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STATISTICAL MECHANICS USING SYMBOLIC DYNAMICS

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Symbolic dynamics[1] provides a means of partitioning phase space so that information concerning the particle orbits is imbedded in the partitioning. Consider a sequence of phase space points $x_0, x_1, x_2 \cdots$ given by the dynamics, which we call an orbit, the subscript refering to time. Define a sequence of integers $s_0, s_1, s_2, \cdots$ associated with this orbit.

Further consider a truncated sequence S of length N and let $D(S)$ be the set of all x_0 which produce the sequence S as time is advanced. Each of the sequences S defines a coarse grain element of phase space. For our purposes the sequences s_t define a symbolic dynamics for the system if for a given S each set $D(S)$ is simply connected.

Now introduce a coarse–grain approximation, i.e. approximate the distribution function $f(x,t)$ as constant in each coarse grain domain $D(S)$. Similarly, let $D(S,S')$ be that part of the set $D(S)$ which maps into coarse grain element $D(S')$ in one time step. The coarse–grain approximation to the branching ratios for transition from state S to state S' is then given by

$$\gamma(S,S') = \frac{D(S,S')}{\sum_{S''} D(S,S'')} \qquad (1)$$

This can be done more formally using the Perron-Frobenius equation.[2] The sets $D(S)$ and the branching ratios $\gamma(S,S')$ are easily found numerically.

The symbolic kinetic equation[2] for the coarse grain probabilities $P_t(S)$ takes the form

$$P_{t+1}(S) = \sum_{S} \gamma(S,S') P_t(S') \qquad (2)$$

and this equation can be readily solved for the steady state probabilities $P(S)$. The number of nonzero matrix elements in $\gamma(S,S')$ is very limited, and as a result Eq. 2 converges extremely rapidly. This is not true using other Markov partitions of phase space, such as coarse grain domains of equal size.

Any statistical quantity can be readily computed from the probabilities $P(S)$ and the branching ratios $\gamma(S,S')$. For example the correlation function is given by

$$C(\tau) = \sum_{S_\tau} x_\tau \Pi_{t=1}^{\tau} \sum x_0 P(S_0) \gamma(S_0, S_t). \qquad (3)$$

The branching ratios $\gamma(S,S')$ define a network, or Markov chain, whose topological properties are related to the physical properties of the system. Systems with long correlation time have networks with relatively few closed loops, whereas systems with short correlation time have networks with high topological connectivity.

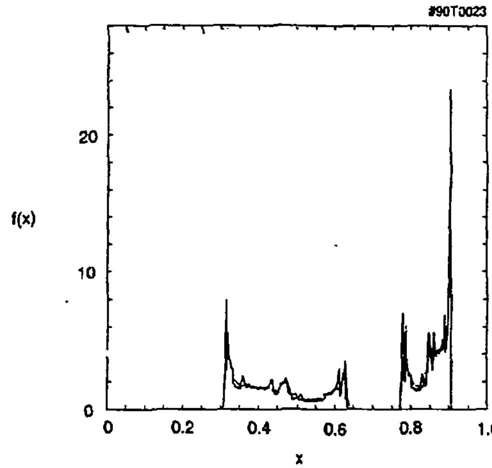

Figure 1: Invariant distribution $f(x)$ for the chaotic attractor produced by the logistic map, $\alpha = 3.62$. Shown is the result from orbit averaging (solid line) and that from solving Eq. 2, $n = 18$

Many examples of symbolic dynamics are known, the simplest being that associated with the logistic map, which exhibits the period doubling route to chaos. This map is given by

$$x_{t+1} = \alpha x_t(1 - x_t) \tag{4}$$

where $0 < x_t < 1$ and $0 < \alpha < 4$.

