MASTER

PREPRINT UCRL-81957

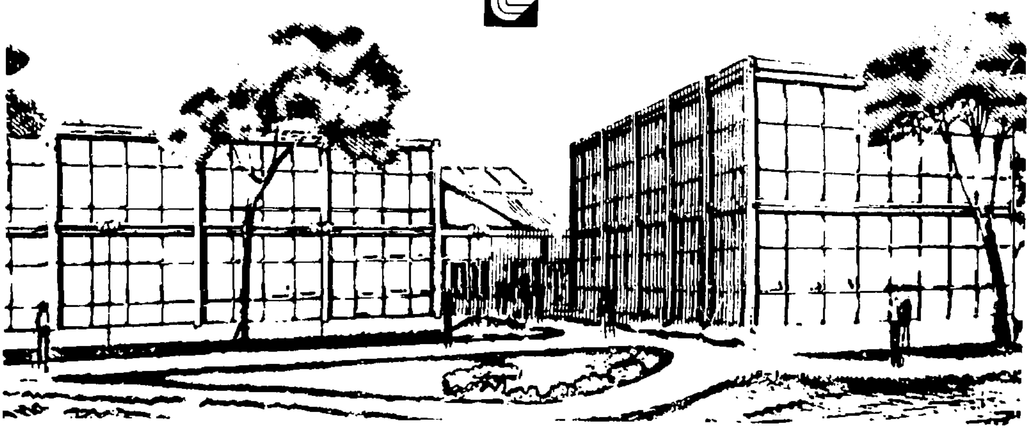
Lawrence Livermore Laboratory

The Effect of Boundary Conditions on a Non-Equilibrium Transient Marshak Wave Problem

C. T. Alonso, LLL

G. C. Pomraning, UCLA

This paper was presented at the Second Topical Conference on Atomic Processes in High Temperature Plasmas, Boulder, Colorado, January 17-19, 1979



The Effect of Boundary Conditions on a Non-Equilibrium Transient Marshak Wave Problem

C. T. Alonso, LLL

G. C. Pomraning, UCLA

January 15, 1979

ABSTRACT

Transient processes in radiative transfer have recently become of interest in the modeling of astrophysical phenomena, particularly with regard to the brightness of novae, supernovae, and perhaps even galactic clouds adjacent to quasars. We present here analytic solutions to a particular class of Marshak wave problem with and without the Marshak (Milne) boundary condition. We find that the choice of boundary condition can have a decisive effect on the coupling of radiative energy to the material energy in the vicinity of a material boundary. The analytic solution we have obtained can be useful as a tool for calibrating numerical calculation techniques.

"Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore Laboratory under contract number W-7405-ENG-48."

I. Introduction

Recently Pomraning[1] reported on analytic solution to the time-dependent non-equilibrium "Marshak" problem, in which an initially cold halfspace of material has radiation incident upon its surface. The problem was made analytically solvable by introducing a specific heat capacity in the material which was proportional to T^3, where T is the material temperature. This specialized heat capacity causes the equations to become linear in T^4, and classical Laplace transform techniques can then be used to solved the problem within the framework of the lowest-order (P1) spherical harmonic approximation.

We subsequently wondered how much the choice of boundary conditions at the material surface affected the solution. In particular we wished to explore how conservation of flux at the boundary (which results in the "Marshak" boundary condition) would affect the rate of energy transfer into the material, and also whether such a flux-conserving condition would lead to a discontinuity in temperature at the material surface, as it is known to do in the complementary case of a source radiating into a transparent medium.[2,3]

In order to assess the importance of a flux-conserving boundary condition we have resolved the problem posed by Pomraning[1] using two different boundary conditions. The first condition is the "equilibrium" condition that would be valid within an infinite medium at equilibrium. The second is the "Marshak" boundary condition which conserves flux across the boundary. Both of these boundary conditions are frequently used in the literature.

In this paper we calculate the total material and radiation energies in the material helfspace as a function of time, and compare these energies as computed with the two types of boundary conditions described above.

II. The Problem

We consider a semi-infinite purely absorbing medium occupying $0 \leq z \leq \infty$. The medium is assumed to be homogeneous and, at $t = 0$, to be at zero temperature with no radiation field present. At $t = 0$ a time-independent radiative flux impinges on the surface at $z = 0$. We wish to compute, as a function of space and time, the material temperature and the radiation field. Hydrodynamic motion and heat conduction will be assumed to be negligible.

The radiative transfer model to be used is the grey P1 diffusion description. The equation of transfer is then

$$\frac{\partial E_r(z,t)}{\partial t} - \frac{\partial}{\partial z}\left[\frac{c}{3K(T)}\frac{\partial E_r(z,t)}{\partial z}\right] = cK(T)\left[aT^4(z,t) - E_r(z,t)\right] \tag{1}$$

where z is the spatial variable, t is time, $T(z,t)$ is the material temperature, $K(T)$ is the absorption cross section (opacity), c is the speed of light, and a is the radiation constant, and $E_r(z,t)$ is the radiation energy density.

The material energy balance equation is

$$C_v(T)\,\frac{\partial T}{\partial t}(z,t) \;=\; cK(T)\left[E_r(z,t) - aT^4(z,t)\right] \tag{2}$$

where C_v, the heat capacity per unit volume, is related to the material energy density E_m by

$$E_m(T) = \int_0^T dT'\; C_v(T') \tag{3}$$

The initial conditions are that no radiation be present:

$$E_r\,(z,0) = 0 \tag{4}$$

and the material temperature be zero:

$$T(z,0) = 0 \tag{5}$$

Our choice of boundary conditions include the "equilibrium" condition

$$E_r(0,t) = \frac{4}{c}\,F_{inc} \tag{6}$$

and the "Marshak" condition

$$E_r(0,t) - \frac{2}{3K[T(0,t)]}\,\frac{\partial E_r(0,t)}{\partial z} = \frac{4}{c}\,F_{inc} \tag{7}$$

where F_{inc} is the flux incident upon the medium at $z = 0$. At $z = \infty$ we have the boundary condition

$$E_r(\infty,t) = 0. \tag{8}$$

Equations (1) through (8) define a nonlinear set of equations for the unknowns $E_r(z,t)$ and $T(z,t)$. In order to make the problem tractable, we assume K to be independent of temperature and we set

$$C_v = \alpha\, T^3 \tag{9}$$

so that the above equations become linear in E_r and T^4.

