where $k_x^2 = (1-n)h^2$ and $k_y^2 = nh^2$ for the $x$ and $y$ motions, respectively. The first two of these equations represent the equations of motion for the first-order monoenergetic terms $s_x$, $c_x$, $s_y$, and $c_y$. That there are two solutions, $c$ and $s$, is a manifestation of the fact that the differential equation is second order; hence the two solutions differ only by the initial conditions of the characteristic $s$ and $c$ functions. The third differential equation for $q$ is a type form which represents the solution for the first-order dispersion $d_x$ and for any one of the coefficients of the second-order aberrations in the system where the driving function $f$ for each aberration is obtained from the substitution of the Taylor expansions of Eqs. (2.38) into the general differential Eqs. (2.31).

The coefficients satisfy the boundary conditions:

$$c(0) = 1\,, \qquad c'(0) = 0\,,$$

$$s(0) = 0\,, \qquad s'(0) = 1\,,$$

$$d(0) = 0\,, \qquad d'(0) = 0\,,$$

$$q(0) = 0\,, \qquad q'(0) = 0\,. \tag{2.40}$$

The driving term $f$ is a polynomial, characteristic of the particular $q$ whose terms are coefficients of order less than that of q, and their derivatives. The coefficients in these polynomials are themselves polynomials in $h$, $h'$, ..., with coefficients that are linear functions of $n$, $\beta$, .... For example, for $q = (x|x_0^2)$, we have

$$f = (2n - 1 - \beta)h^3 c_x^2 + h\,c_x c'_x + \frac{1}{2}h c'^2_x\,. \tag{2.41}$$

In Table I are listed the $f$ functions for the remaining linear coefficient, the momentum dispersion $d_x(s)$ and all of the nonzero quadratic coefficients, shown in Eqs. (2.38), which represent the second-order aberrations of a system.

The coefficients $c$ and $s$ (with identical subscripts) satisfy the same differential equation, which has the form of the homogeneous equation of a harmonic oscillator. Here, the stiffness $k^2$ is a function of s and may be of either sign. In view of their boundary conditions, it is natural to consider $c$ and $s$ as the analogs of the two fundamental solutions of a simple harmonic oscillator, namely $\cos\omega s$ and $(\sin\omega s)/\omega$. The function $q$ is the response of the hypothetical oscillator when, starting at equilibrium and at rest, it is subjected to a driving force $f$.

The stiffness parameters $k_x^2$ and $k_y^2$ represent the converging powers of the field for the two respective coordinates. It is possible for either to be negative, in which case it actually represents a diverging effect. Addition of $k_x^2$ and $k_y^2$ yields

$$k_x^2 + k_y^2 = h^2\,. \tag{2.42}$$

For a specific magnitude of $h$ (within one dipole magnet), $k_x^2$ and $k_y^2$ may be varied by adjusting $n$, but the total converging power is unchanged; any increase in one converging power is at the expense of the other. The total converging power is positive.

A special case of interest is provided by the uniform field; here $h = \text{const}$ and $n = 0$; then $k_x^2 = h^2$ and $k_y^2 = 0$. Thus, there is a converging effect for $x$ resulting in the familiar semicircular focusing, which is accompanied by no convergence or divergence in $y$.

Table I. The Driving Terms for the Coefficients

| $d_x =$ | | | | | |
|---|---|---|---|---|---|
| | $(x\|\delta)$ | $h$ | | | |
| | $(x\|x_0^2)$ | | $+(2n-1-\beta)h^3c_x^2$ | $+h'c_xc'_x$ | $+\frac{1}{2}hc'^2_x$ |
| | $(x\|x_0x'_0)$ | | $+2(2n-1-\beta)h^3c_xs_x$ | $+h'(c_xs'_x+c'_xs_x)$ | $+hc'_xs'_x$ |
| | $(x\|x_0\delta)$ | $(2-n)h^2c_x$ | $+2(2n-1-\beta)h^3c_xd_x$ | $+h'(c_xd'_x+c'_xd_x)$ | $+hc'_xd'_x$ |
| | $(x\|x'_0{}^2)$ | | $(2n-1-\beta)h^3s_x^2$ | $+h's_xs'_x$ | $+\frac{1}{2}hs'^2_x$ |
| | $(x\|x'_0\delta)$ | $(2-n)h^2s_x$ | $+2(2n-1-\beta)h^3s_xd_x$ | $+h'(s_xd'_x+s'_xd_x)$ | $+hs'_xd'_x$ |
| | $(x\|\delta^2)$ | $-h+(2-n)h^2d_x$ | $+(2n-1-\beta)h^3d_x^2$ | $+h'd_xd'_x$ | $+\frac{1}{2}hd'^2_x$ |
| | $(x\|y_0^2)$ | | $\frac{1}{2}(h''-nh^3+2\beta h^3)c_y^2$ | $+h'c_yc'_y$ | $-\frac{1}{2}hc'^2_y$ |
| | $(x\|y_0y'_0)$ | | $(h''-nh^3+2\beta h^3)c_ys_y$ | $+h'(c_ys'_y+c'_ys_y)$ | $-hc'_ys'_y$ |
| | $(x\|y'_0{}^2)$ | | $\frac{1}{2}(h''-nh^3+2\beta h^3)s_y^2$ | $+h's_ys'_y$ | $-\frac{1}{2}hs'^2_y$ |
| | $(y\|x_0y_0)$ | | $2(\beta-n)h^3c_xc_y$ | $+h'(c_xc'_y-c'_xc_y)$ | $+hc'_xc'_y$ |
| | $(y\|x_0y'_0)$ | | $2(\beta-n)h^3c_xs_y$ | $+h'(c_xs'_y-c'_xs_y)$ | $+hc'_xs'_y$ |
| | $(y\|x'_0y_0)$ | | $2(\beta-n)h^3s_xc_y$ | $+h'(s_xc'_y-s'_xc_y)$ | $+hs'_xc'_y$ |
| | $(y\|x'_0y'_0)$ | | $2(\beta-n)h^3s_xs_y$ | $+h'(s_xs'_y-s'_xs_y)$ | $+hs'_xs'_y$ |
| | $(y\|y_0\delta)$ | $nh^2c_y$ | $+2(\beta-n)h^3c_yd_x$ | $-h'(c_yd'_x-c'_yd_x)$ | $+hc'_yd'_x$ |
| | $(y\|y'_0\delta)$ | $nh^2s_y$ | $+2(\beta-n)h^3s_yd_x$ | $-h'(s_yd'_x-s'_yd_x)$ | $+hs'_yd'_x$ |

Another important special case is given by $n = 1/2$; here, $k_x^2 = k_y^2 = h^2/2$. Thus, both coordinates experience an identical positive convergence, and $c_x = c_y$ and $s_x = s_y$; that is, in the linear approximation, the two coordinates behave identically, and if the trajectory continues through a sufficiently extended field, a double focus is produced.

The method of solution of the equations for $c$ and $s$ will not be discussed here, since they are standard differential equations. The most suitable approach to the problem must be determined in each case. In many cases it will be a satisfactory approximation to consider $h$ and $n$, and therefore $k^2$ also, as piecewise constant. Thus, c and $s$ are represented in each interval by a sinusoidal function, a hyperbolic function, a linear function of s or simply a constant. Using Eq. (2.39) it follows for either the $x$ or $y$ motions that

$$\frac{d}{ds}(cs' - c's) = 0 .$$

Upon integrating and using the initial conditions on $c$ and $s$ in Eq. (2.40) we

find

$$cs' - c's = 1 . \tag{2.43}$$

This expression is just the determinant of the first-order transfer matrix representing either the $x$ or $y$ equations of motion. It can be demonstrated that the fact that the determinant is equal to one is equivalent to Liouville's theorem, which states that phase areas are conserved throughout any first-order system in both the $x$ and the $y$ plane motions.

The coefficients $q$ are evaluated using a Green's function integral

$$q = \int_0^s f(\tau) G(s,\tau) d\tau \tag{2.44}$$

where

$$G(s,\tau) = s(s)c(\tau) - s(\tau)c(s) \tag{2.45}$$

and

$$q = s(s) \int_0^s f(\tau)c(\tau)d\tau - c(s) \int_0^s f(\tau)s(\tau)d\tau . \tag{2.46}$$

To verify this result, it should be noted that this equation, in conjunction with Eq. (2.43), reduces the last of Eqs. (2.39) to an identity, and that the last pair of Eqs. (2.40) follow readily from this proposed solution. In particular, if $f = 0$ then $q = 0$. Then it will be seen from Table I that several coefficients are absent, including the linear terms that would represent a coupling between $x$ and $y$. Frequently, the absence of a particular coefficient is obvious from considerations of symmetry.

Differentiation of Eq. (2.46) yields

$$q' = s'(s) \int_0^s f(\tau)c(\tau)d\tau - c'(s) \int_0^s f(\tau)s(\tau)d\tau \tag{2.47}$$

and

$$q'' = f + s''(s) \int_0^s f(\tau)c(\tau)d\tau - c''(s) \int_0^s f(\tau)s(\tau)d\tau . \tag{2.48}$$

The driving terms tabulated in Table I, combined with Eqs. (2.46), (2.47), and (2.48), complete the solution of the general second-order theory. The explicit solutions for specific systems or element of systems can be found in the report

SLAC 75 by Brown.[5] It is useful to integrate the driving terms tabulated in Table I for a separated function lattice, so that the dipole, quadrupole, and sextupole terms are separated. The results are shown in Table II for point to point imaging ($s_{x,y}(s) = 0$), and in Table III for parallel to point imaging ($c_{x,y}(s) = 0$). The fringing field terms containing $h'(s)$ have been dropped. This is a reasonable approximation at high energies where $\rho(s)$ is very large compared with the beam dimensions.

Tables I, II, and III are especially useful for determining the symmetry conditions needed to make a given aberration or group of aberrations vanish, or to determine the coupling coefficient of the terms with respect to the multipole strengths $S_{nj}$ for the $j^{th}$ element. Where $n$ is the order of the multipole and $j$ identifies distinct multipole magnets.

Table II. The integrated values of the second-order matrix elements for a separated function lattice for point to point imaging ($s_{x,y}(s) = 0$); fringing field terms are not included.

| | Dipole | Sextupole | Quadrupole |
|---|---|---|---|
| $(x\|x_0^2)$ | $\cong -\frac{1}{2}c_x(s)\int_0^s c_x'^2 s_x d\alpha$ | $+c_x(s)\sum_j S_{2j}c_{xj}^2 s_{xj}$ | |
| $(x\|x_0x_0')$ | $\cong -c_x(s)\int_0^s c_x' s_x' s_x d\alpha$ | $+2c_x(s)\sum_j S_{2j}c_{xj}s_{xj}^2$ | |
| $(x\|x_0\delta)$ | $\cong -c_x(s)\int_0^s c_x' d_x' s_x d\alpha$ | $+2c_x(s)\sum_j S_{2j}c_{xj}s_{xj}d_{xj} - c_x(s)\sum_j S_{1j}c_{xj}s_{xj}$ | |
| $(x\|x_0'^2)$ | $\cong -\frac{1}{2}c_x(s)\int_0^s s_x'^2 s_x d\alpha$ | $+c_x(s)\sum_j S_{2j}s_{xj}^2$ | |
| $(x\|x_0'\delta)$ | $\cong -c_x(s)\int_0^s s_x' d_x' s_x d\alpha$ | $+2c_x(s)\sum_j S_{2j}s_{xj}^2 d_{xj}$ | $-c_x(s)\sum_j S_{1j}s_{xj}^2$ |
| $(x\|\delta^2)$ | $\cong -\frac{1}{2}c_x(s)\int_0^s d_x'^2 s_x d\alpha$ | $+c_x(s)\sum_j S_{2j}s_{xj}d_{xj}^2$ | $-c_x(s)\sum_j S_{1j}s_{xj}d_{xj}$ |
| $(x\|y_0^2)$ | $\cong \frac{1}{2}c_x(s)\int_0^s c_y'^2 s_x d\alpha$ | $-c_x(s)\sum_j S_{2j}c_{yj}^2 s_{xj}$ | |
| $(x\|y_0y_0')$ | $\cong c_x(s)\int_0^s c_y' s_y' s_x d\alpha$ | $-2c_x(s)\sum_j S_{2j}c_{yj}s_{yj}s_{xj}$ | |
| $(x\|y_0'^2)$ | $\cong \frac{1}{2}c_x(s)\int_0^s s_y'^2 s_x d\alpha$ | $-c_x(s)\sum_j S_{2j}s_{yj}^2 s_{xj}$ | |
| $(y\|x_0y_0)$ | $\cong -c_y(s)\int_0^s c_x' c_y' s_y d\alpha$ | $-2c_y(s)\sum_j S_{2j}c_{xj}c_{yj}s_{yj}$ | |
| $(y\|x_0y_0')$ | $\cong -c_y(s)\int_0^s c_x' s_y' s_y d\alpha$ | $-2c_y(s)\sum_j S_{2j}c_{xj}s_{yj}^2$ | |
| $(y\|x_0'y_0)$ | $\cong -c_y(s)\int_0^s s_x' c_y' s_y d\alpha$ | $-2c_y(s)\sum_j S_{2j}s_{xj}c_{yj}s_{yj}$ | |
| $(y\|x_0'y_0')$ | $\cong -c_y(s)\int_0^s s_x' s_y' s_y d\alpha$ | $-2c_y(s)\sum_j S_{2j}s_{xj}s_{yj}^2$ | |
| $(y\|y_0\delta)$ | $\cong -c_y(s)\int_0^s c_y' d_y' s_y d\alpha$ | $-2c_y(s)\sum_j S_{2j}c_{yj}d_{yj}s_{yj} + c_y(s)\sum_j S_{1j}c_{yj}s_{yj}$ | |
| $(y\|y_0'\delta)$ | $\cong -c_y(s)\int_0^s s_y' d_y' s_y d\alpha$ | $-2c_y(s)\sum_j S_{2j}d_{yj}s_{yj}^2 + c_y(s)\sum_j S_{1j}s_{yj}^2$ | |