The symbolic sequence associated with x_t is defined by

$$s_t = \begin{cases} 0 & \text{when} \quad x_t < \frac{1}{2} \\ 1 & \text{when} \quad x_t > \frac{1}{2} \end{cases} . \tag{5}$$

A convenient means of labeling[2] sequences S, and hence the coarse grain domains is through the integer

$$l = \sum_{t=0}^{N-1} [\sum_{k=0}^{t} s_k] \bmod (2) 2^{n-t-1} \tag{6}$$

An example of the invariant distribution function is shown in Fig. 1, which shows the solution to Eq. (2) as well as the result of directly averaging over a long orbit. The result for the correlation function is shown in Fig. 2. An example of a network diagram is shown in Fig. 3 for the logistic map for $\alpha = 3.71$, $n = 6$. This network has high topological connectivity, corresponding to the short correlation time apparent from Fig. 2.

In Fig. 4 is shown the power spectrum $P(w)$, which is the Fourier transform of the correlation function, for the logistic map, $\alpha = 3.94$. The "fluctuations" have a broad spectrum with two peaks.

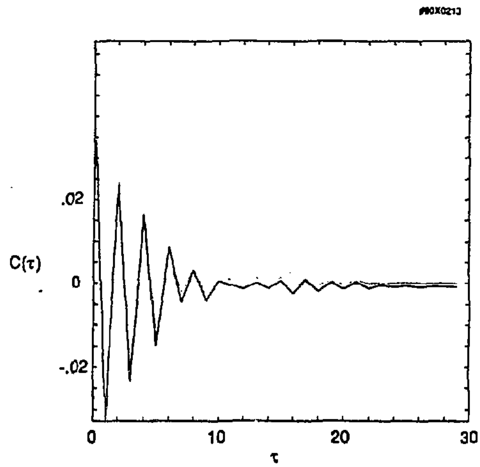

Figure 2: Correlation function for the logistic map, $\alpha = 3.71$. Shown is the result of orbit averaging (solid line) and the coarse grain result from Eq. (3).

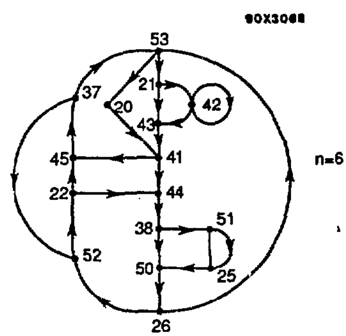

Figure 3: Network diagram for the logistic map, $\alpha = 3.71$, $n = 6$. The points indicate coarse grain elements, and the arrows show the effect of advancing in time.

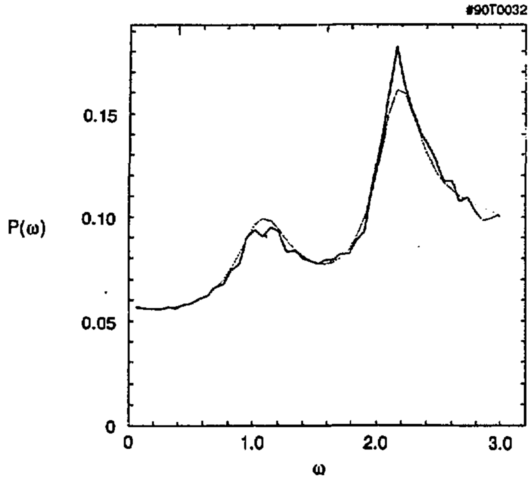

Figure 4: The power spectrum $P(w)$ obtained from the correlation function for the logistic map, $\alpha = 3.94$. The line is the result of an orbit average and the dotted line the coarse-grain result from Eq. (3).