Before proceeding with the solutions using the two different boundary conditions, we recast the equations in dimensionless form. The incident flux is written in terms of an effective temperature Θ_{inc} as

$$F_{inc} = \sigma\, \Theta_{inc}^4 \tag{10}$$

where σ is the Stefan-Boltzmann constant, $\sigma = ac/4$. We also define a radiation temperature $\Theta(z,t)$ by

$$E_r\,(z,t) = a\, \Theta^4(z,t) \tag{11}$$

Introduce the dimensionless variables

$$x \equiv \sqrt{3}\, Kz \tag{12}$$

$$\tau \equiv \frac{16\, \sigma\, K}{\alpha}\, t = \frac{4acK}{\alpha}\, t \tag{13}$$

and define the new independent variables

$$U(x,t) \equiv \left[\frac{\Theta(z,t)}{\Theta_{inc}}\right] \quad (14)$$

$$v(x,t) \equiv \left[\frac{T(z,t)}{\Theta_{inc}}\right] \quad (15)$$

Now equations (1) and (2) take the dimensionless form

$$\varepsilon \frac{\partial u}{\partial \tau}(x,\tau) - \frac{\partial^2 u}{\partial x^2}(x,\tau) = v(x,\tau) - u(x,\tau) \quad (16)$$

$$\frac{\partial v}{\partial \tau}(x,\tau) = u(x,\tau) - v(x,\tau) \quad (17)$$

where we have defined the parameter

$$\varepsilon \equiv \frac{16\sigma}{c\alpha} = \frac{4a}{\alpha} \quad (18)$$

The condition at $z = \infty$ and the initial conditions are now

$$u(\infty,\tau) = u(x,0) = v(x,0) = 0. \quad (19)$$

The equilibrium boundary condition becomes

$$u(0,\tau) = 1 \quad (20)$$

and the Marshak boundary condition is now

$$u(0,\tau) - \frac{2}{\sqrt{3}} \frac{\partial u}{\partial x}(0,\tau) = 1. \quad (21)$$

Equations (16) through (21) are the equations we shall solve.

Setting $\varepsilon = 0$ is equivalent to assuming no retardation, i.e. an infinite speed of light. This implies that the radiation field instantly comes into a steady state distribution with the material temperature at any time t. Note that $\varepsilon = 0$ does <u>not</u> imply $u = v$ ($E_r = E_m$) because of the spatial streaming term in equation (16). Only in complete thermodynamic equilibrium does $E_r = E_m$.

III. The General Solution

Introduce the Laplace transform $\bar{f}(s)$ of a function $f(t)$ by the definition

$$\bar{f}\,(s) \equiv \int_0^\infty d\tau\; e^{-s\tau}\; f(\tau) \tag{22}$$

Taking the Laplace transform of equations (16), (17), and (19) gives

$$\varepsilon s\bar{u}(x,s) - \frac{\partial^2 \bar{u}(x,s)}{\partial x^2} = \bar{v}(x,s) - \bar{u}(x,s) \tag{23}$$

$$s\bar{v}(x,s) = \bar{u}(x,s) - \bar{v}(x,s) \tag{24}$$

$$\bar{u}\,(\infty,s) = 0. \tag{25}$$

The transform of the equilibrium boundary condition is

$$\bar{u}\,(0,s) = \frac{1}{s} \tag{26}$$

and the transform of the Marshak condition is

$$\bar{u}(0,s) - \frac{2}{\sqrt{3}} \frac{\partial \bar{u}}{\partial x} (0,s) = \frac{1}{s} . \tag{27}$$

Equation (24) gives

$$\bar{v}(x,s) = \frac{1}{s+1} \bar{u}(x,s) \tag{28}$$

and using this in (23) gives

$$\frac{\partial^2 \bar{u}}{\partial x^2}(x,s) = \beta^2(s)\, \bar{u}(x,s) \tag{29}$$

where

$$\beta^2(s) \equiv \frac{s}{s+1} [1 + \varepsilon(s+1)] \tag{30}$$

The solution to Eq. (29), subject to the boundary condition at $x = \infty$, Eq. (25) is

$$\bar{u}(x,s) = A(s)\, e^{-\beta(s)x}. \tag{31}$$

The constant A(s) is determined from the condition at $X = 0$. In the case of the equilibrium boundary conditions we find

$$\bar{u}(x,s) = \frac{e^{-\beta x}}{s} \tag{32}$$

and

$$\bar{v}(x,s) = \frac{e^{-\beta x}}{s(s+1)} \tag{33}$$

In the case of the Marshak condition we have

$$\bar{u}\,(x,s) = \frac{\sqrt{3}\,e^{-\beta(s)x}}{s[\sqrt{3} + 2\beta(s)]} \tag{34}$$

and

$$\bar{v}\,(x,s) = \frac{\sqrt{3}\,e^{-\beta(s)x}}{s(s+1)[\sqrt{3} + 2\beta(s)]} \tag{35}$$

The solutions for $u(x,T)$ and $v(x,T)$ follow by use of the Laplace inversion theorem

$$f(\tau) = \frac{1}{2\pi i}\int_C ds\; e^{s\tau}\,\bar{f}(s) \tag{36}$$

where the integration contour C is a line parallel to the imaginary S axis to the right of all singularities of $\bar{f}(s)$.

From the large and small limits of s one can immediately deduce that

$$u\,(x,0) = v\,(x,0) = 0 \tag{37}$$

and

$$u\,(x,\tau) \xrightarrow[\tau\to\infty]{} v\,(x,\tau) \xrightarrow[\tau\to\infty]{} 1 \tag{38}$$

where we have used the two theorems

$$\lim_{s\to\infty}\left[s\bar{f}(s)\right] = \lim_{\tau\to 0}\left[f(\tau)\right] \tag{39}$$

$$\lim_{s\to 0}\left[s\bar{f}(s)\right] = \lim_{\tau\to\infty}\left[f(\tau)\right] \tag{40}$$

Equation (38) states that at infinite time the radiation and material temperatures approach a constant equal to the temperature of the impinging flux.

IV. The Solution For $\varepsilon = 0$ With "Equilibrium" and "Marshak" Boundary Conditions

Pomraning[1] has already reported the solution of equations (16) through (19) with the Marshak boundary condition (21). We outline here a similar solution using the equilibrium boundary condition (20). For comparison the corresponding Marshak boundary condition solution equations will be included in brackets after the equilibrium boundary condition solution equations.

It is useful to first examine the case $\varepsilon = 0$, corresponding to no retardation. In the limit $\tau = \infty$ we find

$$u\ (x,t) \xrightarrow[t\to\infty]{} v\ (x,t) \xrightarrow[\tau\to\infty]{} 1, \tag{41}$$

just as in the general solution, for both boundary conditions. That is, at late times the system equilibrates to the correct constant even when $\varepsilon = 0$.

However, for $\tau = 0$ we find that $\beta(\infty) = 1$ for $\varepsilon = 0$ and this gives

$$u\ (x,0) = e^{-x} \qquad \left[u\ (x,0) = \frac{3}{3+2}\ e^{-x} \right] \tag{42}$$

and

$$v\ (x,0) = 0 \qquad \left[v\ (x,0) = 0 \right] \tag{43}$$

That is, the material field is still zero at $\tau = 0$, consistent with the initial condition, but the radiation field with $\varepsilon = 0$ is not zero and its value depends on the choice of boundary condition. Thus the radiation field with no retardation comes to a steady state consistent with a zero material temperature but corresponding to an incoming flux of radiation. The radiation field at the surface is larger in the case of a Marshak boundary condition by a factor of 0.46. Since u is proportional to Θ^4 this implies that radiation temperatures at the surface differ by a factor of 0.82 when $\varepsilon = 0$, with the temperature resulting from a Marshak boundary condition greater than the surface temperature from an equilibrium boundary condition.