**Table III. The integrated values of the second-order matrix elements for a separated function lattice for parallel to point imaging ($c_{x,y}(s) = 0$); fringing field terms are not included.**

| | Dipole | Sextupole | Quadrupole |
|---|---|---|---|
| $(x\|x_0^2)$ | $\cong +\frac{1}{2}s_x(s)\int_0^s c'^2_x c_x d\alpha$ | $-s_x(s)\sum_j S_{2j}c_x^3$ | |
| $(x\|x_0x_0')$ | $\cong +s_x(s)\int_0^s c'_x s'_x c_x d\alpha$ | $-2s_x(s)\sum_j S_{2j}c_x^2 s_x$ | |
| $(x\|x_0\delta)$ | $\cong +s_x(s)\int_0^s c'_x d'_x c_x d\alpha$ | $-2s_x(s)\sum_j S_{2j}c_x^2 d_x$ | $+s_x(s)\sum_j S_{1j}c_x^2$ |
| $(x\|x_0'^2)$ | $\cong +\frac{1}{2}s_x(s)\int_0^s s'^2_x c_x d\alpha$ | $-s_x(s)\sum_j S_{2j}s_x^2 c_x$ | |
| $(x\|x_0'\delta)$ | $\cong +s_x(s)\int_0^s s'_x d'_x c_x d\alpha$ | $-2s_x(s)\sum_j S_{2j}s_x c_x d_x$ | $+s_x(s)\sum_j S_{1j}s_x c_x$ |
| $(x\|\delta^2)$ | $\cong +\frac{1}{2}s_x(s)\int_0^s d'^2_x c_x d\alpha$ | $-s_x(s)\sum_j S_{2j}c_x d_x^2$ | $+s_x(s)\sum_j S_{1j}c_x d_x$ |
| $(x\|y_0^2)$ | $\cong -\frac{1}{2}s_x(s)\int_0^s c'^2_y c_x d\alpha$ | $+s_x(s)\sum_j S_{2j}c_y^2 c_x$ | |
| $(x\|y_0y_0')$ | $\cong -s_x(s)\int_0^s c'_y s'_y c_x d\alpha$ | $+2s_x(s)\sum_j S_{2j}c_y s_y c_x$ | |
| $(x\|y_0'^2)$ | $\cong -\frac{1}{2}s_x(s)\int_0^s s'^2_y c_x d\alpha$ | $+s_x(s)\sum_j S_{2j}s_y^2 c_x$ | |
| $(y\|x_0y_0)$ | $\cong +s_y(s)\int_0^s c'_x c'_y c_y d\alpha$ | $+2s_y(s)\sum_j S_{2j}c_x c_y^2$ | |
| $(y\|x_0y_0')$ | $\cong +s_y(s)\int_0^s c'_x s'_y c_y d\alpha$ | $+2s_y(s)\sum_j S_{2j}c_x c_y s_y$ | |
| $(y\|x_0'y_0)$ | $\cong +s_y(s)\int_0^s s'_x c'_y c_y d\alpha$ | $+2s_y(s)\sum_j S_{2j}s_x c_y^2$ | |
| $(y\|x_0'y_0')$ | $\cong +s_y(s)\int_0^s s'_x s'_y c_y d\alpha$ | $+2s_y(s)\sum_j S_{2j}s_x c_y s_y$ | |
| $(y\|y_0\delta)$ | $\cong +s_y(s)\int_0^s c'_y d'_x c_y d\alpha$ | $+2s_y(s)\sum_j S_{2j}c_y^2 d_x$ | $-s_y(s)\sum_j S_{1j}c_y^2$ |
| $(y\|y_0'\delta)$ | $\cong +s_y(s)\int_0^s s'_y d'_x c_y d\alpha$ | $+2s_y(s)\sum_j S_{2j}d_x c_y^2$ | $-s_y(s)\sum_j S_{1j}s_y c_y$ |

# 3 FIRST-ORDER OPTICS

## 3.1 Notations and Definitions

This chapter is devoted to the detailed study of first-order optics. The results are derived from those obtained in Chapter 2, and in particular from Eqs. (2.35) to (2.40).

In order to simplify the notation, the following convention is adopted:

The variables $x$, $x'$, $y$, $y'$, $l$, $\delta$ will be denoted by $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $x_6$.

Using this notation and restricting ourselves to first order, Eq. (2.35) can

be written in the following form:

$$x_i = \sum_{j=1}^{6} R_{ij} x_{0j} , \qquad i = 1, 2, \ldots, 6 \tag{3.1}$$

where we have adopted for $(x_i|x_{0j})$ the notation $R_{ij}$ as used in linear algebra. Equation (3.1) can also be rewritten in compact matrix notation as

$$X = RX_0 .$$

### 3.1.1 *Geometric Terms, Chromatic Terms, and Phase Space*

In optics studies it is customary first to study the properties of a set of optical elements by restricting the momentum of the test particles to one value (called the reference momentum), and then to study the properties as the momentum is changed. The elements $R_{ij}$ of the matrix $R$ that contain one subscript with the value 6 are called chromatic terms. The elements $R_{ij}$ for which no subscript is equal to 6 are referred to as geometric terms.

The condition expressed by Eqs. (2.36) and (2.37) simplify the matrix $R$ to the following expression:

$$R = \begin{pmatrix} c_x(s) & s_x(s) & 0 & 0 & 0 & d_x(s) \\ c_x'(s) & s_x'(s) & 0 & 0 & 0 & d_x'(s) \\ 0 & 0 & c_y(s) & s_y(s) & 0 & 0 \\ 0 & 0 & c_y'(s) & s_y'(s) & 0 & 0 \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} \end{pmatrix} .$$

Because there is no coupling between the variables $x_1, x_2$ and $x_3, x_4$ (resulting from midplane symmetry) it is convenient, when one is not considering the chromatic terms, to retain only the two by two matrices describing the motion in the planes defined by the coordinates $x_1, x_2$ or $x, x'$ and by the coordinates $x_3, x_4$ or $y, y'$. The first plane is the horizontal phase space or the $(x, x')$ phase plane, and the second is the vertical phase plane or the $(y, y')$ phase plane. The matrix $R$, called the transfer matrix, then reduces to the simple $2 \times 2$ form for each plane:

$$R(s) = \begin{bmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{bmatrix} = \begin{bmatrix} c(s) & s(s) \\ c'(s) & s'(s) \end{bmatrix} .$$

## 3.2 GREEN'S FUNCTION INTERPRETATION

Consider two points in a beam line defined by the positions $r$ and $s$ (assumed in increasing order) of the longitudinal coordinate, as illustrated in Fig. 9. An interesting problem is to determine at the point $s$ the effect of a punctual (zero length) magnetic element located at coordinate $r$. It is understood that only angular kicks can be achieved by such elements.

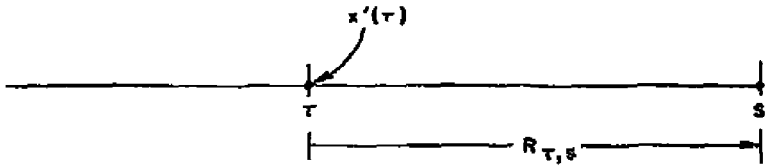


Fig. 9. Green's function interpretation: an angular kick at position $r$ results in an effect at position $s$.

Let $x'(r)$ denote the angular kick produced at position $r$. The values of $x(s)$ and $x'(s)$ denoting the effect at position $s$ are given by

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = R_{r,s} \begin{pmatrix} 0 \\ x'(r) \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}_{r,s} \begin{pmatrix} 0 \\ x'(r) \end{pmatrix}$$

where $R$ is the linear transformation matrix between positions $r$ and $s$.

Since the matrix

$$R(r) = \begin{pmatrix} c(r) & s(r) \\ c'(r) & s'(r) \end{pmatrix}$$

transfers data from position 0 to position $r$, the transfer matrix $R_{r,s}$ between $r$ and $s$ is obtained as follows:

$$R(s) = R_{r,s}\, R(r)$$

from which one gets

$$R_{r,s} = P(s)R(r)^{-1}$$

or, in explicit form,

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}_{r,s} = \begin{pmatrix} c(s) & s(s) \\ c'(s) & s'(s) \end{pmatrix} \begin{pmatrix} s'(r) & -s(r) \\ -c'(r) & c(r) \end{pmatrix}$$

from which

$$(R_{12})_{\tau,s} = -c(s)s(\tau) + s(s)c(\tau)$$

and

$$(R_{22})_{\tau,s} = -c'(s)s(\tau) + s'(s)c(\tau) .$$

$(R_{12})_{\tau,s}$ is the Green's function for $x(s)$ and $(R_{22})_{\tau,s}$ is the Green's function for $x'(s)$.

### 3.2.1 *Example of the Use of the Green's Function*

**Find the expression for the first-order dispersion in terms of the $c$ and $s$ functions and find the condition (in terms of the same functions) for first-order achromaticity (a dispersion-free system). The dispersion function $d_x(s)$ is the solution of the equation**

$$d_x'' + k_x^2 d_x = h(s) = (1/\rho_0) .$$

In this equation $h(s)$ is the driving term $f$ for the dispersion $d_x$; then, from Eqs. (2.46) and (2.47), we have

$$d_x(s) = \int_0^s f(\tau)G(\tau,s)d\tau = \int_0^s h(\tau)G(\tau,s)d\tau$$

$$= s_x(s)\int_0^s c_x(\tau)h(\tau)d\tau - c_x(s)\int_0^s s_x(\tau)h(\tau)d\tau .$$

Denoting $d\tau/\rho_0$ by $d\alpha$, the differential angle of bend of the central trajectory, we obtain

$$d_x(s) = s_x(s)\int_0^s c_x(\tau)d\alpha - c_x(s)\int_0^s s_x(\tau)d\alpha$$

and

$$d_x'(s) = s_x'(s)\int_0^s c_x(\tau)d\alpha - c_x'(s)\int_0^s s_x(\tau)d\alpha .$$

From these two formulas defining the dispersion and its derivative in terms of

the $c$ and $s$ functions we get the achromaticity conditions as

$$\int_0^a c_x(\tau)d\alpha = 0 = \int_0^a R_{21}(\tau)d\alpha ,$$

$$\int_0^a s_x(\tau)d\alpha = 0 = \int_0^a R_{12}(\tau)d\alpha .$$

## 3.3 Linear Beam Optics

By the word *beam* we mean a set of n particles where $n$ is a large integer. The behavior of beams can be studied by the tracing of a large number of individual particles or by studying the transfer properties of algebraic curves which are assumed to bound the particles contained in the beam. It is a property of linear algebra that the only curves that are simple to transfer are the conics (second-degree curves). Therefore, as a simplification decision, it will be assumed that beams restricted to two dimensions are adequately described in linear optics by an ellipse.

### *3.3.1 Elliptical Beam Envelopes*

Let us first consider the two-dimensional case in the horizontal phase plane $x, x'$.

The general equation of an ellipse, centered on the origin, is

$$ax^2 + 2bxx' + cx'^2 = m$$

which can be written in matrix form as

$$X^t B X = m$$

where $B$ is a positive definite symmetric matrix defined by the coefficients $a, b, c$ as follows:

$$B = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x \\ x' \end{pmatrix}$$

and $X^t$ is the transpose of $X$. The multiplication of all four coefficients by a common factor does not change the ellipse. One has then the choice of either letting $m = 1$ or $\det B = 1$. In the first instance the area of the ellipse is given by $\pi/\sqrt{\det B}$ and in the second instance the area equals $\pi m$.

Let us adopt the definition

$$X^t B X = 1 \tag{3.2}$$

and denote $\sigma$ as the inverse of matrix $B$

$$\sigma = B^{-1} = \begin{pmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} . \tag{3.3}$$

One can prove, by using techniques of dual spaces in linear algebra, that $\sqrt{\sigma_{11}}$ and $\sqrt{\sigma_{22}}$ are the tangential projections of the extreme points of the ellipse on the axes $x$ and $x'$ respectively, as shown in Fig. 10.