Another example is provided by the Hénon map, which possesses a strange attractor. The Hénon map is defined by

$$x_{t+1} = a - by_t - x_t^2$$
$$y_{t+1} = x_t \tag{7}$$

Shown in Fig. 5 is the strange attractor which results for $a = 1.4$, $b = -0.3$. The symbolic dynamics is constructed by defining

$$s_t = \begin{cases} 0 & \text{for} \quad x_t < 0 \\ 1 & \text{for} \quad x_t > 0 \end{cases} \tag{8}$$

for $t = 0, 1, 2, \cdots, n-1$ and

$$s_{t-1} = \begin{cases} 0 & \text{for} \quad y_t < 0 \\ 1 & \text{for} \quad y_t > 0 \end{cases} \tag{9}$$

for $t = 0, -1, -2, \cdots, -n$. The sequence can be characterized by a single integer l

$$l = \sum_{t=-n}^{n-1} 2^{n-1-t} s_t \tag{10}$$

The resulting coarse grain domains for $n = 3$ are shown in Fig. 6. In Fig. 7 is shown the correlation function $C(\tau)$ obtained using $n = 4$.

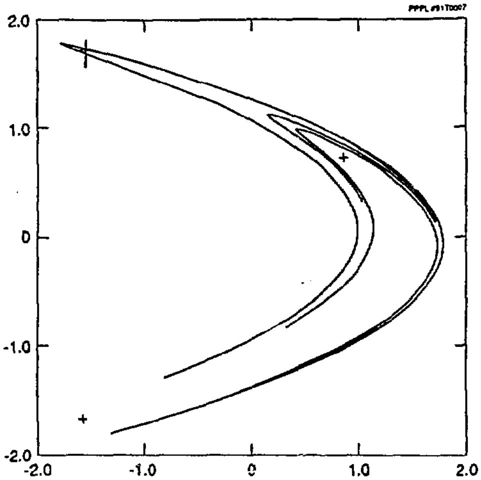

Figure 5: The strange attractor given by the Hénon map, $a = 1.4$, $b = -0.3$.

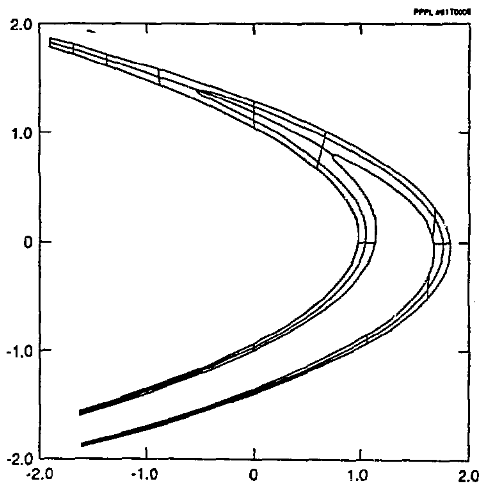

Figure 6: Symbolic domains for the Hénon map, $a = 1.4$, $b = -.3$, $n = 3$.

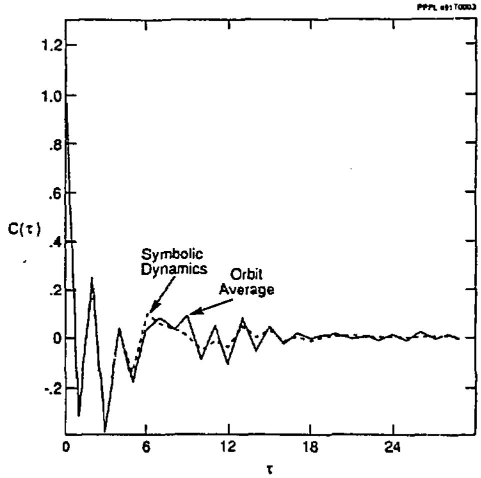

Figure 7: The correlation function for the Hénon map, $a = 1.4$, $b = -0.3$, $n = 4$.

This technique has also been applied to the Chrikov–Taylor map (Hamiltonian), with similar agreement found between orbit averages and the solution to the symbolic kinetic equation. The results indicate that the use of symbolic dynamics can lead to a rapid calculation of the statistical properties of chaotic systems, and give insight into their dynamical properties.

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References

[1]R.L. Devaney, **An Introduction to Chaotic Dynamical Systems**, (Addison Wesley, 1989), p.184, 189.

[2]A.B. Rechester and R.B. White, *Symbolic Kinetic Equation for a Chaotic Attractor*, preprint available from authors.