At the surface x = 0 we have

$$\bar{u}\,(0,s) = \frac{1}{s} \qquad \left[\bar{u}\,(0,s) = \frac{\sqrt{3}\;\sqrt{s+1}}{s\,[\sqrt{3}+\sqrt{s+1} + 2\sqrt{s}\;]}\right] \tag{44}$$

$$\bar{v}\,(0,s) = \frac{1}{s(s+1)} \qquad \left[\bar{v}\,(0,s) = \frac{\sqrt{3}\quad\sqrt{s+1}}{s(s+1)\,[\sqrt{3}\,\sqrt{s+1} + 2\sqrt{s}]}\right] \tag{45}$$

The inverses of Eq. (44) and Eq. (45) are tabulated[4] to be

$$u(0,\tau) = 1 \qquad \left[u(0,\tau) = 1 - \frac{4\sqrt{3}}{\pi}\int_0^1 d\eta\,\frac{(1-\eta^2)^{1/2}\;e^{-\eta^2\tau}}{(3-\eta^2)}\right] \tag{46}$$

and

$$v\,(0,\tau) = 1 - e^{-\tau} = u\,(0,\tau) - e^{-\tau}$$

$$\left[v\,(0,\tau) = u\,(0,\tau) - \frac{4\sqrt{3}}{\pi}\int_0^1 d\eta\,\frac{(1-\eta^2)^{1/2}\;e^{-\tau(1-\eta^2)}}{(4-\eta^2)}\right] \tag{47}$$

These equations show that the material field always lags behind the radiation field, as one expects it should.

The flux of radiation is given by

$$\Gamma(z,t) = -\frac{c}{3\,K}\,\frac{\partial Er(z,t)}{\partial z} \tag{48}$$

Define the dimensionless flux

$$W(x,\tau) \equiv \frac{F(z,t)}{F_{inc}} \tag{49}$$

or in terms of the dimensionless variables x and τ

$$W(x,\tau) = -\frac{4}{\sqrt{3}} \frac{\partial u}{\partial x}(x,\tau) \tag{50}$$

The Laplace transform of the surface flux is

$$\bar{W}(0,s) = \frac{4}{\sqrt{3}} \frac{1}{\sqrt{s(s+1)}} \qquad \left[\bar{W}(0,\tau) = \frac{4}{\sqrt{3s(s+1)} + 2s} \right] \tag{51}$$

From the small and large s behavior we find

$$W(0,\tau) \xrightarrow[\tau \to \infty]{} 0 \qquad \left[W(0,\tau) \xrightarrow[\tau \to \infty]{} 0 \right] \tag{52}$$

$$W(0,0) = \frac{4}{\sqrt{3}} \qquad \left[W(0,0) = \frac{4}{\sqrt{3} + 2} \right] \tag{53}$$

Equation (52) shows that at infinite time equilibrium has been reached throughout the material and the net flux is zero. At $\tau = 0$ one physically expects that all of the incoming radiation is absorbed because the material is cold. However, instead of $W(0,0) = 1$, Eq. (53) gives

$$W(0,0) = 2.3094 \qquad [1.0718] \tag{54}$$

We note that while the Marshak boundary condition gives a 7% error in flux, the "equilibrium" boundary condition gives a 231% error, when applied to the P1 approximation.

For a general value of τ the tabulated solution to

$$\bar{W}(0,s) = \frac{4}{\sqrt{3}} \frac{1}{\sqrt{s\ (s+1)}} \tag{55}$$

is given by

$$W(0,\tau) = \frac{4}{\sqrt{3}} e^{-\tau/2} I_o(-\tau/2)$$

$$= \frac{4}{\sqrt{3}} e^{-\tau/2} \left(1 + \frac{\tau^2}{16} + \frac{\tau^4}{1024} + \ldots \right) \tag{56}$$

$$\left[W(0,\tau) = \frac{8\ 3}{\pi} \int_0^1 d\eta \frac{(1-\eta^2)^{1/2} e^{-\eta^2 \tau}}{(3+\eta^2)} \right] \tag{57}$$

where I_o is a modified Bessel function of the first kind with order zero.

V. The Integrated Energies for $\epsilon = 0$

Next we calculate the integrated energies for the case $\epsilon = 0$. We define

$$P_r(\tau) \equiv \int_0^\infty dz\ E_r(z,t) = \frac{a\ \theta^4_{inc}}{\sqrt{3}\ K} \int_0^\infty dx\ u(x,\tau) \tag{58}$$

$$P_m(\tau) \equiv \int_0^\infty dz\ \frac{\alpha}{4} T^4(z,t) = \frac{\alpha\ \theta^4_{inc}}{4\sqrt{3}\ K} \int_0^\infty dx\ v(x,\tau) \tag{59}$$

The dimensionless integral energies are defined as

$$\Psi_r(\tau) \equiv \frac{P_r(\tau)}{P_r(0)} \quad \text{and} \quad \Psi_m(\tau) \equiv \frac{P_m(\tau)}{P_m(0)} \tag{60}$$

where

$$P_r(0) \equiv \frac{a\,0^4_{inc}}{\sqrt{3}\;\;K} \text{ and } P_m(0) \quad \frac{\alpha\,0^4_{inc}}{4\sqrt{3}\,K} \tag{61}$$

Thus we have

$$\Psi_r(\tau) = \int_0^\infty dx \quad u(x,\tau) \tag{62}$$

$$\Psi_m(\tau) = \int_0^\infty dx \; v(x,\tau) \tag{63}$$

Taking the Laplace transforms of these integrals and integrating gives

$$\overline{\Psi}_r(s) = \frac{\sqrt{s+1}}{s^{3/2}} \qquad \left[\overline{\Psi}_r(s) = \frac{\sqrt{3}\quad(s+1)}{s(\sqrt{3s(s+1)} + 2s)} \right] \tag{64}$$

$$\overline{\Psi}_m(s) = \frac{1}{s^{3/2}\;(s+1)^{1/2}} \qquad \left[\overline{\Psi}_m(s) = \frac{\sqrt{3}}{s(\sqrt{3s(s+1)} + 2s)} \right] \tag{65}$$