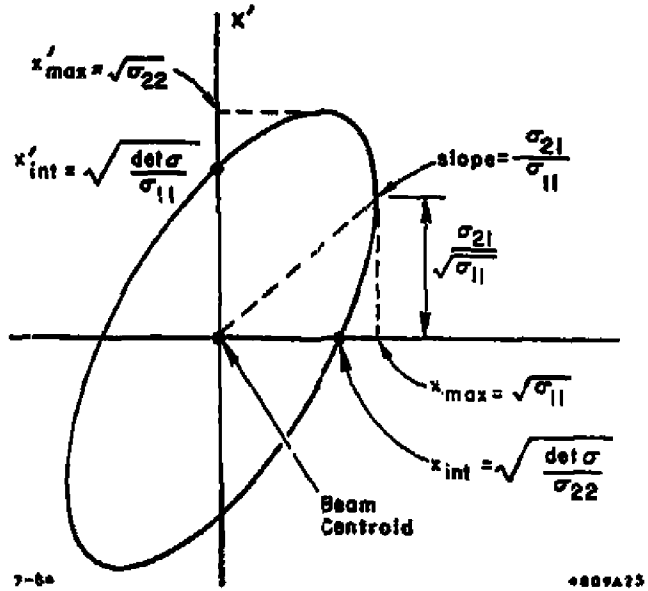


Fig. 10. A beam ellipse based on the $\sigma$ matrix. The maximum extent of the ellipse and its orientation are shown as a function of the matrix elements.

With this definition the equation of the ellipse may also be expressed as

$$X^t \sigma^{-1} X = 1$$

or alternatively as

$$\sigma_{22} x^2 - 2\sigma_{21} x x' + \sigma_{11} x'^2 = \det \sigma$$

and its area is $A = \pi\sqrt{\det \sigma}$.

This definition may be extended in a straightforward way to $n$-dimensions. An $n$-dimensional ellipsoid is defined by $\sigma$, a positive definite symmetric matrix, and has the equation

$$X^t\sigma^{-1}X = 1$$

where now $X$ stands for an $n$-dimensional vector. The volume enclosed by the ellipsoid is given by

$$\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\sqrt{\det\sigma}\,.$$

In particular for dimensions 4 and 6 the volume of the ellipsoid is given by $(\pi^2/2)\sqrt{\det\sigma}$ and by $(\pi^3/6)\sqrt{\det\sigma}$ respectively.

### 3.3.2 *Beam Ellipse Transformation*

Assume a beam to be defined at a longitudinal position $s_1$ by the matrix $\sigma_1$. Its equation is

$$X_1^t\sigma_1^{-1}X_1 = 1\,.$$

Consider the point $s_2$ and assume that $R$ is the linear transfer matrix from $s_1$ to $s_2$. The coordinate transformation satisfies the following relations:

$$X_2 = RX_1 \qquad \text{and} \qquad X_1 = R^{-1}X_2\,.$$

This linear transformation will change the ellipse $\sigma_1$ at point $s_1$ into another ellipse $\sigma_2$ at point $s_2$. The equation of the second ellipse is

$$X_2^t\sigma_2^{-1}X_2 = 1\,.$$

We need to find the relation between $\sigma_2$ and $\sigma_1$. To do so we express $X_2^t$ and $X_2$ in terms of $X_1^t$ and $X_1$. We obtain

$$X_1^tR^t\sigma_2^{-1}RX_1 = 1$$

from which we conclude

$$\sigma_1^{-1} = R^t\sigma_2^{-1}R$$

and, by inverting,

$$\sigma_1 = R^{-1}\sigma_2(R^t)^{-1}$$

or equivalently

$$\sigma_2 = R\sigma_1R^t\,.$$

Note that since $\det R = 1$ (Liouville's theorem) it follows that

$$\det\sigma_1 = \det\sigma_2\,,$$

showing that the transformation has preserved the phase volume of the beam.

Before pursuing the study of the transmission of beams through a system we shall look at the formulas that govern single-particle motion in closed machines.

## 3.4 Single-Particle Optics for a Closed Machine

The first-order equations of motions are given by

$$\frac{d^2x}{ds^2} + k_x^2(s)x = \frac{\Delta p/p}{\rho(s)} = \delta h(s) ,$$

$$\frac{d^2y}{ds^2} + k_y^2(s)y = 0 .$$

In a closed machine the functions $k_x(s), k_y(s)$ and $\rho(s)$ are periodic functions of $s$ . Let us consider solutions for the nondispersive ($\delta = 0$) stable case. The theorem of Floquet states that there exist two periodic functions $\beta(s)$ and $\psi(s)$ in terms of which the general solution $x(s)$ can be expressed:

$$x(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s) + \phi)$$

where $\epsilon$ and $\phi$ are two arbitrary constants and the two functions $\beta(s)$ and $\psi(s)$ are not independent, but are linked by the relation

$$\psi(s) = \int_0^s \frac{d\tau}{\beta(\tau)} .$$

$\psi(s)$ is called the "machine phase shift" between points 0 and $s$ . Differentiation of $x(s)$ with respect to $s$ yields

$$x'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}\frac{\beta'(s)}{2}\cos(\psi(s) + \phi) - \sqrt{\epsilon\beta(s)}\left(\sin(\psi(s) + \phi)\frac{1}{\beta(s)}\right)$$

$$= -\sqrt{\frac{\epsilon}{\beta(s)}}\Big(\alpha(s)\cos(\psi(s) + \phi) + \sin(\psi(s) + \phi)\Big)$$

where we define the function $\alpha(s)$ by

$$\beta'(s) = -2\alpha(s) .$$

Alternatively $x'(s)$ can be written in the form

$$x'(s) = \sqrt{\epsilon\gamma(s)}\cos(\chi(s) + \phi)$$

where $\chi(s)$ satisfies the relation

$$\tan(\psi(s) - \chi(s)) = \frac{1}{\alpha(s)}$$

or equivalently

$$\sin(\psi(s) - \chi(s)) = -\frac{1}{\sqrt{\beta(s)\gamma(s)}}$$

and the function $\gamma(s)$ is defined by

$$\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)} .$$

Let us note once again that ALL the functions $x(s)$, $x'(s)$, $\beta(s)$, $\alpha(s)$, $\gamma(s)$, and $\psi(s)$ are periodic with the period $L$ where $L$ is the length of the closed machine. Consider now the values of the solution $x$ and its derivative at successive revolutions at a fixed point $s$. We can describe the motion at position $s$ by plotting the values of $x$ and $x'$ in the "$x$-phase plane". Eliminating the trigonometric functions from the expressions of $x(s)$ and $x'(s)$ yields, after some manipulation,

$$\gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 = \epsilon ,$$

which shows that the positions $(x, x')$ of a particle at the coordinate $s$ upon successive turns lie on an ellipse. This ellipse can also be written in matrix form.

The parameters $\alpha$, $\beta$, and $\gamma$ are sometimes referred to in the literature as the Twiss parameters.

### *3.4.1 The Machine Ellipse*

Let $T$ denote the matrix

$$T = \begin{pmatrix} \beta(s) & -\alpha(s) \\ -\alpha(s) & \gamma(s) \end{pmatrix} \tag{3.4}$$

where $T$ has a determinant equal to 1. The equation of the ellipse characteristic of the machine may then also be written in the matrix form

$$X^t T^{-1} X = \epsilon . \tag{3.5}$$

The area of this ellipse is $\pi\epsilon$. As was shown for the beam ellipse in a previous paragraph, we can compute the maximum $x$ excursion $x_{max}$ and the maximum

$x'$ excursion $x'_{max}$. They are given by the expressions :

$$x_{max} = \sqrt{\beta\epsilon} \quad , \quad x'_{max} = \sqrt{\gamma\epsilon} \, .$$

From the explicit equation of the ellipse one can also obtain the coordinates of the intercepts with the axes:

$$x_{inter} = \sqrt{\frac{\epsilon}{\gamma}} \quad , \quad x'_{inter} = \sqrt{\frac{\epsilon}{\beta}} \, .$$

From these expression one can deduce alternative expressions for the area of the ellipse:

$$\text{Area} = \pi\epsilon = \pi x_{max} x'_{inter} = \pi x_{inter} x'_{max} \, .$$

This result can also be generalized to dimension $n$. For $n$ dimensions $\epsilon$ is the product of one intercept ,one maximum and $(n-2)$ maxima of subspace intercepts. Figure 11 illustrates these points in two dimensions.

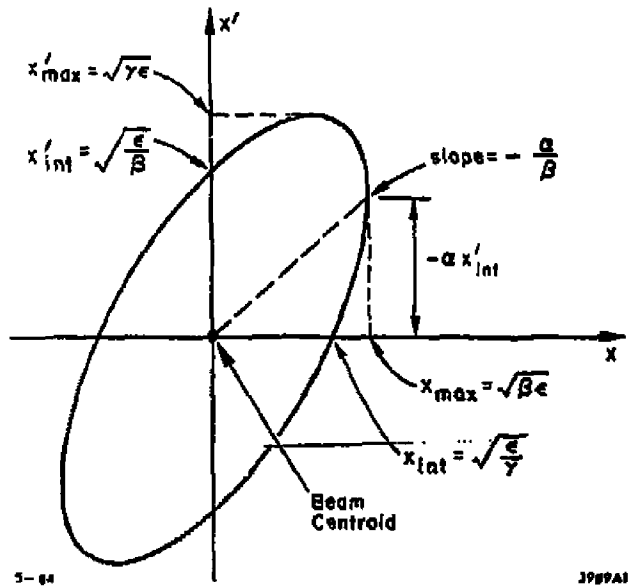


**Fig. 11. An ellipse based on the machine parameters $\beta$, $\alpha$, $\gamma$, illustrating single-particle motion in a closed machine. The area of the ellipse is $A = \pi\epsilon$.**

Consider now two points $S_1$ and $S_2$ on the reference orbit of the closed machine. Let $T_1$ and $T_2$ denote the machine ellipse matrices at these two points and $R$ the transfer matrix from point $S_1$ to point $S_2$. As for the beam ellipses, we have the following transformation relating $T_2$ to $T_1$:

$$T_2 = RT_1R^t$$

or

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} R_{11}^2 & -2R_{11}R_{12} & R_{12}^2 \\ -R_{11}R_{21} & 1+2R_{12}R_{21} & -R_{12}R_{22} \\ R_{21}^2 & -2R_{21}R_{22} & R_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} . \qquad (3.6)$$

## 3.5 The Relationship Between the Beam Ellipse and the Machine Ellipse

Having defined ellipses both for beams of particles and for single-particle motion in closed machines, we now turn our attention to the relationship between the two. Consider a closed machine that is characterized by the ellipse $E_1$ with emittance $\epsilon$ and area $A_1$, as shown in Fig. 12. Let $P_1$ denote a point on that ellipse and let $O$ denote the origin of the axes. After successive turns around the machine the point $P_1$ will reappear at $P_2$, $P_3$, etc. Since the transformation $R$ governing this motion is linear and area preserving, the area $OP_1P_2$ is equal

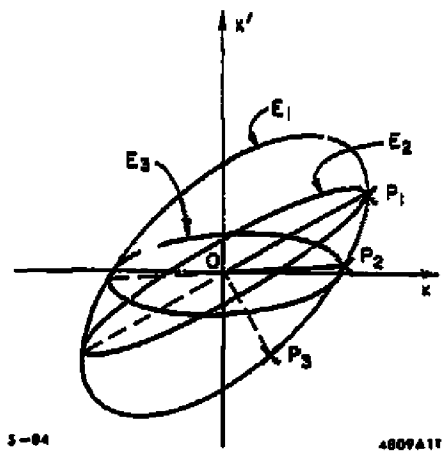


Fig. 12. The superposition of *beam* ellipses $E_2$ and $E_3$ with a *machine* ellipse $E_1$.

to area $OP_2P_3$, etc. It also follows that these areas and $A_1$ satisfy the following relation:

$$\frac{OP_1P_2}{A_1} = mod(\frac{\mu}{2\pi}, 1)$$

where $mod(x, 1)$ denotes the fractional part of $x$. Here we have introduced the new notation $\mu$, defined as

$$\mu = \psi(L) ,$$

which is the phase shift for one complete revolution of a closed machine, where $L$ is the length of the closed machine. The ratio $\mu/2\pi$ is denoted by $\nu$ and is called the tune of the machine. Consider now an ellipse $E_2$ inscribed in $E_1$ with a contact point at $P_1$. Let the ellipse $E_2$ represent a beam of particles circulating in the machine. Ellipse $E_2$ becomes, after one turn, ellipse $E_3$ with contact point $P_2$. Ellipses $E_2$ and $E_3$ have the same area.

When the beam ellipse $E_2$ is concentric and similar to the machine ellipse $E_1$, the beam is said to be matched to the machine. In this instance the beam reappears on successive turns as the same ellipse, but the individual particles in the beam rotate around the ellipse as did the points $P_1$ etc. This observation shows that the phase space area (and consequently the physical aperture) needed to accommodate a given beam is minimum when the beam is matched to the machine. We shall now use the above properties to define beamlines in various ways which prove useful in practice.

Let us find the transfer matrix which transforms the ellipse defined by the input values $\beta_1$ and $\alpha_1$ at position $s_1$ into an ellipse with the values $\beta_2$ and $\alpha_2$ at position $s_2$.

Consider the solutions as given by the Floquet theorem:

$$x(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s) + \phi) ,$$

$$x'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\Big(\alpha(s)\cos(\psi(s) + \phi) + \sin(\psi(s) + \phi)\Big) .$$

Expanding the trigonometric functions and simplifying the notation gives

$$x = \sqrt{\epsilon\beta}(\cos\psi\cos\phi - \sin\psi\sin\phi) ,$$

$$x' = -\sqrt{\frac{\epsilon}{\beta}}(\alpha\cos\psi\cos\phi - \alpha\sin\psi\sin\phi + \sin\psi\cos\phi + \cos\psi\sin\phi) .$$

The point having $\psi = 0$ is assumed to be associated with the values $\beta_1$ and $\alpha_1$

and $x_1$ and $x_1'$; these values then satisfy the following relations:

$$x_1 = \sqrt{\epsilon\beta_1}\cos\phi\ ,$$

$$x_1' = -\sqrt{\frac{\epsilon}{\beta_1}}(\alpha_1\cos\phi + \sin\phi)\ .$$

Denoting by $\beta_2$, $\alpha_2$, $x_2$, and $x'_2$ the values associated with $\psi$ nonzero, and eliminating $\cos\phi$ and $\sin\phi$ from the previous four equations, one gets

$$x_2 = x_1\sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi + \alpha_1\sin\psi) + x_1'\sqrt{\beta_2\beta_1}\sin\psi\ ,$$

$$x_2' = x_1\frac{-\alpha_2\cos\psi - \sin\psi - \alpha_2\alpha_1\sin\psi + \alpha_1\cos\psi}{\sqrt{\beta_2\beta_1}} + x_1'\sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi - \alpha_2\sin\psi)\ .$$

From the above equations we deduce the transfer matrix between position 1 and position 2 to be

$$R = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\psi + \alpha_1\sin\Delta\psi) & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\frac{(1+\alpha_1\alpha_2)\sin\Delta\psi + (\alpha_2 - \alpha_1)\cos\Delta\psi}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\psi - \alpha_2\sin\Delta\psi) \end{pmatrix} \tag{3.7}$$

where $\Delta\psi$ is the phase shift between position $s_1$ and $s_2$.

In the particular case where the input values $(\beta_1, \alpha_1)$ are equal to the output values $(\beta_2, \alpha_2)$ the transfer matrix becomes

$$R = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

where we have defined

$$\beta = \beta_1 = \beta_2\ , \qquad \alpha = \alpha_1 = \alpha_2\ , \qquad \mu = \Delta\psi\ ,$$

and

$$\gamma = \frac{1+\alpha^2}{\beta}\ .$$

Formula (3.7) expresses the elements of the transfer matrix $R$ in terms of the input parameters $\beta_1$, $\alpha_1$, the output parameters $\beta_2$, $\alpha_2$, and the phase advance $\Delta\psi$ between positions $s_1$ and $s_2$.

It is also possible to express the output Twiss parameters and the phase advance in terms of the input Twiss parameters and the matrix elements. The first part of this *inversion process* is achieved in formula (3.6) which we reproduce here:

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} R_{11}^2 & -2R_{11}R_{12} & R_{12}^2 \\ -R_{11}R_{21} & 1+2R_{12}R_{21} & -R_{12}R_{22} \\ R_{21}^2 & -2R_{21}R_{22} & R_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} . \qquad (3.8)$$

The phase shift $\Delta\psi$ is derived from formula (3.7) as

$$\tan \Delta\psi = \frac{R_{12}}{R_{11}\beta_1 - R_{12}\alpha_1} \qquad (3.9)$$

or

$$\sin \Delta\psi = \frac{R_{12}}{\sqrt{\beta_1\beta_2}} \qquad (3.10)$$

or equivalently by the formulas relating $\psi(s)$ and $\beta(s)$:

$$\Delta\psi = \int_1^2 \frac{ds}{\beta(s)} .$$

Let us look at some elementary configurations and determine their phase shifts:

a) A thin lens is characterized by $s_1 = s_2$ so that $\Delta\psi = 0$.

b) If $R_{12} = 0$ (point to point imaging) then $\Delta\psi = n\pi$.

c) If $R_{..} = 0$ (parallel to point imaging) then $\tan \Delta\psi = -1/\alpha_1$.

d) For a drift of length $L$, $R_{12} = L$ and $\sin \Delta\psi = L/\sqrt{\beta_1\beta_2}$.

### 3.5.1 *Introduction to an Alternative Notation for Beam Definition*

In obtaining Eq. (3.3) we have shown that a beam contained in an ellipse can be characterized by the matrix

$$\sigma = B^{-1} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} .$$

Let us recall that the square roots of the diagonal terms give the maximum extent of the beam, and that the number $\epsilon = \sqrt{\det \sigma}$ is called the emittance

of the beam. The area of the beam ellipse is then $\pi\epsilon$. With this notation the equation (3.2) of the boundary ellipse is

$$X^t B X = 1$$

where $B$ is the inverse of $\sigma$ and its determinant is equal to $\epsilon^{-2}$. Let us multiply each element of the matrix $B$ by the scalar $\epsilon$ and let us denote the newly obtained matrix by $E$ and let

$$E = \begin{pmatrix} c & a \\ a & b \end{pmatrix} . \tag{3.11}$$

Observe that

$$\det E = bc - a^2 = 1$$

and the Eq. (3.2) of the ellipse becomes

$$X^t E X = \epsilon . \tag{3.12}$$

The above results show that a beam contained in an ellipse may be defined in two equivalent modes, either by the four parameters:

$$\sigma_{11} \quad \sigma_{12} \quad \sigma_{22} \quad \epsilon$$

and the relation

$$\sqrt{\det \sigma} = \epsilon$$

or by the parameters

$$a \quad b \quad c \quad \epsilon$$

and the relation

$$bc - a^2 = 1 .$$

Let us now turn our attention to Eqs. (3.4) and (3.5), in which we defined an ellipse associated with the transfer matrix for single-particle motion in a closed machine. This ellipse has the form

$$X^t T^{-1} X = \epsilon \tag{3.13}$$

where

$$T = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \tag{3.14}$$

and

$$\beta\gamma - \alpha^2 = 1 .$$

The mathematical similarity between the relations (3.14), (3.13) and (3.11), (3.12) is clear. This has led many designers to use the parameters $\beta$, $\alpha$, $\gamma$, and $\epsilon$

to define a beam ellipse as well as to define a transfer ellipse for a closed machine. This habit can lead to some confusion. It is certainly mathematically correct, but the interpretation of the physics is clearly different.

To illustrate this mathematical equivalence let us consider an optical cell characterized by the machine transfer matrix

$$R = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

and an input beam characterized by the matrix

$$E_1 = \begin{pmatrix} c_1 & a_1 \\ a_1 & b_1 \end{pmatrix} .$$

$E_1$ will be transformed as follows:

$$E_2 = (R^t)^{-1} E_1 R^{-1} .$$

With the explicit multiplication of the above matrix relation one can show that $E_2 = E_1$ when the beam defined by $E_1$ is such that

$$b_1 = \beta , \qquad a_1 = \alpha , \qquad c_1 = \gamma .$$

When these conditions are met one says that the beam is matched to the cell. This alternate notation for a beam defining ellipse gives a very simple form to the matching conditions.

This fact alone justifies the usefulness of the alternative notation presented here.

## 4 OPTICAL BUILDING BLOCKS

Having studied the behavior of beams in a general context, we shall now turn our attention to the study of special elements or sets of elements which can be used to design modules with specific functions in beam optics. The following sections are devoted to the study of a few simple and practical modules that occur frequently as lattice building blocks.

### 4.0.1 *A Drift Space or Field-Free Region*

The transfer matrix of a drift is

$$R = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

from which one derives

$$\Delta x = x_2 - x_1 = L x_1' \qquad \text{and} \qquad x_2' = x_1' = \text{a constant} .$$

The Twiss parameters transform as follows according to formula (3.8):

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 1 & -2L & L^2 \\ 0 & 1 & -L \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

From this relation one obtains

$$\Delta\alpha = \alpha_2 - \alpha_1 = -L\gamma_1 \qquad \text{and} \qquad \gamma_2 = \gamma_1 = \text{ a constant} .$$

The relation (3.10) applied to the drift gives

$$\sin\Delta\psi = \frac{R_{12}}{\sqrt{\beta_1\beta_2}} = \frac{L}{\sqrt{\beta_1\beta_2}}$$

showing the relation between the phase advance and the length. The relation (3.9) gives

$$\tan\Delta\psi = \frac{R_{12}}{R_{11}\beta_1 - R_{12}\alpha_1} = \frac{L}{\beta_1 - L\alpha_1} .$$

Consider the extreme point on the beam ellipse shown in Fig. 13.

As the beam travels through the drift space, this point will be displaced by $\Delta x$ given by

$$\Delta x = L\sqrt{\gamma\epsilon} = L\sqrt{\frac{\epsilon}{\beta_w}}$$

where $\beta_w$ is the $\beta$ value achieved at the point where the beam has a waist.

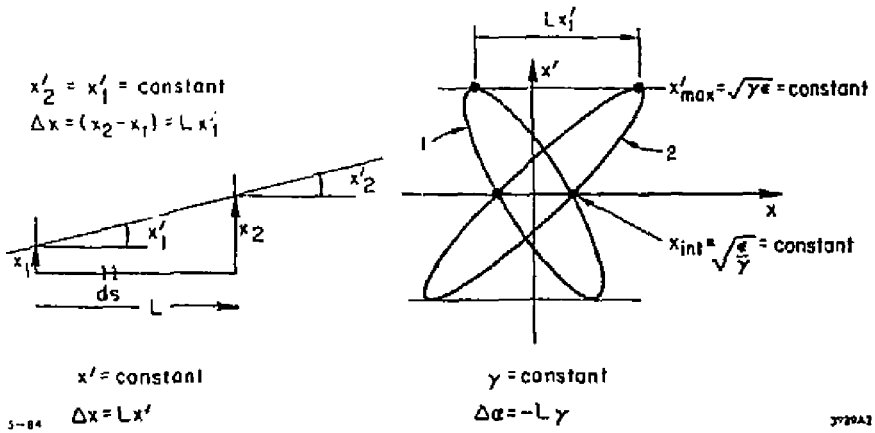


Fig. 13. The transformation of an ellipse through a drift (field-free) space.

*4.0.2 A Thin Lens*

A focusing thin lens has the following transfer matrix:

$$R = \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix}$$

from which one derives

$$x_2 = x_1 = \text{a constant} \qquad \text{and} \qquad \Delta x' = x_2' - x_1' = -\frac{x_1}{F} .$$

The Twiss parameters transform according to formula (3.8),

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/F & 1 & 0 \\ 1/F^2 & 2/F & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

which gives

$$\beta_2 = \beta_1 = \text{a constant} \qquad \text{and} \qquad \Delta\alpha = \alpha_2 - \alpha_1 = \frac{\beta_1}{F} .$$

The relation (3.9) gives

$$\tan \Delta\psi = \frac{R_{12}}{R_{11}\beta_1 - R_{12}\alpha_1} = 0$$

and so $\Delta\psi = 0$ because the integral

$$\Delta\psi = \int \frac{ds}{\beta(s)} = 0$$

since the thin lens has a length equal to zero. The transformation of an ellipse through a focusing thin lens is illustrated in Fig. 14.

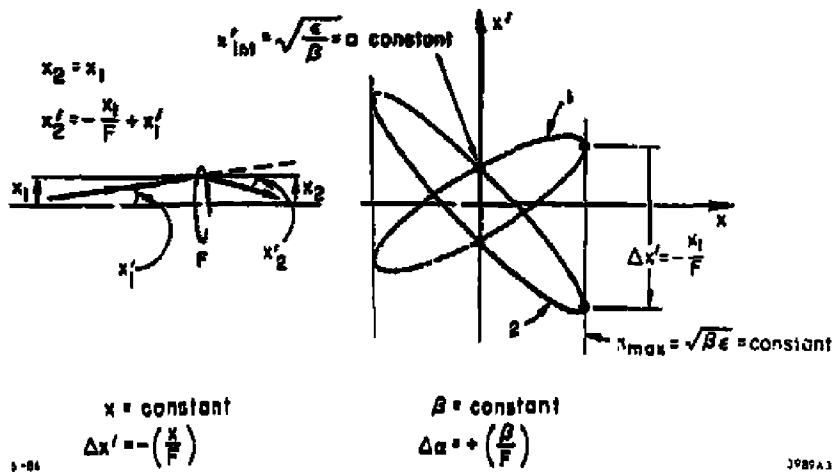


Fig. 14. The transformation of an ellipse through a focusing *thin* lens.

*4.0.3 A Quadrupo'a*

The thin lens quadrupole behaves in each phase plane $(x, x')$ and $(y, y')$ like a thin lens of opposite signs. If the lens is focusing in the $x$-plane, the matrices can be written as follows:

$$R = \begin{pmatrix} 1 & 0 \\ \mp 1/F & 1 \end{pmatrix} .$$

We have assumed here that the quantity $F$ is positive.

The phase advance is zero in both planes, and $\beta$ is constant in both planes. The change in $\alpha$ is given by

$$\Delta\alpha = \pm\frac{\beta}{F} .$$

In these expressions the upper sign applies to the $(x, x')$ focusing plane and the lower sign to the $(y, y')$ defocusing plane.