The inverse of these transforms is tabulated to give

$$\Psi_r(\tau) = (\tau + 1)e^{-\tau/2} \; I_o\,(\tau/2) + \tau e^{-\tau/2} \; I_1\,(\tau/2)$$

$$= (\tau+1)e^{-\tau/2}\left(1 + \frac{\tau^2}{16} + \frac{\tau^4}{1024}\right) + \ldots \quad + \tau e^{-\tau/2}\left(\frac{\tau}{4} + \frac{\tau^3}{128} + \ldots\right) \tag{66}$$

$$\left[\Psi_r(\tau) = \frac{2}{\pi}\int_o^1 d\eta \left\{2\tau + \left(\frac{7 + 4\ \ 1 - \eta^2 - 3\eta^2}{1 + \ \ 1 - \eta^2}\ \frac{1}{3 + \eta^2}\right)\right\} e^{-\tau\eta^2} + \underline{2e^{-\tau}} - \frac{2}{3}\right]$$

for the radiation energy, and for the material energy

$$\Psi_m(\tau) = \tau e^{-\tau/2}\left(I_o(\tau/2) + I_1\ (\tau/2)\right)$$

$$= \tau e^{-\tau/2}\left(1 + \frac{\tau^2}{16} + \frac{\tau^4}{1024} + \ldots + \frac{\tau}{4} + \frac{\tau^3}{128} + \ldots\right) \qquad (67)$$

$$\left[\ \Psi_m(\tau) = \frac{2}{\pi}\int_o^1 d\eta \left\{2\tau + \left(\frac{4 + \sqrt{1-\eta^2}}{1 + \sqrt{1-\eta^2}}\right)\left(\frac{1}{3 + \eta^2}\right)\right\} e^{-\tau\eta^2} + \frac{2e^{-\tau}}{\pi} - \frac{2}{\sqrt{3}}\ \right]$$

Note that

$$\Psi_r(\tau) = e^{-\tau/2}\left(1 + \frac{\tau^2}{16} + \frac{\tau^4}{1024} + \ldots\right) + \Psi_m(\tau) \qquad (68)$$

or that

$$\Psi_r(\tau) - \Psi_m(\tau) = \frac{\partial\Psi_m(\tau)}{\partial\tau} \qquad (69)$$

which is just the energy balance equation for the material, obtained by integrating Eq. (17) over all x.

For large values of the argument I_o and I_1 have the asymptotic approximation[5]

$$I_o(Z) \underset{Z \to \infty}{} \frac{1}{\sqrt{2\pi Z}} \; e^Z \left(1 + \frac{1}{8Z} + \frac{1}{128Z^2} + \dots\right) \tag{70}$$

$$I_1(Z) \underset{Z \to \infty}{} \frac{1}{\sqrt{2\pi Z}} \; e^Z \left(1 - \frac{3}{8Z} - \frac{23}{128Z^2} + \dots\right) \tag{71}$$

thus for large τ we get

$$\Psi_m(\infty) = \Psi_r(\infty) = 2\sqrt{\frac{\tau}{\pi}} \qquad \left[\Psi_m(\infty) = \Psi_r(\infty) = 2\sqrt{\frac{\tau}{\pi}}\right] \tag{72}$$

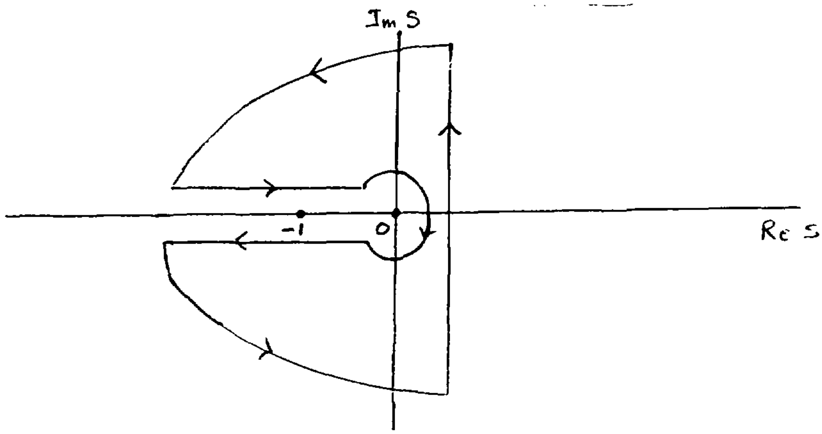
VI. The Distributions $u(x,\tau)$ and $u(x,\tau)$ for $\epsilon = 0$

Finally we integrate according to Eq. (36) to find the inverses of $\bar{u}$ and $\bar{v}$ for the case $\varepsilon = 0$. That is, we transform Eq. (32) and (33) with $\beta^2 = s/(s + 1)$. Thus

$$u(x,\tau) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} ds \; e^{s\tau} \frac{e^{-\beta x}}{s} \tag{73}$$

$$v(x,\tau) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} ds \; e^{s\tau} \frac{e^{-\beta x}}{s(s+1)} \tag{74}$$

These Laplace inversion integrals are vertical lines in the right half s-plane, as shown in the contour below. Closing the contour with a large semicircle in the left half plane gives a zero contribution except for contributions from the pole at s = 0 and from the branch cut, with branch points at s = 0 and s = -1.



Proper branches are defined as those which give a positive square root for s lying on the real positive axis.

Omitting the algebraic detail, we find

$$u(x,\tau) = 1 - \frac{2}{\pi} \int_0^1 d\eta \; \frac{e^{-\eta^2 \tau}}{\eta} \; \sin\left(\frac{\eta x}{\sqrt{1 - \eta^2}}\right) \tag{75}$$

$$\left[u(x,\tau) = 1 - \frac{4\sqrt{3}}{\pi} \int_0^1 d\eta \; \frac{\sqrt{1 - \eta^2}}{(3 + \eta^2)} \; \cos\left(\frac{\eta x}{\sqrt{1 - \eta^2}}\right) e^{-\eta^2 \tau} \right.$$

$$\left. - \frac{6}{\pi} \int_0^1 d\eta \left(\frac{1 - \eta^2}{3 + \eta^2}\right) \sin\left(\frac{\eta x}{\sqrt{1 - \eta^2}}\right) e^{-\eta^2 \tau} \right]$$

and

$$v(x,\tau) = 1 - \frac{2}{\pi} \int_0^1 d\eta \; \frac{e^{-\eta^2 \tau}}{\eta(1-\eta^2)} \; \sin\left(\frac{\eta x}{\sqrt{1-\eta^2}}\right)$$

$$\left[v(x,\tau) = 1 - \frac{4\sqrt{3}}{\pi} \int_0^1 d\eta \; \frac{\sqrt{1-\eta^2}}{(3+\eta^2)} \; \cos\left(\frac{\eta x}{\sqrt{1-\eta^2}}\right) e^{-\eta^2 \tau} \right. \tag{76}$$

$$\left. - \frac{4\sqrt{3}}{\pi} \int_0^1 d\eta \; \frac{\sqrt{1-\eta^2}}{(4-\eta^2)} \; \cos\left(\frac{\sqrt{1-\eta^2}\, x}{\eta}\right) e^{-\tau(1-\eta^2)} \right]$$

$$- \frac{6}{\pi} \int_0^1 d\eta \left(\frac{1}{\eta(3 + \eta^2)} \right) \sin\left(\frac{\eta x}{1 - \eta^2} \right) e^{-\tau\eta^2} \Bigg]$$

It can be verified that these equations yield the previously derived results in the limits $\tau = \infty$ and $x = 0$.