#### 4.0.4 *A thin Dipole*

A wedge dipole with the field index $n$ equal to 0 (i.e. a uniform field) can be simulated as a thin element (having zero length), located at its middle, and having the following transfer matrix:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ -\sin\alpha/\rho & 1 & \sin\alpha \\ 0 & 0 & 1 \end{pmatrix}$$

where $\alpha$ is the deflection angle of the central trajectory and where the third row and column describe the part of the transformation associated with the energy-dependent parameter $\delta = (\Delta p/p)$. The wedge dipole behaves like a thin lens of focal length $F = \rho/\sin\alpha$ in the $(x, x')$ plane. In the $(y, y')$ plane the wedge dipole behaves like a drift ,or a sharp cutoff field boundary. The matrix $R$ gives us

$$x_2 = x_1 = \text{a constant} \qquad \text{and} \qquad \Delta x' = x_2' - x_1' = -\frac{x_1 \sin\alpha}{\rho} + \delta \sin\alpha \,.$$

The formula (3.8) becomes

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \sin\alpha/\rho & 1 & 0 \\ \sin^2\alpha/\rho^2 & 2\sin\alpha/\rho & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

which gives

$$\beta_2 = \beta_1 = \text{a constant} \qquad \text{and} \qquad \Delta\alpha = \alpha_2 - \alpha_1 = \frac{\beta_1 \sin\alpha}{\rho} \,.$$

As for the thin lens, the relation (3.10) shows that $\Delta\psi = 0$ for the zero length dipole. The transformation of an ellipse through a wedge dipole magnet is illustrated in Fig. 15.

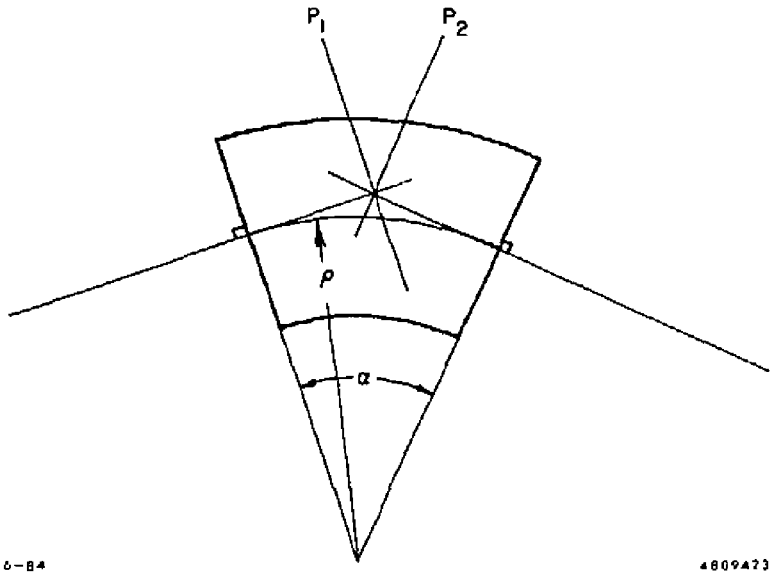


Fig. 15. A wedge dipole magnet from input to output principal planes.

## 4.1 STUDY OF SIMPLE USEFUL COMPOSITE MODULES

Using the basic elements discussed in the previous section we shall now explore some typical composite modules.

### *4.1.1 Basic Focusing Module*

If a focusing thin lens of focal length $F$ is placed between two drifts of length $F$, the transfer matrix for the composite system is

$$R = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & F \\ -1/F & 0 \end{pmatrix} .$$

From the matrix $R$ we observe that angles are transformed to displacements and displacements to angles as follows:

$$x_2 = F x_1' \qquad \text{and} \qquad x_2' = -\frac{x_1}{F} .$$

From the relation (3.8) we have

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & F^2 \\ 0 & -1 & 0 \\ 1/F^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

from which

$$\beta_2 = F^2\gamma_1 \qquad \text{and} \qquad \alpha_2 = -\alpha_1 \ .$$

Relations (3.9) and (3.10) yield

$$\tan \Delta\psi = -\frac{1}{\alpha_1} \qquad \text{and} \qquad \sin \Delta\psi = \frac{F}{\sqrt{\beta_1\beta_2}}$$

from which we can conclude the following result: If $\alpha_1 = \alpha_2 = 0$ then, since $\sin \Delta\psi > 0$, we must have $\Delta\psi = \pi/2$ and $F = \sqrt{\beta_1\beta_2}$.

This relation links the lens focal length $F$ and the length $L = 2F$ of the module to the magnitude of the $\beta$ values.

Practical two-dimensional modules based on this concept are typically achieved by symmetric triplets or by quadruplets, as shown in Fig. 16.

For the triplet, the focal length is different in the two phase planes $(x, x')$ and $(y, y')$ because of basic properties of triplets.

If it is required that $F_x = F_y$, then a symmetric quadruplet array of quadrupoles may be used as illustrated in Fig. 16.

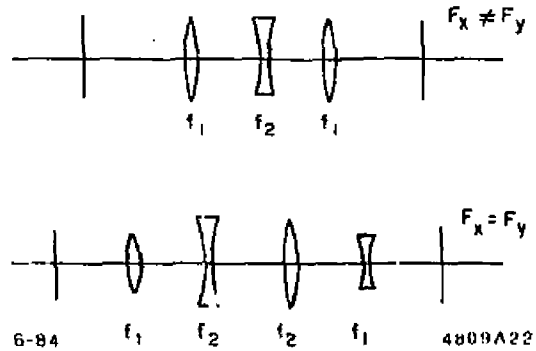


Fig. 16. A triplet and a quadruplet lens system possessing parallel to point and point to parallel imaging in both planes.

### 4.1.2 The *FODO Array*

The FODO array is perhaps the most common building block used in the design of machine lattices and beam lines. Its structure is illustrated in Fig. 17.

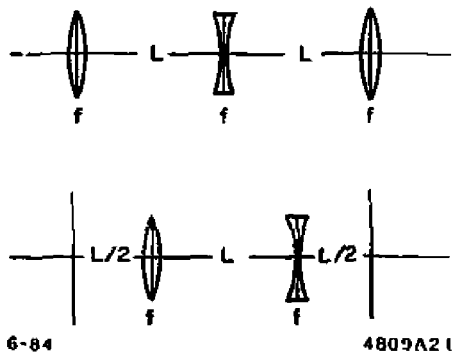


Fig. 17. A FODO array as a building block for lattices. 1) The transformation for one cell between the centers of the lenses. 2) The transformation for one cell between the centers of the drift regions.

It is informative to study the FODO array at two different observation points in order to better understand its basic properties.

1) First case: The cell begins and ends at the center of a lens, then the transfer matrix for the $x$ and $y$ planes is obtained by the following multiplication:

$$R = \begin{pmatrix} 1 & 0 \\ \mp 1/2f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mp 1/2f & 1 \end{pmatrix}$$

where again the upper sign applies to the $(x, x')$ plane and the lower sign to the $(y, y')$ plane.

If we assume that $\beta_1 = \beta_2 = \beta$ and $\alpha_1 = \alpha_2 = \alpha$, then

$$R = \begin{pmatrix} c + \alpha s & \beta s \\ -\gamma s & c - \alpha s \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{L^2}{2f^2}\right) & 2L\left(1 \pm \frac{L}{2f}\right) \\ -\frac{L}{2f^2}\left(1 \mp \frac{L}{2f}\right) & \left(1 - \frac{L^2}{2f^2}\right) \end{pmatrix}$$

from which

$$\cos\mu = c = \left(1 - \frac{L^2}{2f^2}\right) ,$$

$$\sin\left(\frac{\mu}{2}\right) = \sqrt{\frac{(1-\cos\mu)}{2}} = \frac{L}{2f} ,$$

$$\beta_{x,y} = 2L\frac{1 \pm \sin(\mu/2)}{\sin\mu} ,$$

and

$$\alpha_{x,y} = 0 ,$$

and, using symmetry arguments, the ratio of the beta functions in the focusing and defocusing lenses is given by

$$\frac{\beta_{\max}}{\beta_{\min}} = \frac{1+\sin(\mu/2)}{1-\sin(\mu/2)} .$$

Note that this ratio is independent of the length of the cell.

2) Second case: If we now begin the FODO array in the middle of one of its drifts, the transfer matrix for one cell is given by

$$R = \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \pm 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mp 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} ;$$

then

$$R = \begin{pmatrix} c+\alpha s & \beta s \\ -\gamma s & c-\alpha s \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{L^2}{2f^2}\right) \mp \frac{L}{f} & 2L - \frac{L^3}{4f^2} \\ -\frac{L}{f^2} & \left(1 - \frac{L^2}{2f^2}\right) \pm \frac{L}{f} \end{pmatrix}$$

from which we obtain

$$\cos\mu = \left(1 - \frac{L^2}{2f^2}\right) ,$$

$$\sin\left(\frac{\mu}{2}\right) = \frac{L}{2f} ,$$

which is the same as in case 1, but

$$\beta_{x,y} = \frac{L}{\sin\mu}\left(2 - \sin^2(\mu/2)\right)$$

and

$$\alpha_{x,y} = \mp \frac{2\sin(\mu/2)}{\sin\mu} .$$

The last two relations show that at this location we have the interesting relations

$$\beta_x = \beta_y \qquad \text{and} \qquad \alpha_x = -\alpha_y ,$$

which is the same property possessed by a thin lens quadrupole.

A particular case of interest is obtained when $\mu = \pi/2$. This corresponds to $(L/f) = \sqrt{2}$. This FODO cell is then often referred to as a 'quarter-wave' or $\lambda/4$ transformer and is shown schematically in Fig. 18.

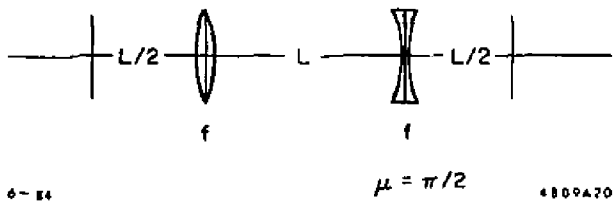


Fig. 18. The $\lambda/4$ transformer.

The transfer matrix $R$ of this quarter-wave transformer is

$$R_{x,y} = \begin{pmatrix} \mp\sqrt{2} & 3L/2 \\ -2/L & \pm\sqrt{2} \end{pmatrix}$$

and we have the interesting property $R_{11} = -R_{22}$ and $R_{11}$, and $R_{22}$ both change signs between the $x$ and $y$ planes.

We shall use all of the above properties later when we discuss the problem of matching between two dissimilar FODO arrays.

### 4.1.3 *A Telescopic System*

The optical system illustrated in Fig. 19 is called telescopic.

Its transfer matrix is given by

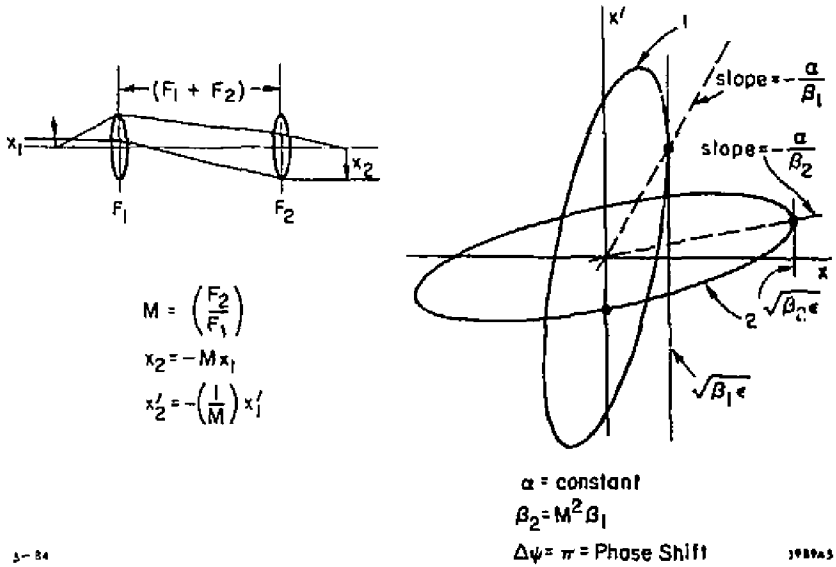


Fig. 19. A one-dimensional telescopic system.

$$R = \begin{pmatrix} 1 & F_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/F_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & F_1 + F_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/F_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & F_1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -F_2/F_1 & 0 \\ 0 & -F_1/F_2 \end{pmatrix} = \begin{pmatrix} -M & 0 \\ 0 & -1/M \end{pmatrix} . \tag{4.1}$$

From the $R$ matrix we obtain

$$x_2 = -Mx_1 \qquad \text{and} \qquad x_2' = -\frac{x_1'}{M} .$$

The relation (3.8) becomes:

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} M^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/M^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

which shows that

$$\beta_2 = M^2\beta_1 \qquad \text{and} \qquad \alpha_2 = \alpha_1 = \text{a constant} .$$

Since $R_{12} = 0$, the relations (3.9) and (3.10) reduce to

$$\tan \Delta\psi = 0 \qquad \text{and} \qquad \sin \Delta\psi = 0 .$$

Using the formula (3.7) rewritten as

$$R = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}} \cos \Delta\psi & 0 \\ 0 & \sqrt{\frac{\beta_1}{\beta_2}} \cos \Delta\psi \end{pmatrix}$$

we deduce that $\cos \Delta\psi < 0$, and consequently that $\Delta\psi = \pi$.