VII. The Integrated Energies for $\epsilon \neq 0$

The general problem with $\epsilon \neq 0$ was given by equations (30) through (35) for both types of boundary condition. Again we shall give solution equations for the case of equilibrium boundary conditions, with the corresponding solutions using Marshak boundary conditions in brackets. For the equilibrium boundary conditions the integrated energy transforms are given by

$$\overline{\Psi}_r(s) = \frac{1}{\beta s} = \frac{1}{s \left(\frac{s}{s+1}\right)^{1/2} \left(1 + \epsilon (s+1)\right)^{1/2}} \tag{77}$$

$$\overline{\Psi}_m(s) = \frac{1}{\rho s(s+1)} = \frac{1}{s(s+1) \left(\frac{s}{s+1}\right)^{1/2} \left(1 + \epsilon (s+1)\right)^{1/2}} \tag{78}$$

Noting that $\Psi_r(\tau) = \Psi_m(\tau) + \frac{\partial\Psi_m(\tau)}{\partial\tau}$, we need only invert $\overline{\Psi}_m$ to get Ψ_m and Ψ_r.

The $s = 0$ limit gives the long-time behavior of Ψ_m.

$$\lim_{s \to 0} \overline{\Psi}_m = \lim_{s \to 0} \overline{\Psi}_r = 2 \sqrt{\frac{\tau}{\pi(1+\epsilon)}} = \Psi_m(\infty) = \Psi_r(\infty) \tag{79}$$

This limit is the same in the case of Marshak boundary conditions. For small times one takes the limit $s \to \infty$ and transforms to get

$$\Psi_r\ (\tau \sim 0) = 2\sqrt{\frac{\tau}{\pi\varepsilon}} \tag{80}$$

$$\Psi_m\ (\tau \sim 0) = \frac{4\ \tau^{3/2}}{3\sqrt{\pi\varepsilon}} \tag{81}$$

Note that these expressions are not well-behaved in the limit $\varepsilon = 0$. At $\tau = 0$ both boundary conditions give

$$\Psi_r(0) = \Psi_m(0) = 0. \tag{82}$$

To get Ψ_m we must perform the contour integration given by

$$\Psi_m(\tau) = \frac{1}{2\pi i} \int_c ds \quad e^{s\tau} \frac{1}{s(s+1) \left(\frac{s}{s+1}\right)^{1/2} \left(1 + \varepsilon(s+1)\right)^{1/2}} \tag{83}$$

$$\left[\Psi_m(\tau) = \frac{1}{2\pi i} \int_c ds \quad e^{s\tau} \frac{\sqrt{3}\quad (s+1)}{s^{3/2}\ (1+\varepsilon(s+1))^{1/2} \left\{\sqrt{3(s+1)} + 2\sqrt{s}\ \ (1+\varepsilon(s+1))^{1/2}\right\}}\right.$$

This integrand has a simple pole at $s = 0$ and branch points at $s = 0$, $s = -1$, and $s = -(1 + \varepsilon)/\varepsilon$. The branches to be used in the integration are again defined as those which give positive square roots for s lying on the real positive axis. All three branch cuts are extended along the negative real axis. The contour of integration is very similar to that sketched previously. Omitting the lengthy algebraic detail, the results are given by the following equations in the case of each boundary condition as noted:

Equilibrium Boundary Conditions:

$$\psi_r^E(\tau) = \frac{2}{\pi\sqrt{1+\varepsilon}} \int_0^1 d\eta\ G_r(\tau,\eta)\ e^{-\tau\eta^2} - \frac{2}{\pi} e^{-\tau} \int_0^1 d\eta\ H_r(\eta)\ e^{-\tau/\varepsilon(1-\eta^2)} \tag{84}$$

$$\psi_m^E(\tau) = \frac{2}{\pi\sqrt{1+\varepsilon}} \int_0^1 d\eta\ G_m(\tau,\eta)\ e^{-\tau\eta^2} + \frac{2e^{-\tau}}{\pi\sqrt{1+\varepsilon}} \tag{85}$$

$$- \frac{2}{\pi} e^{-\tau} \int_0^1 d\eta\ H_m(\eta)\ \exp\left(\frac{-\tau}{\varepsilon(1-\eta^2)}\right)$$

$$H_r(\eta) = \frac{1}{(1-\eta^2)^{1/2}\ [1+\varepsilon(1-\eta^2)]^{3/2}} \tag{86}$$

$$H_m(\eta) = (1-\eta^2)\ H_r(\eta) \tag{87}$$

$$G_m(\tau,\eta) = 2\tau + \frac{\varepsilon\ \eta^2 - 2\varepsilon - 1}{(1-\eta^2)(1+\varepsilon(1-\eta^2)) + \sqrt{1-\eta^2}\sqrt{1+\varepsilon}(1+\varepsilon(1-\eta^2))^{1/2}} \tag{88}$$

$$G_r(\tau,\eta) = (1-\eta^2)\ G_m(\tau,\eta) + 2 \tag{89}$$

Marshak Boundary Conditions:

$$\psi_r^M(\tau) = \frac{2}{\pi\sqrt{1+\varepsilon}} \int_0^1 d\eta\ g_r(\tau,\eta) e^{-\tau\eta^2}\ \frac{-2}{\sqrt{3}} \tag{90}$$

$$- \frac{6}{\pi} e^{-\tau} \int_0^1 d\eta\ h_r(\eta)\ \exp\ \frac{-\tau}{\varepsilon(1-\eta^2)}$$

$$\psi_m^M(\tau) = \frac{2}{\pi\sqrt{1+\varepsilon}} \int_0^1 d\eta\ g_m(\tau,\eta) e^{-\tau\eta^2} + \frac{2e^{-\tau} - \frac{2}{\sqrt{3}}}{\pi\sqrt{1+\varepsilon}} \tag{91}$$

$$+ \frac{6\varepsilon}{\pi} e^{-\tau} \int_0^1 d\eta\ h_m(\eta) \exp \frac{-\tau}{\varepsilon(1-\eta^2)}$$

$$h_r(\eta) = \frac{(1-\eta^2)^{1/2}}{[1+\varepsilon(1-\eta^2)]^{3/2}\ [3+(1+4\varepsilon)\eta^2 - 4\varepsilon\eta^4]} \tag{92}$$

$$h_m(\eta) = (1-\eta^2)\ h_r(\eta) \tag{93}$$

$$g_m(\tau,\eta) = 2\tau + \frac{1 + 4\varepsilon(1-\eta^2)}{3 + (4\varepsilon+1)\eta^2 - 4\varepsilon\eta^4} \tag{94}$$

$$+ \frac{1}{3} \left\{ [1+\varepsilon(1-\eta^2)]^{1/2}\ [3+(4\varepsilon+1)\eta^2 - 4\varepsilon\eta^4] \left[\sqrt{1+\varepsilon(1-\eta^2)} + \sqrt{1+\varepsilon}\sqrt{1-\eta^2}\right] \right\}^{-1}$$

$$g_r(\tau,\eta) = (1-\eta^2)\ g_m(\tau,\eta) + 2 \tag{95}$$

These quantities have been computed on a CDC 7600 using 16-point Gaussian quadrature on N intervals in the range $0 \leq \eta \leq 1$. N was successively doubled until the desired accuracy was achieved.

The time dependence of ψ_m^E, ψ_r^E, ψ_m^E, and ψ_r^M is shown in Figures 1 3 for various values of ε. We find, as shown in Figure 1, that for values of τ greater than about 10, $\psi_m^E(\tau) = \psi_r^E(\tau)$ and $\psi_m^M(\tau) = \psi_r^M(\tau)$, but the Marshak solutions lag the equilibrium solutions by a difference of the order of unity. The equilibrium solution reaches the analytical late-time value of Eq. (79)

quite early, but the Marshak solution of Pomraning[1] appears to converge very slowly to its infinite-time analytical value. As ϵ is increased the total amount of energy in the half-space is decreased.

Figure 2 shows the early time behavior of Ψ_m. The equilibrium condition energy is greater than the Marshak condition energy at all times. The highest energy absorption occurs for $\epsilon = 0$, the no retardation situation.

The different behavior at $\tau = 0$ in Ψ_r is shown in Figure 3. The equilibrium condition energy and the Marshak condition radiation energy for $\epsilon = 0$ in $\tau = 0$ are given by

$$\Psi_r^E(0) = 1 \qquad \text{and} \qquad \Psi_r^M(0) = \frac{\sqrt{3}}{\sqrt{3} + 2} = 0.46410 \qquad (96)$$

that is, there is more than a factor of two difference in the amount of energy entering the surface at early times.

The differences between integrated energies using the two different boundary conditions are summarized in Figures 4 and 5 as percentages. For larger values of ϵ the discrepancies become enormous at early times, and even with $\epsilon = 0$ the differences become less than a few percent only at times of the order of $\tau = 1000$.

VIII. Conclusion

We have attempted to compare the results of two different analytic solutions of the nonequilibrium transient Marshak problem, one using a flux-conserving (Marshak) boundary condition and the other using the equilibrium infinite-medium type of boundary condition which is often implicitly employed in radiative transfer.

We find that very substantial differences result from the use of these boundary conditions at early times. In particular we have compared the total radiation and material energies in the half-space as functions of time and boundary condition. In the case of no retardation the differences can be as much as a factor of 2, and for the retarded cases the energies can be orders of magnitude different.

The implications are that unless great care is attached to applying the correct boundary condition to a transient non-equilibrium problem such as this one, the early time solution could be grossly miscalculated. This has a particular impact on such astrophysical phenomena as the brightness of galactic clouds adjacent to strong sources like quasars or the luminosity of clouds or nebulae receiving radiation from supernovae. It may also have some technological significance in space science where the radiative transfer involved with rocket re-entry is a critical part of nosecone design.

In addition to the general warning that these results underline concerning the choice and use of boundary conditions, the analytic solutions themselves provide a means for testing and calibrating computer codes which calculate time dependent radiative transfer. Such calculations are beginning to be of great interest in astrophysics. Since analytic solutions such as these are relatively rare, we hope that these equations will be treated as reference solutions in the development of computer codes of this type.

References

1. G. C. Pomraning, "The Non-Equilibrium Marshak Wave Problem", J. Quant. Spectry. Radiative Transfer. (To be published)

2. M. A. Heaslet and R. F. Warming, "Radiative Transport and Wall Temperature Slip in an Absorbing Planar Medium", Int. J. Heat Mass Transfer 8, 9795, (1965).

3. G. C. Pomraning, "Radiative Transfer Via Spherical Harmonics", submitted to J. Quant. Spectry. Radiative Transfer.

4. See for example A. V. Luikov, "Analytical Heat Diffusion Theory", Appendix 5, Academic Press, (1968).

5. Ibid. p. 143.

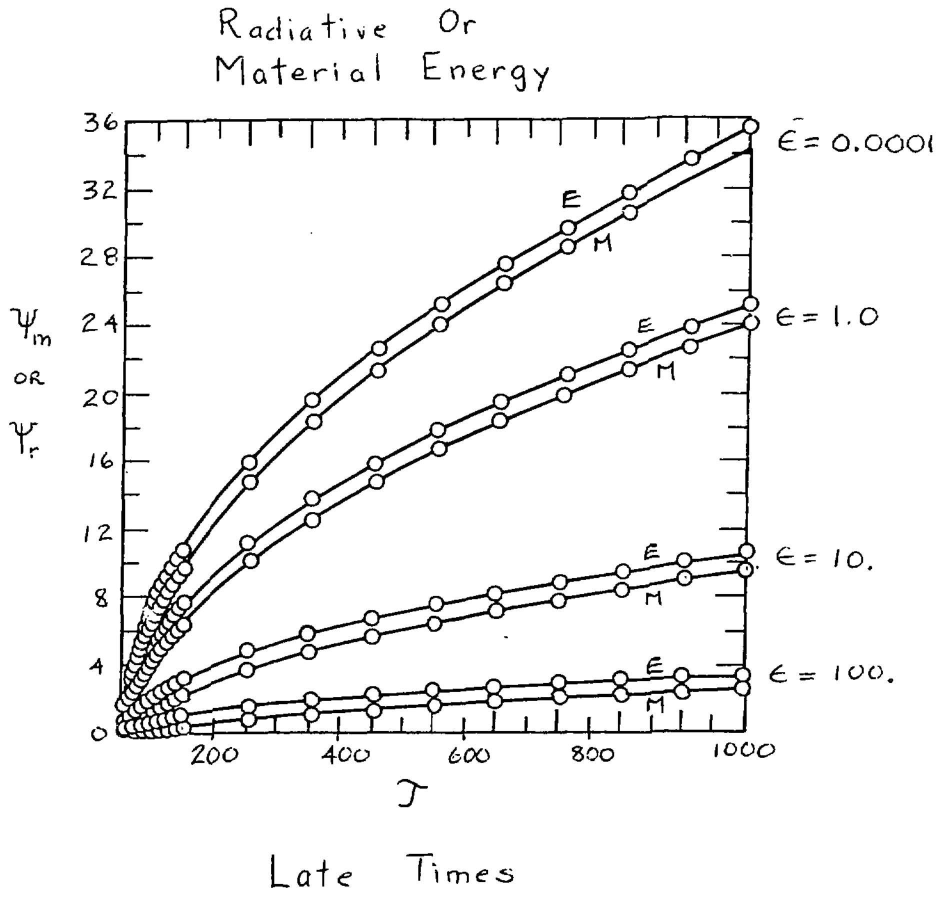

Radiative Or
Material Energy
36
32
28
24
20
16
12
8
4
0
Ψ_m
OR
Ψ_r
E
M
ϵ = 0.0001
ϵ = 1.0
ϵ = 10.
ϵ = 100.
200
400
600
800
1000
τ
Late Times