A telescopic system has an optical magnification $M$ given by

$$M = \frac{F_2}{F_1} .$$

It also has the property that the transfer matrix $R$ is an invariant if a drift length situated to the right of the lenses is transported to the front with the multiplication factor $M^2$. To prove and illustrate this property, consider the telescopic system having the transfer matrix of Eq. (4.1) and let it be preceded by a drift of length $l_1$ and followed by a drift of length $l_2$. The total matrix is

$$R_T = \begin{pmatrix} 1 & l_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -M & 0 \\ 0 & -(1/M) \end{pmatrix} \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -M & -Ml_1 - l_2/M \\ 0 & -(1/M) \end{pmatrix} .$$

The matrix $R_T$ is equal to the matrix of the original telescopic system if and only if the following condition holds:

$$Ml_1 + l_2/M = 0$$

or equivalently

$$l_2 = -M^2 l_1 .$$

In practice, to achieve a telescope in both planes one needs at least two quadrupoles to simulate each lens of the telescope. Figure 20 shows such a solution.

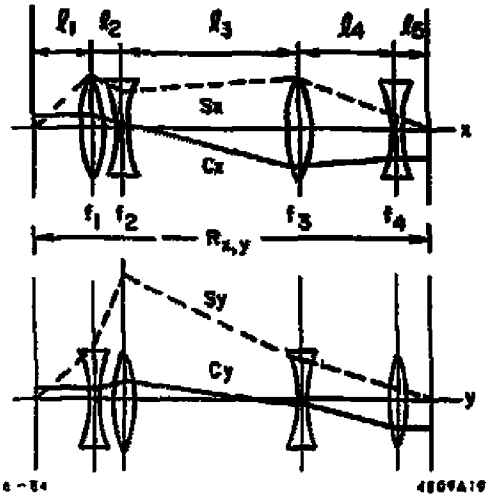


Fig. 20. A two-dimensional telescopic system.

The magnification may be different in each plane; therefore, the general $1 \times 4$ transfer matrix of the system becomes

$$R = \begin{pmatrix} -M_x & 0 & 0 & 0 \\ 0 & -1/M_x & 0 & 0 \\ 0 & 0 & -M_y & 0 \\ 0 & 0 & 0 & -1/M_y \end{pmatrix}$$

or

$$R = \begin{pmatrix} -\sqrt{\frac{\beta_{2x}}{\beta_{1x}}} & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{\beta_{1x}}{\beta_{2x}}} & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{\beta_{2y}}{\beta_{1y}}} & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{\beta_{1y}}{\beta_{2y}}} \end{pmatrix}.$$

### 4.1.4 *Repetitive Cells*

Consider the following transfer matrix:

$$R_c = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} .$$

If the cell characterized by this transfer matrix is repeated $n$ times, the total transfer matrix becomes

$$R_c^n = \begin{pmatrix} \cos(n\mu) + \alpha\sin(n\mu) & \beta\sin(n\mu) \\ -\gamma\sin(n\mu) & \cos(n\mu) - \alpha\sin(n\mu) \end{pmatrix}$$

such that after each successive cell

$$\beta_1 = \beta_2 = \cdots = \beta_n = \beta$$

and

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha .$$

This system is nonmagnifying in $\beta$.

### 4.1.5 *Repetitive Magnifying Cells*

How can one obtain a set of magnifying cells that would have the properties

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n$$

and

$$\sqrt{\frac{\beta_2}{\beta_1}} = \sqrt{\frac{\beta_3}{\beta_2}} = \cdots = \sqrt{\frac{\beta_n}{\beta_{n-1}}} = r ,$$

in other words each cell has a transverse magnification equal to $r$? Since phase space areas must be preserved, the transverse slopes will be subjected to a magnification of $1/r$. Consider then the following matrices $R_A$ and $M$:

$$R_A = \begin{pmatrix} \cos\mu + \alpha_1\sin\mu & \beta_1\sin\mu \\ -\gamma_1\sin\mu & \cos\mu - \alpha_1\sin\mu \end{pmatrix} , \qquad M = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} .$$

If the first cell in such a sequence has the matrix

$$R_1 = MR_A = \begin{pmatrix} r(c + \alpha_1 s) & r\beta_1 s \\ -\gamma_1 s/r & (c - \alpha_1 s)/r \end{pmatrix}$$

and the successive cells are defined by a similarity transformation

$$R_m = M R_{m-1} M^{-1} = M^m R_A M^{-(m-1)}$$

then the total matrix for a sequence of $n$ cells becomes

$$R_T = R_n \cdots R_1 = M^n R_A^n$$

or

$$R_T = \begin{pmatrix} r^n(\cos n\mu + \alpha_1 \sin n\mu) & r^n \beta_1 \sin n\mu \\ -\dfrac{\gamma_1 \sin n\mu}{r^n} & \dfrac{\cos n\mu - \alpha_1 \sin n\mu}{r^n} \end{pmatrix}$$

In the particular case where $\mu$ is chosen so that $n\mu = \pi$, the total matrix $R_T$ becomes

$$R_T = \begin{pmatrix} -r^n & 0 \\ 0 & -1/r^n \end{pmatrix} = \begin{pmatrix} -\sqrt{\beta_n/\beta_1} & 0 \\ 0 & -\sqrt{\beta_1/\beta_n} \end{pmatrix}$$

and the set of $n$ magnifying cells becomes a telescopic system with an optical magnification of $r^n$.

## 4.2 CELL MATCHING

In a previous paragraph we derived the conditions under which a beam contained in an ellipse is matched to a cell. It was also indicated in paragraph 3.5 that when the matching conditions were satisfied between a beam and a set of repetitive cells the aperture required to contain the beam is minimized.

All lattices, be they beamlines or segments of circular machines, are made by the juxtaposition of a series of cells having different transfer properties. One important problem facing the designer can be expressed in the following way:

Consider a section $S_2$ which is to follow a section $S_1$. Suppose the beam is matched to the section $S_1$. Generally this beam will not be matched to section $S_2$. Is it possible to design an intermediate section $S_{12}$ so that the beam is matched from $S_1$ to $S_2$? The problem of finding such a section $S_{12}$ is called the section matching problem.

Many design programs help the designer in solving this problem in its generality. It is, however, important to have some rational guidelines on how this matching can be achieved. The following paragraphs indicate two general methods for matching one FODO array to another FODO array.

### 4.2.1 General Considerations on FODO Cell Matching

Consider the matched symmetric FODO cell that was described in paragraph 4.1.2. If we choose the beginning of the cell to be halfway between the two quadrupoles, the following conditions hold at this point in every cell:

$$\beta_x = \beta_y \qquad \text{and} \qquad \alpha_x = -\alpha_y \ .$$

Consider now two sets of FODO cells characterized by the two sets of relations

$$\beta_{1x} = \beta_{1y} \qquad \text{and} \qquad \alpha_{1x} = -\alpha_{1y} \ ,$$

$$\beta_{2x} = \beta_{2y} \qquad \text{and} \qquad \alpha_{2x} = -\alpha_{2y} \ .$$

What properties should a matching section have in order to transform the values $\beta_1$, $\alpha_1$, $\gamma_1$ into the values $\beta_2$, $\alpha_2$, $\gamma_2$? If the transfer matrix of the matching section for the $x, x'$ plane is

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

then the following relation exists:

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} R_{11}^2 & \underline{-2R_{11}R_{12}} & R_{12}^2 \\ \underline{-R_{11}R_{21}} & R_{11}R_{22} + R_{12}R_{21} & \underline{-R_{12}R_{22}} \\ R_{21}^2 & \underline{-2R_{21}R_{22}} & R_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} \ . \qquad (4.2)$$

Let us note the following:

If at the input of the matching cell we have

$$\beta_{1x} = \beta_{1y} \qquad \text{and} \qquad \alpha_{1x} = -\alpha_{1y} \qquad (4.3)$$

and if the transfer matrix $R$ of the matching cell is such that the underlined elements in Eq. (4.2) change sign from the $(x, x')$ plane to the $(y, y')$ plane and the other elements do not change sign, then it follows from the Twiss transformation that:

$$\beta_{2x} = \beta_{2y} \qquad \text{and} \qquad \alpha_{2x} = -\alpha_{2y} \ .$$

When such a situation is created, then the values of a FODO cell are matched to the values of another FODO cell. This, however, does not mean that the above procedure matches any FODO cell to another arbitrarily chosen FODO cell. The following procedure will exemplify and extend the preceding one.

The first condition can be realized generally in two ways: either the matrix $R$ is such that

$$R = \begin{pmatrix} \underline{R_{11}} & R_{12} \\ R_{21} & \underline{R_{22}} \end{pmatrix}$$

or it is such that

$$R = \begin{pmatrix} R_{11} & \underline{R_{12}} \\ \underline{R_{21}} & R_{22} \end{pmatrix}$$

where we have underlined the elements that must change sign as one switches from plane $(x, x')$ to plane $(y, y')$. An example of a practical matching system is the following.

### 4.2.2 *Beam Matching with a Quarter-Wave Transformer*

Consider the quarter-wave transformer defined in the FODO array section of paragraph 4.1.2 and illustrated in Fig. 21.

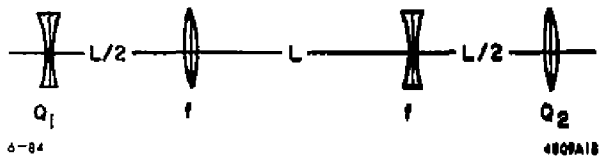


Fig. 21. A quarter-wave matching transformer.

The matrix element of this cell can be written as

$$R = \begin{pmatrix} \underline{a} & b \\ -c & \underline{-a} \end{pmatrix}$$

where, according to our convention, the underlined elements change sign when switching from the $(x, x')$ plane to the $(y, y')$ plane.

The transformation of this cell satisfies the condition of the previous paragraph, and this cell will match pairs of FODO cells whose parameters both satisfy the relation

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} a^2 & \underline{-2ab} & b^2 \\ \underline{ac} & -a^2 - bc & \underline{ba} \\ c^2 & \underline{-2ac} & a^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} .$$

Using the $\lambda/4$ cell, which matches specific pairs of FODO cells, one can obtain, by the addition of two elements, a cell which will match any two pairs of FODO cells (with some constraint on the range of $\beta_2$).

Consider a quarter-wave transformer to which we add a quadrupole $Q_1$ at its entrance and another quadrupole $Q_2$ at its exit.

The insertion of quadrupole $Q_2$ does not change the exit value $\beta_2$ but will change the value $\alpha_2$ of the planes $(x, x')$ and $(y, y')$ in opposite directions and so preserves the condition $\alpha_{2x} = -\alpha_{2y}$.

The insertion of quadrupole $Q_1$ at the entrance does not change the value $\beta_1$ or the relation $\alpha_{1x} = -\alpha_{1y}$ but it does change the absolute values of $\alpha_{1x}$ and $\alpha_{1y}$. The Twiss transformation, Eq. (3.8), for the quarter-wave transformer shows that this variation of $Q_1$ will change the values of both $\beta_2$ and $\alpha_2$ while preserving the conditions $\beta_{1x} = \beta_{1y}$ and $\alpha_{1x} = -\alpha_{1y}$.

Using the transformation matrix of the quarter-wave transformer and considering $\alpha_1$ to be variable (via variation of the strength of $Q_1$), one can show that the value $\beta_2$ that can be matched by the preceding cell has a minimum value equal to $b^2/\beta_1$, as follows:

The expression for $\beta_2$ is

$$\beta_2 = a^2\beta_1 - 2ab\alpha_1 + b^2\gamma_1$$

$$= a^2\beta_1 - 2ab\alpha_1 + \frac{b^2(1+\alpha_1^2)}{\beta_1} .$$

The first and second derivatives with respect to $\alpha_1$ are

$$\frac{d\beta_2}{d\alpha_1} = -2ab + \frac{2b^2\alpha_1}{\beta_1}$$

and

$$\frac{d^2\beta_2}{d\alpha_1^2} = \frac{2b^2}{\beta_1} > 0 .$$

Therefore, a minimum will be achieved if

$$\alpha_1 = \frac{a}{b}\beta_1$$

and the value of this minimum is

$$\beta_{2\min} = \frac{b^2}{\beta_1} .$$

The procedure of adjustment of the matching cell then becomes:

The quadrupole $Q_1$ is adjusted so that, given the input values $\beta_1$, $\alpha_1$, the required output value $\beta_2$ is achieved at the exit. Quadrupole $Q_2$ is then adjusted to obtain the required $\alpha_2$, and the match is accomplished.

### 4.2.3 Beam Matching with Half-Wave Transformers

Telescopic systems which have a phase shift of $\pi$ may also be used as matching transformers with the restriction that $\alpha_2 = \alpha_1$ and $\beta_2 = M^2\beta_1$, where $M$ is the optical magnification of the transformer. Their most obvious application is to match between two points where $\alpha_1 = \alpha_2 = 0$ (the location of an erect phase ellipse). They have the advantage that $M_x$ does not have to equal $M_y$. They also have the property of minimizing the higher-order optical distortions because of their optical symmetry. Half-wave matching transformness are illustrated schematically in Fig. 22.

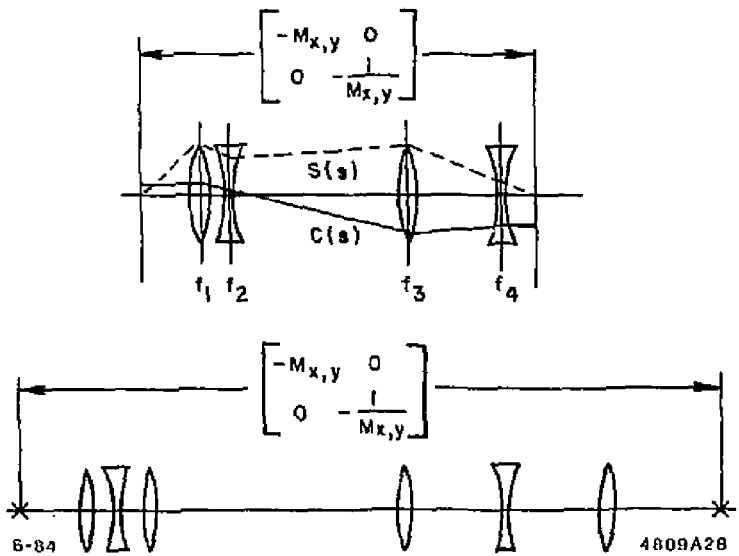


Fig. 22. A half-wave matching transformer: 1) using doublets; 2) using triplets.

# 5 SECOND-ORDER OPTICS

In Chapter 2 we introduced a general notation for the coefficients of the Taylor expansion of the solution of the equations of motion. The notation of the first-order terms was simplified in order to conform with the standard matrix notation. For example,

$$R_{11} = (x|x_0) , \qquad R_{21} = (x'|x_0) , \qquad R_{34} = (y|y'_0) .$$

In order to ease the writing, a similar simplification of notation is introduced for the second-order terms: the tensor $T_{ijk}$ can be defined in a similar way. For example,

$$T_{112} = (x|x_0x'_0) , \qquad T_{246} = (x'|y'_0\delta) .$$

The following discussion will frequently use the adjectives geometric and chromatic to describe the optical properties of beam lines.

All terms for which no subscript is equal to 6 will be referred to as geometric terms or alternatively as geometric aberrations because they depend only upon the central momentum $p_0$.

Any term $R_{ij}$ or $T_{ijk}$ where one subscript is equal to 6 will be referred to as a chromatic term (or chromatic aberration) by virtue of the fact that its effect depends on the momentum deviation $\delta = \Delta p/p_0$ of the particle.

## 5.1 SECOND-ORDER PROPERTIES OF BASIC ELEMENTS

In Chapter 2, table I gives the driving terms which generate the various second-order coefficients for a general magnetic element . From this table it is possible to deduce the contributions to second-order terms from the following basic elements: dipole, quadrupole, and sextupole.

For a pure dipole $n = 0$ and $\beta = 0$ with $h(s) = (1/\rho_0) \neq 0$.

For a pure quadrupole let $\beta = 0$ and take the $\lim(h) = 0$ and $\lim(nh^2) = -k_1^2 = -K_1$.

For a pure sextupole let $h = 0$, $nh^2 = 0$ and take the $\lim(\beta h^3) = K_2$.

A careful examination of Tables I, II, and III will show that the following statements are correct.

Dipoles introduce both second-order geometric and second-order chromatic aberrations.

Quadrupoles do not introduce second-order geometric aberrations.

Quadrupoles do introduce chromatic aberrations.

Sextupoles introduce both second-order geometric and chromatic aberrations.

The chromatic aberrations of a quadrupole can be interpreted simply in the following way:

The magnetic induction of a quadrupole is a linear function of either variable $x$ or $y$. In other words the gradient of the induction is constant within a quadrupole. A particle with momentum $p$ will be affected differently than a particle with momentum $p_0$. The corresponding strengths of the quadrupole $K_1(p)$ and $K_1(p_0)$ satisfy the relation

$$\frac{K_1(p)}{K_1(p_0)} = \frac{p_0}{p} .$$

In other words, the focal strength of a quadrupole decreases as the momentum of the particle increases.

The chromatic properties of a sextupole may be interpreted in a similar fashion.

The geometric properties of a sextupole may be understood in the following simplified way:

The magnetic field of a sextupole varies quadratically with the variable $x$. Or the gradient of the field varies linearly with $x$. In the immediate neighborhood of $x$ the sextupole can be considered to be a quadrupole whose gradient varies linearly with the variable $x$. This intuitive view of the sextupolar field will be helpful in the understanding of the underlying principle of chromatic corrections in beam lines to be discussed later.

## 5.2 Objectives in Second-Order Optics Studies

Second-order optics studies come as a complement to the first-order studies and serve to detect and correct deviations from the results obtained in first-order studies. These deviations, generally called aberrations, are of two types: chromatic and geometric. In some cases, as for example in resonant extraction, the second-order properties of a lattice are an essential feature.

## 5.3 Chromatic Corrections

As described in a previous section, chromatic effects occur because particles with different momenta respond differently to a given magnetic field. Consider a lattice made up entirely of quadrupoles, as illustrated schematically in Fig. 23.


**Fig. 23. Schematic representation of a lattice with quadrupoles.**

A particle with nominal momentum $p_0$ and initial coordinates $x_i = 0$ follows the axis of the system. The linear motion of neighboring particles of the same momentum is described by the functions $\psi(s)$ and $\beta(s)$. These two functions determine the amplitude of oscillation and the phase advance of individual particles. A particle with the same input coordinates but a different momentum $p_1$ (its sixth coordinate $\delta \neq 0$) will follow the same central axis in the quadrupoles, and the motion of neighboring particles having this new momentum $p_1$ is described by the functions $\psi(s,\delta)$ and $\beta(s,\delta)$. The difference in the values of the functions is the result of the fact that the particles with momentum $p_1$ see the quadrupoles with strengths different from the strengths experienced by the particles with momentum $p_0$.

To compensate for this chromatic difference a lattice may be designed where particles of greater momentum encounter an extra *quadrupolar* field to compensate for the increased momentum. This is achieved by the introduction of dipoles and sextupoles into the lattice structure.

Figure 24 shows a lattice made up of quadrupoles and dipoles which has the potential for chromatic corrections because particles of different momenta follow different trajectories.

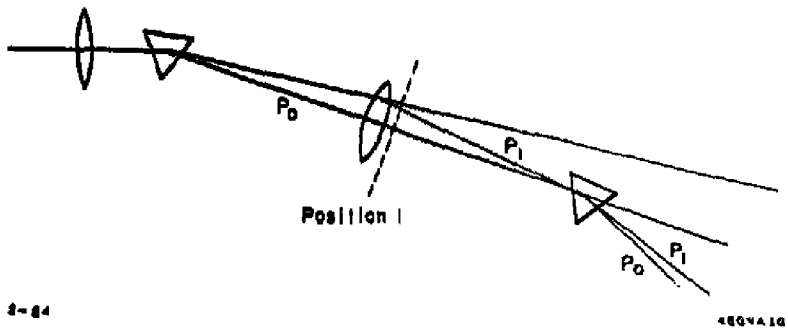


**Fig. 24. A chromatic correction lattice.**

A particle $P_0$ with zero initial coordinates and nominal momentum follows the central trajectory (axis of the figure). The particle $P_1$ with zero initial coordinates but with $\delta \neq 0$ will follow the trajectory defined by the function $\delta d_x(s)$.

This function is nonzero after the first dipole of the lattice. Consider position 1 in the lattice. At this point particle $P_1$ will have encountered quadrupolar strengths that are slightly different from the strengths encountered by particle $P_0$. Let us introduce at point $P_1$ a sextupolar element with its axis coincident to the reference axis of the lattice. Particle $P_0$ and neighboring particles will not experience any first-order disturbance of their motion from the sextupole (a sextupole field has zero gradient on its axis). However, particle $P_1$ will experience a gradient that is proportional to its displacement and therefore is proportional to the quantity $\delta$. If the strength of the sextupole is appropriately chosen, the extra gradient will exactly compensate the difference in gradient experienced by particles with different momenta in the preceding quadrupoles. By progressing along the lattice it seems feasible to set up a family of sextupoles that would exactly compensate for the chromatic aberrations arising from the quadrupolar fields.

However, in this process, the sextupoles will in general introduce geometric distortions. In the next section we describe a simple procedure that eliminates the chromatic aberrations to second order without introducing second-order geometric aberrations.

### 5.3.1 *Module for Sextupolar Chromatic Correction*

Consider two FODO cells set up as in paragraph 4.1.2 and tuned so that $\mu_{x,y} = 90$ degrees for each cell. Such a setup is often referred to as a $-I$ telescopic transformer because its transfer matrix in both the $x$ and $y$ transverse planes is

$$R_{x,y} = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} .$$

In Fig. 25 is a schematic representation of such a $-I$ transformer. Let 1 and 2 denote the entrance and exit positions.

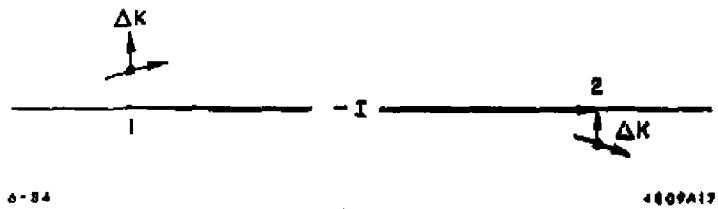


Fig. 25. Principle of a $-I$ transformer.

A particle at position 1 with coordinates $x_1, x_1'$ will emerge at position 2 with coordinates $x_2, x_2'$ given by

$$x_2 = -x_1 \qquad \text{and} \qquad x_2' = -x_1' .$$

Imagine now that we place at position 1 a thin magnetic element that produces

an angle kick to the particle, say $\Delta K$. The particle of momentum $p_0$ will now arrive at position 2 with the coordinates

$$x_2 = -x_1 \qquad \text{and} \qquad x_2' = -x_1' - \Delta K \ .$$

If we now submit the particle to another angle kick equal to $\Delta K$ at position 2, we see that the exit coordinates are the same as they were without kicks. In conclusion, when particles are submitted to equal angle kicks at the entrance and exit points of a $-I$ transformer, there is no visible effect on their behavior outside the $-I$ transformer for monoenergetic particles having momentum $p_0$.

Let us apply this principle, using some of our elementary building blocks.

1) Dipoles: Dipoles are even-order elements in the sense that the angle kick they deliver to a particle is an even function of the lateral displacement (in this case a constant function). Thus, if we place two identical dipole magnets (one at the entrance and one at the exit) of a $-I$ transformer, there will be no net angular deflection experienced by particles of momentum $p_0$ outside of the $-I$ transformer.

2) Quadrupoles: The angular displacement produced by a quadrupole is an odd function of the lateral position $x$. (In this case the angle kick is proportional to $x$.) Consequently two identical quadrupoles of opposite polarity placed at the entrance and exit of a $-I$ transformer will have no net effect outside the transformer.

3) Sextupoles: Sextupoles are even-order elements . The angular kick they produce is proportional to $x^2$. In this instance pairs of equal strength sextupoles will have no net effect outside the $-I$ transformer.

Thus, in summary, all odd-order elements (quadrupoles, octupoles, etc.) will have to be introduced in pairs of opposite polarity, and all even-order elements (dipoles, sextupoles, etc.) have to be introduced in pairs with the same polarity in order for the cancellation to be effective.

### *5.3.2 A $-I$ Transform Sextupolar Chromatic Correction*

Consider now a $-I$ transformer with two sextupoles of equal strength placed at the entrance and exit, and suppose that dipoles have been inserted in each cell of the $-I$ transformer. From the previous discussion we know that the sextupoles will not introduce geometric aberrations. The presence of the dipoles between the sextupoles ensures that there will be coupling between the sextupole strengths and the chromatic behavior of particles. Having thus demonstrated the principle of the chromatic correction, let us analyze its feasibility in greater detail.

In practice one must do at least one chromatic correction per phase plane, and sometimes two or more per plane. The ideal situation, from the point of view of the second-order geometric aberrations, is to assemble enough $-I$ transformers so that the different sextupole pairs (placed $-I$ apart) do not interfere with each other. This condition is often prohibitive in its space requirement and in its cost. So let us analyze the effect of interlacing sextupole pairs used in chromatic corrections.

Consider, as shown in Fig. 26, two consecutive $-I$ transformers containing two interlaced pairs of sextupoles $S_1$ and $S_2$.

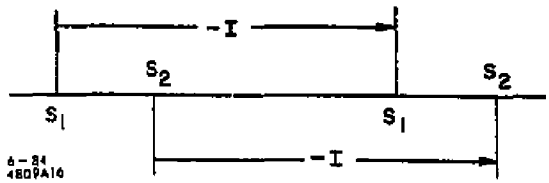


Fig. 26. Interlaced sextupole pairs.

If the sextupoles are pure second-order elements, no additional second-order aberrations are introduced by the coupling between the sextupoles of the two pairs.

Suppose a particle arrives at the first sextupole $S_1$ with displacement $x_1$. As it reaches the first sextupole of the pair $S_2$, its motion, within the $-I$ transformer that separates the pair $S_1$, is perturbed, and the particle will reach the second sextupole of the $S_1$ pair with a displacement that is not equal to $-x_1$. Consequently the second sextupole of the $S_1$ pair will not exactly compensate the geometrics introduced by the the first sextupole. However since the disturbance introduced by the sextupole $S_2$ is of order two, the uncorrected geometric aberration of the pair $S_1$ is of order three and four.