Figure 1

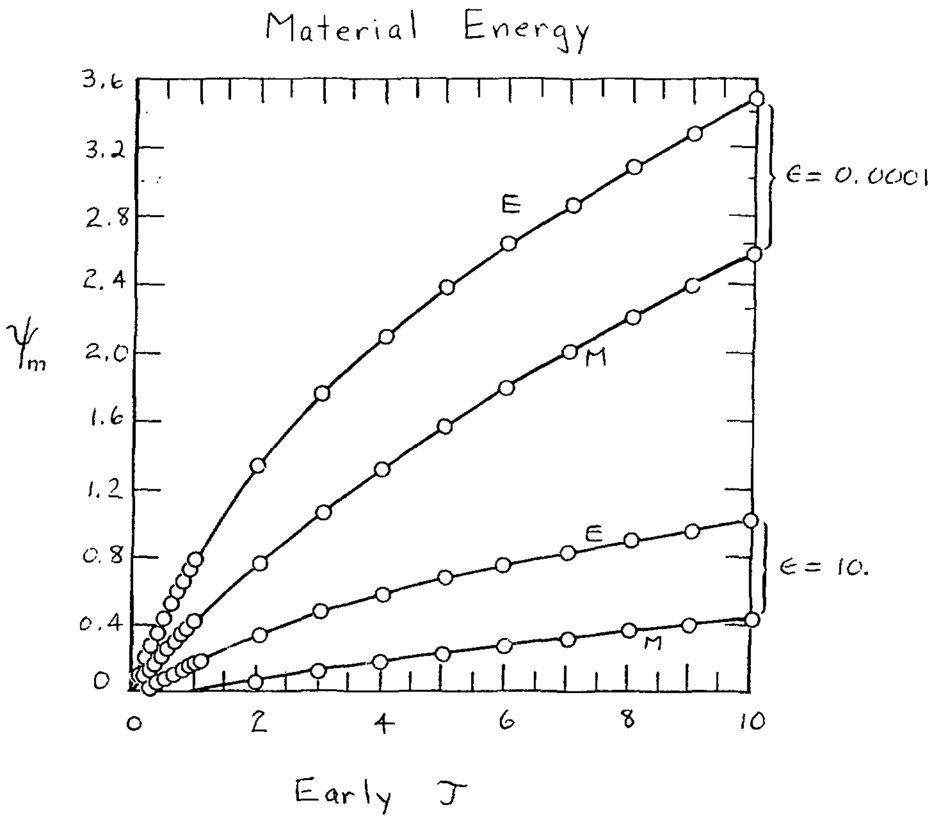

Material Energy
3.6
3.2
2.8
2.4
2.0
1.6
1.2
0.8
0.4
0
Ψm
E
M
E
M
ϵ= 0.0001
ϵ= 10.
0
2
4
6
8
10
Early J

Figure 2

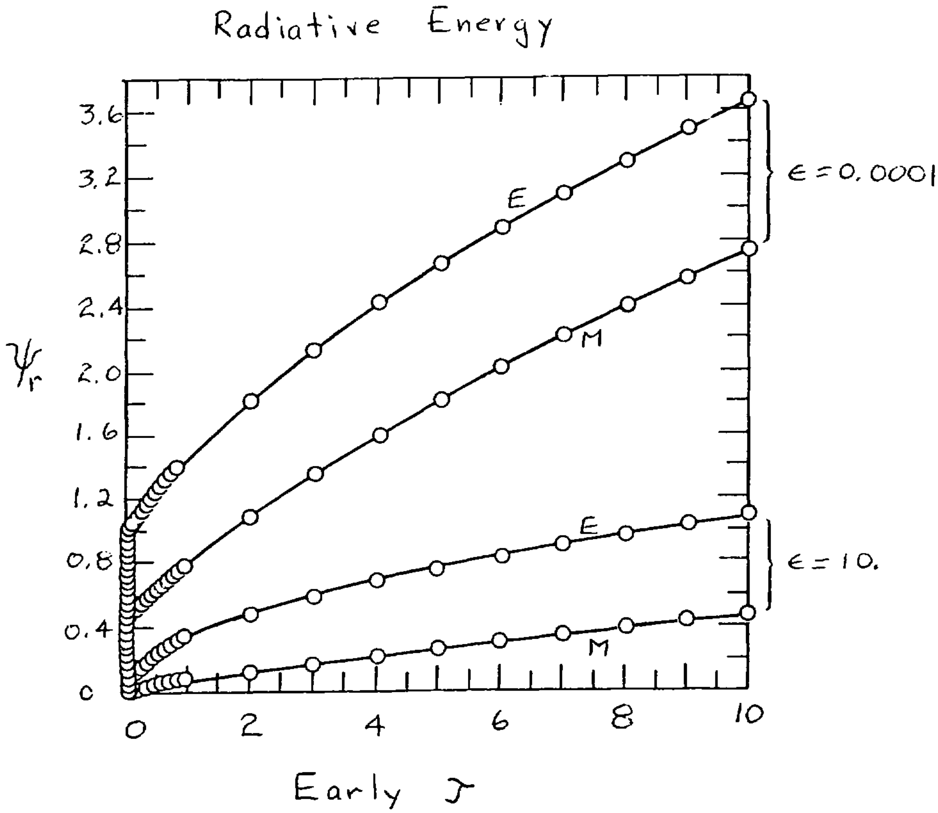

Radiative Energy
3.6
3.2
2.8
2.4
2.0
1.6
1.2
0.8
0.4
0
ψ_r
E
M
E
M
$\epsilon = 0.0001$
$\epsilon = 10.$
0
2
4
6
8
10
Early τ