In a following paragraph we shall show a complete practical setup of a correction scheme using interlaced families of sextupoles.

## 5.4 Geometric Correction Using Repetitive Symmetry

The second-order aberrations are obtained by the computation of integrals containing the sinelike and cosinelike functions (see, for example, Tables II and III). The first-order condition that a lattice be stable implies that the sinelike and cosinelike functions oscillate in a manner similar to the circular functions. Symmetries introduced in the design of a lattice may have the desirable effect of canceling some second-order aberrations. Among the important symmetries

to be considered are mirror symmetry, rotational symmetry, $(x,y)$ symmetry, repetitive symmetry, and repetitive symmetry with magnification.

Let us look at a general approach to the study of the effect of some of these symmetries on the second-order aberrations.

Inspection of Tables I, II, and III shows that the second-order geometric aberration terms can be expressed as

$$T_{ijk} = \int_0^L K_p (R_{ij}(s))^n (R_{ik}(s))^m ds \qquad \text{where} \qquad (n+m) = 2$$

where $K_0$ is the dipole strength per unit length and $K_2$ is the sextupole strength per unit length. Pure quadrupoles do not generate second-order geometric aberrations.

Since the $R_{ij}(s)$ are linear combinations of $\sin \Delta\psi$ and $\cos \Delta\psi$, we can write

$$T_{ijk} = \int_0^L F_p \sin^n(\Delta\psi) \cos^m(\Delta\psi) ds$$

where the functions $F_p$ are equal to the strengths $K_p$ multiplied by some power of the $\beta(s)$ functions. Adopting a complex variable notation, we obtain the condition for having all second-order geometric terms $T_{ijk}$ vanish, namely,

$$\int_0^L F_p e^{\pm i\psi} ds = 0 \qquad \text{and} \qquad \int_0^L F_p e^{\pm 3i\psi} ds = 0 .$$

The integral of the expressions $F_p e^{\pm i\psi}$ and $F_p e^{\pm 3i\psi}$ for each separate element of a lattice can be represented geometrically as a vector in the complex plane, as shown in Fig. 27. The integrals over the total lattice become the vector sums of all the complex vectors representing the geometric aberrations of the individual elements, namely,

$$\sum_1^N F_k e^{i\psi_k} \qquad \text{and} \qquad \sum_1^N F_k e^{3i\psi_k} .$$

For reasons that should appear clear in the next paragraph, one generally places the vectors corresponding to $i\psi$ . one diagram and the vectors corresponding to $3i\psi$ in another. The second-order geometric aberrations are zero if both these sums are zero.

We shall now restrict our study to the case of repetitive symmetry.

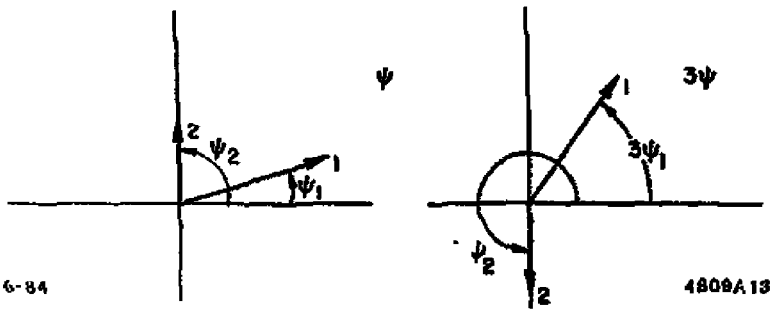


Fig. 27. Complex plane diagram for second-order aberrations.

### 5.4.1 *First-Order Repetitive Geometric Correction*

For repetitive symmetry (i.e. when the lattice is made of the juxtaposition of equal cells), the beta functions are equal from cell to cell and so are the element strengths.

In this case the functions $F_j(s)$ are equal in value at the same location from cell to cell. Let us analyze two special cases: a lattice containing four identical cells and a lattice containing three identical cells, and such that the total phase advance for the lattice is $2\pi$ in both cases.

Consider the $\psi$ plot of Fig. 28. The vectors correspond to the number of the cell to which they belong. In the $\psi$ plane they appear in consecutive order with an angle of 90 degrees. Their sum obviously is zero. In the $3\psi$ plane the angle between consecutive vectors becomes 270 degrees, and their sum will also be zero.

In conclusion, in a lattice made of four equal cells with total phase shift of $2\pi$, the second-order geometric aberrations originating in individual elements will cancel.

Consider now the $\psi$ plot of Fig. 29. The three vectors display an angle of 120 degrees, and so their sum is also zero. However, in the $3\psi$ plot they will have an angle of 360 degrees and will all coincide. Their sum is not zero unless their amplitude is zero.

In conclusion, for a lattice with three cells and a total phase shift of $2\pi$, some geometric aberrations do not cancel.

We can now formulate the following important theorem:

In a lattice made of $n$ identical cells with $n > 3$ and having a total phase shift of $2m\pi$, all second-order geometric aberrations will cancel.

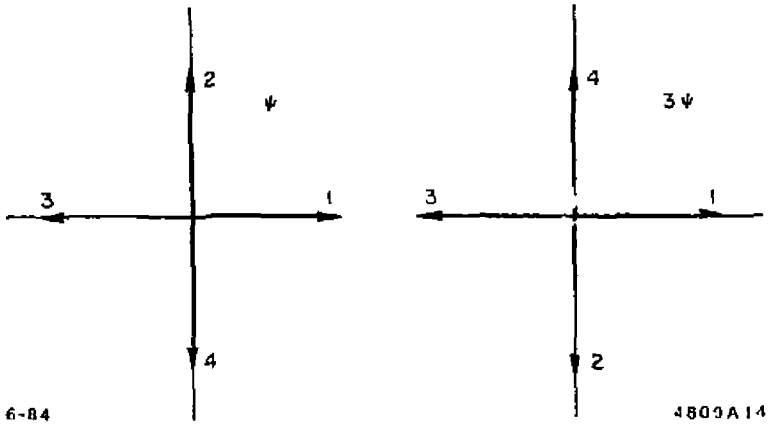


Fig. 28. Complex plane diagram for second-order aberrations in a four-cell lattice with repetitive symmetry and a $2\pi$ phase shift.

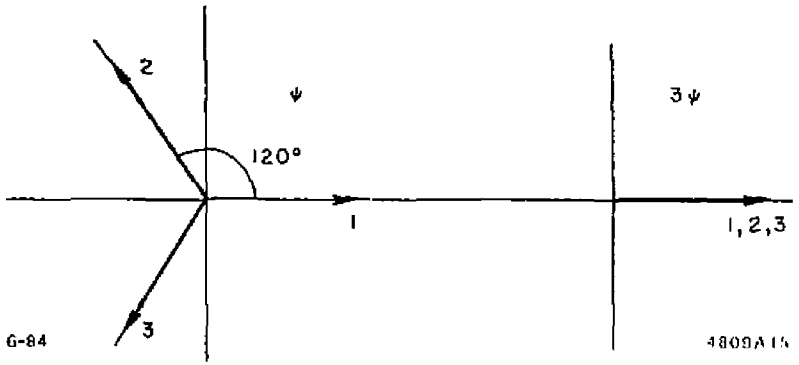


Fig. 29. Complex plane diagram for second-order aberrations in a three-cell lattice with repetitive symmetry and a $2\pi$ phase shift.

## 5.5 THE SECOND-ORDER ACHROMAT

Modern high energy machines require long beam lines to transport the beam from one region to another or to perform specific functions within a lattice. These beam lines are expected to transport the beam achromatically to as high an order as possible without introducing appreciable geometric aberrations. We call an achromat a line that would meet that goal perfectly. We qualify the name with nth order when the goal is met up to the order $n$ in the Taylor expansion of the motion.

In the following sections we restrict our analysis to repetitive cell structures.

### 5.5.1 *The First-Order Achromat*

Consider a lattice made of $n$ identical cells having the following transfer matrix:

$$R = \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} M & \vec{w} \\ 0 & 1 \end{pmatrix} .$$

The total transfer matrix $T$ will be

$$T = \begin{pmatrix} M^n & M^{n-1}\vec{w} + M^{n-2}\vec{w} + \cdots + \vec{w} \\ 0 & 1 \end{pmatrix} .$$

The dispersive vector of the total transfer matrix $T$ can be written in the following form:

$$\vec{d} = (M^{n-1} + M^{n-2} + \cdots + I)\vec{w} = (M^n - I)(M - I)^{-1}\vec{w} .$$

From the above expression one can deduce the following theorem:

A lattice made of $n$ identical cells is achromatic to first order if and only if

1) $M^n = I$

or

2) $\vec{w} = 0$ .

In other words, it is achromatic if and only if each cell is achromatic, or the total transfer matrix is the identity matrix (equivalently if the total phase advance is $2m\pi$ for any integer m).

This first-order result is the basis for the building of achromatic beam lines.

### 5.5.2 *A Practical Second-Order Achromat*

Figure 30 shows a possible layout for a four-cell second-order achromat. The labels BD stand for bending dipoles. The labels QF and QD stand for horizontally focusing quadrupoles and horizontally defocusing quadrupoles. We assume that the quadrupoles have been tuned to provide a total phase advance of $2\pi$.

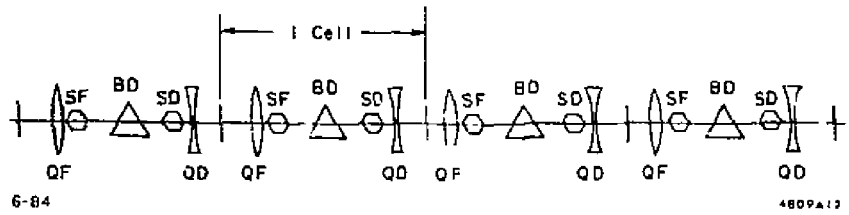


Fig. 30. Example of a practical second-order achromat with four cells.

Sextupoles have been introduced so that the chromatic correction procedure can be performed in both the $(x, x')$ and the $(y, y')$ plane.

The sextupoles of the family SF will couple predominantly with the $x$ plane because they are located close to the focusing quadrupoles, where the values of the $\beta_x$ function are greater.

Similarly the sextupoles of the family SD will couple predominantly to the $y$ motion, where $\beta_y$ is larger.

Once the quadrupoles have been tuned to provide a $2\pi$ phase shift, the second-order geometric aberrations introduced by the dipoles and by the sextupoles cancel exactly.

One then tunes the sextupoles SF and SD so that one of the second-order chromatic terms $T_{1j6}$ or $T_{2j6}$ and one of $T_{3j6}$ or $T_{4j6}$ are zero. It has been shown previously[8] that all the second-order chromatic terms except $T_{566}$ then become simultaneously zero.

We now have a system that is completely achromatic to second order with the only exception being the momentum dependence of the path length.

### 5.5.3 *Application of the Achromat Concept to Chromatic Corrections*

The second-order achromat as described above is an optical system whose transformation matrix is the identity matrix to a precision of second order in all of the phase space variables $x$, $x'$, $y$, $y'$, $l$, and $\delta$ except for the second order matrix element relating the path length to the square of the momentum.

While the second-order achromat may not be directly applicable to the design of circular machines, the optical principles evolved for its development are definitely useful when formulating the sextupole configurations necessary for the chromatic corrections in circular machines and in particular for [illegible] where the interaction regions have very small beta functions. Let us review the salient features of the second-order correction theory developed above that are applicable to this problem.

1) Any family of sextupoles inserted into a lattice such that their vector sums cancel in the $\psi$ and $3\psi$ diagrams described above will not introduce second-order geometric aberrations.

2) The interlacing of two or more sextupole families, each of which satisfies criterion 1), does not introduce second-order geometric aberrations.

3) Interlacing of one sextupole family with another sextupole family will introduce third- and higher-order distortions to the lattice.

4) It should be noted that in order for the sextupoles not to introduce second-order geometric distortions, the tune shift per cell of the lattice in the region of the sextupoles must remain fixed and must be equal in both the $x$ phase plane and the $y$ phase plane. The quadrupoles in this region must not be used to vary the tune of the machine. The variation in tune must be achieved in a 'sextupole-free' region.

It follows from the above that a simple recipe for the introduction of sextupole families to correct for chromatic effects and at the same time minimize the optical distortions at the interaction regions may be evolved by following the guide lines contained in the preceding paragraphs. This has been discussed in previously,[9] and has been implemented in the design of many of the new machines in the last decade. Some examples are the LEP machine at CERN, the EROS ring at Saskatoon, the SLC at SLAC, and the CEBAF ring at SURA. All of these machines have a lattice design that permits these important principles to be implemented.

The basic procedure is the following: The sextupole families are chosen according to the above rules, and their strengths are then adjusted to correct for the second-order chromaticity introduced by the quadrupoles in the lattice and to correct locally for the momentum dependence of the beta functions. If there are enough (at least three) independent families in each phase plane, the strengths of the families can be adjusted relative to each other so as to minimize the optical distortions caused by the cross coupling between families. This is not a trivial exercise, and special programs have been written to handle this particular problem. Examples are the program HARMON developed by Donald[14] and the program PATRICIA developed by Wiedemann.[15]

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