Figure 3

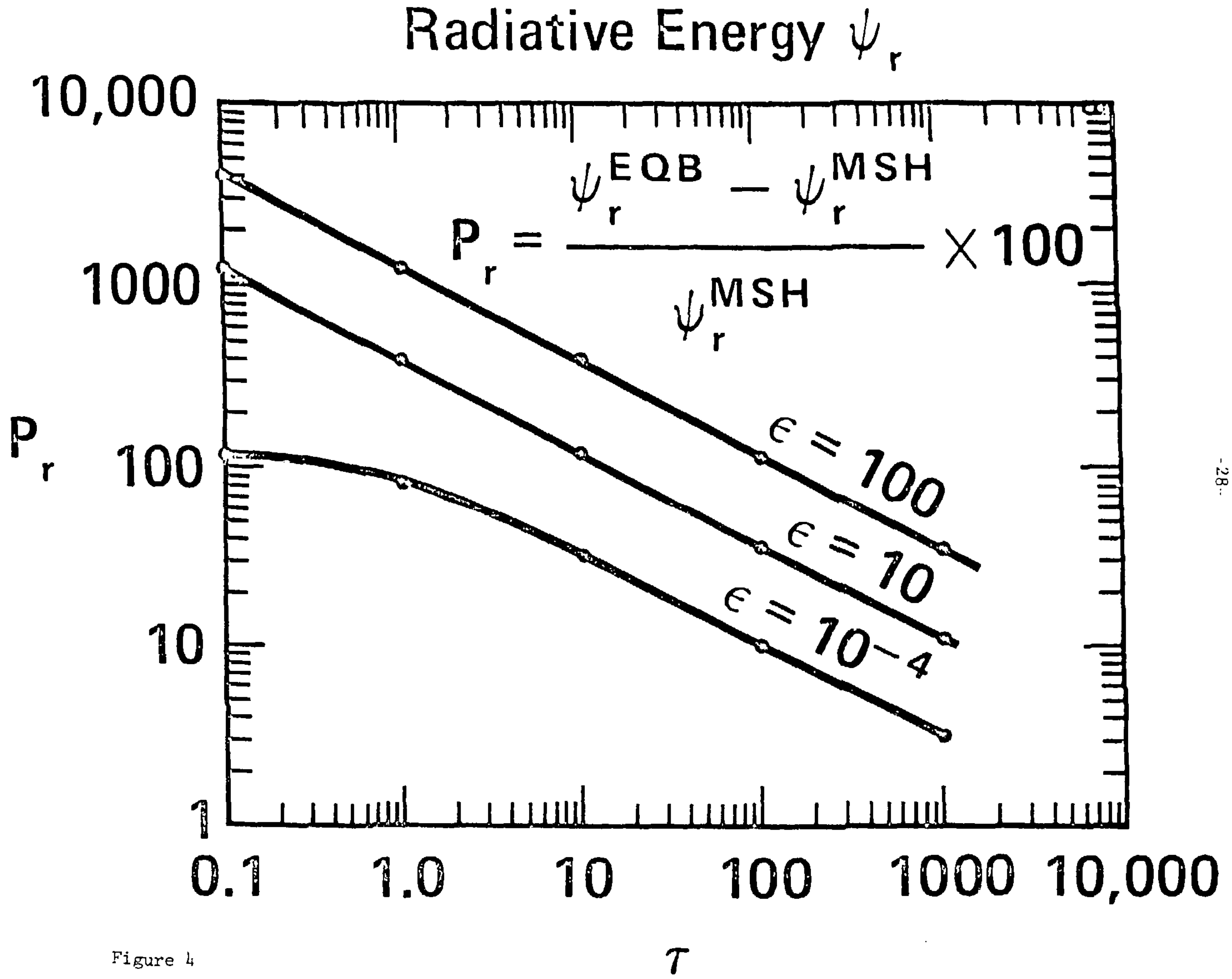
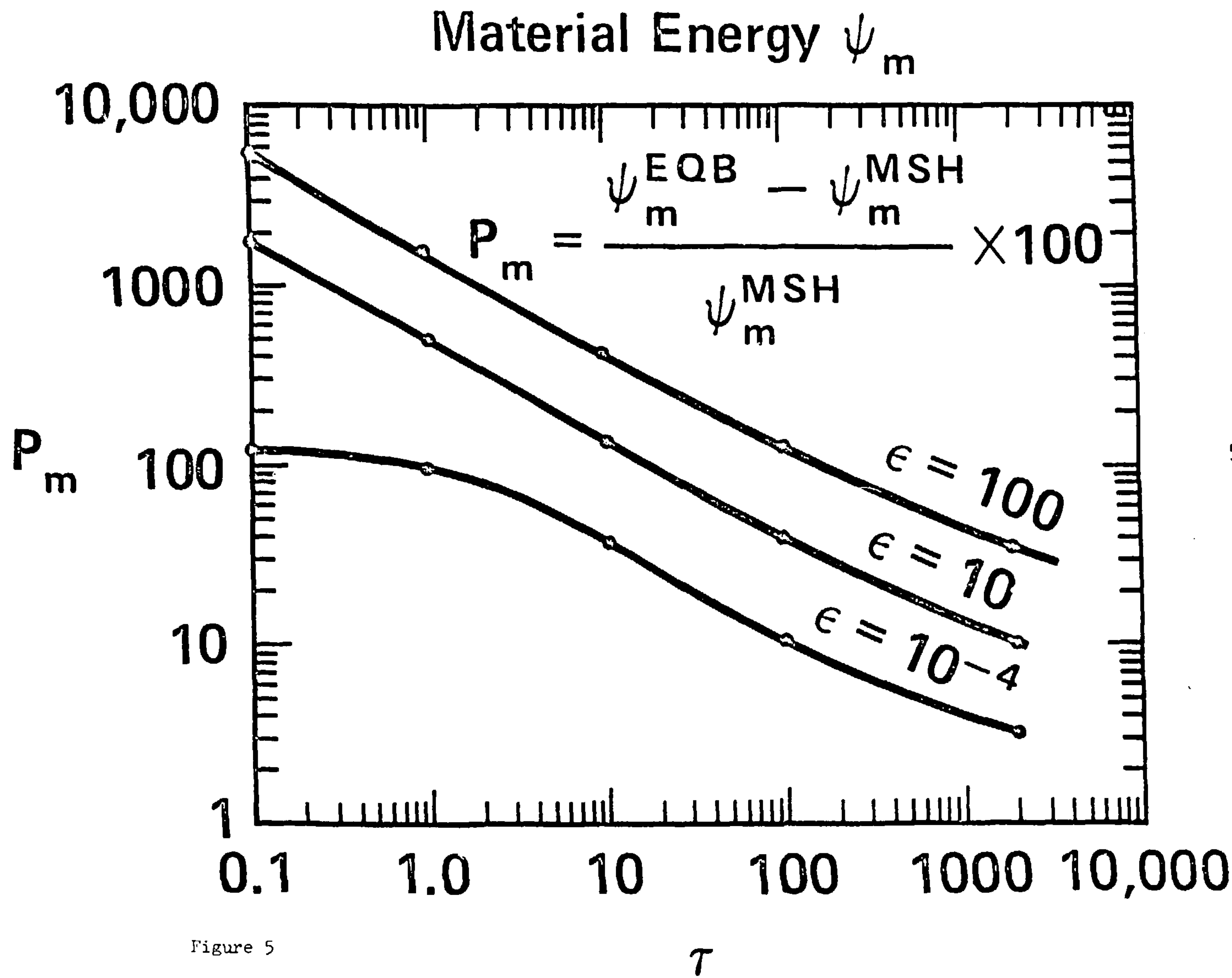

Radiative Energy ψ_r
$P_r = \frac{\psi_r^{EQB} - \psi_r^{MSH}}{\psi_r^{MSH}} \times 100$
$\epsilon = 100$
$\epsilon = 10$
$\epsilon = 10^{-4}$
P_r
10,000
1000
100
10
1
0.1
1.0
10
100
1000
10,000
τ

Figure 4



Material Energy ψ_m
$P_m = \frac{\psi_m^{EQB} - \psi_m^{MSH}}{\psi_m^{MSH}} \times 100$
10,000
1000
100
10
1
P_m
$\epsilon = 100$
$\epsilon = 10$
$\epsilon = 10^{-4}$
0.1
1.0
10
100
1000
10,000
τ

Figure 5