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A SHORTCUT TO HARD THERMAL LOOPS

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ABSTRACT

I review the generating functional of hard thermal loops derived by Taylor and Wong.

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1. Introduction

Dynamical processes in gauge theories at high temperature exhibit an unexpectedly rich structure. To compute consistently in perturbation theory it is necessary to resum a certain subclass of one loop diagrams, termed "hard thermal loops", which are the dominant loop diagrams at high temperature.[1-5] (For pedagogical discussions see Ref. 6.) With g the gauge coupling constant and T the temperature, hard thermal loops arise from one loop diagrams, proportional to g^2T^2 times functions of the momenta. They appear in the amplitudes between N gluons, and between a quark pair and $N-2$ gluons, for all $N \geq 2$.

The central miracle which occurs for hard thermal loops is that they are *all* gauge invariant. By this I mean that the hard thermal loop in any amplitude is independent of the choice of gauge. This was established between axial and Feynman gauges by Frenkel and Taylor,[3] and between arbitrary Coulomb and covariant gauges by Braaten and I.[2] It is crucial to establish gauge independence for arbitrary covariant gauges, for away from Feynman gauge individual diagrams have gauge dependent terms which are powers of $1/g$ times hard thermal loops. Yet when all diagrams which contribute to a given amplitude are summed together, the gauge dependence cancels, leaving the hard thermal loop in Feynman gauge.

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In their work Frenkel and Taylor[3] showed than an identity can be derived to simplify the hard thermal loop in the four–gluon amplitude. Braaten and I[4] generalized this to the hard thermal loops in arbitrary amplitudes. Using special properties of hard thermal loops, we derived identities which reduce the color structure of N–point functions to an elementary form: the amplitude is equal to the Casimir of the virtual field in the loop times a sum over terms, each of which has a color structure typical of an N–point function at tree level.

Taylor and Wong[5] demonstrated that these results can be summarized elegantly by means of a generating functional for hard thermal loops. They showed that if one starts with the hard thermal loop in the two–point functions (the self energies), and takes the gauge invariance of hard thermal loops for granted, then the complete generating functional is determined solely by gauge invariance.

In this article I review the generating functional of Taylor and Wong. For the gluon action I obtain a form which is a bit different from theirs, Eq. 15 below. While the derivation is almost elementary, it does take the gauge invariance of hard thermal loops for granted. This gauge invariance is unexpected, for hard thermal loops are off shell Green's functions which in general are gauge dependent. The proof of gauge invariance in covariant gauges[2] is not particularly difficult, but it is roundabout, relying upon induction and the Ward identities obeyed by hard thermal loops. Surely there is a more direct understanding of this apparent miracle.

2. Quark and gluon amplitudes

I start with the hard thermal loops between a quark pair and any number of gluons, for they are simpler than purely gluonic amplitudes.

In Feynman gauge the self energy of the quark is

$$\tilde{\Gamma}^2(P) \;=\; 2ig^2 C_f\, T \sum_{j=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3}\, \frac{\not{K}}{K^2(P-K)^2}\,. \tag{1}$$

P and K are four momenta; *e.g.*, $K^\mu = (k^0, \vec{k})$, with $\vec{k} = k\,\hat{k}$. $C_f = (N_c^2 - 1)/(2N_c)$ is the Casimir for a quark field in the fundamental representation of an $SU(N_c)$ gauge theory. I employ the imaginary time formalism at nonzero temperature, so the loop momenta $k^0 = (2j+1)\pi T$. The sum over k^0 can be done either by brute force, using contour integral representations, or by the

"Saclay trick", using fourier transforms of the propagators in imaginary time.[7] Either way, after the sum over k^0 is done there remains a sum over energy denominators:

$$\tilde{\Gamma}^2(P) = \frac{i g^2 C_f}{2} \int \frac{d^3k}{(2\pi)^3} \frac{(\tilde{n}(E_k) + n(E_{p-k}))}{E_k E_{p-k}} \left(\frac{k \; \hat{K}\!\!\!/}{ip^0 + E_k - E_{p-k}} + c.c. \right) + \ldots . \tag{2}$$

$E_k = k$ is the energy of the virtual quark, and $E_{p-k} = |\vec{p} - \vec{k}|$ the energy of the virtual gluon; $\tilde{n}(E)$ and $n(E)$ are the appropriate statistical distribution functions. I introduce the vector

$$\hat{K}^\mu = (i, \hat{k}) ,$$

which is lightlike, $\hat{K}^2 = 0$; note that $\hat{K}$ depends only upon the direction, and not the magnitude, of $\vec{k}$. I analytically continue $p^0 \to -iE$, and assume that the energy E and the momentum p of the external quark are both "soft," on the order of gT. The dominant term in the quark self energy occurs when the loop momentum k is "hard," on the order of T, with the energy denominator $E + E_k - E_{p-k} \sim P^\mu \hat{K}^\mu$. Then the hard thermal loop in the quark self energy, $\delta\tilde{\Gamma}^2$, is given by

$$\delta\tilde{\Gamma}^2(P) = - \frac{i g^2 T^2 C_f}{2} \int \frac{d\Omega}{4\pi} \frac{\hat{K}\!\!\!/}{P \cdot \hat{K}} . \tag{3}$$

For convienience I introduce the notation $P \cdot \hat{K} = P^\mu \hat{K}^\mu$, *etc.*

In Eq. 3 the integral over $\int dk$ decouples from the angular integral over $\hat{k}$, $\int d\Omega$. The integral over $\int dk$ is proportional to T^2, which defines the hard thermal loop; for soft $P \sim gT$, Eq. 3 is of order $g^2T^2/P{\cdot}\hat{K} \sim gT$. I remark that while there are other energy denominators besides those written in Eq. 2, Eq. 16c of Ref. 7, they do not contribute to the hard thermal loop. For example, there is an energy denominator $ip^0 + E_k + E_{p-k}$: after analytic continuation this is proportional to $2k \sim T$ at hard k, and only contributes to the quark self energy through terms of order g^2T, down by g from the hard thermal loop.

Although the angular integral in Eq. 3 can be done directly, it is best to leave it as is. Since the direction of the unit vector $\hat{k}$ is just an integration variable, if Eq. 3 is sandwiched between quark fields, to obtain $\delta\tilde{S}^2$, the effective action for the hard thermal loop in the quark self energy, we just replace the quark momentum, P_μ, with the operator $i\partial_\mu$:

$$\delta\tilde{S}^2 = \overline{\psi}\,\delta\tilde{\Gamma}^2\,\psi = - \frac{g^2 T^2 C_f}{2} \int \frac{d\Omega}{4\pi}\, \overline{\psi}\, \frac{\hat{K}\!\!\!/}{\partial \cdot \hat{K}}\, \psi . \tag{4}$$

Eq. 4 is very suggestive. Let the hard thermal loop between a quark pair and $N-2$ gluons be $\delta\tilde{\Gamma}^N$, so the generating functional of all hard thermal loops between a quark pair and any number of gluons is, schematically,

$$\delta\tilde{S} = \sum_{N=2}^{\infty} \overline{\psi}\, \delta\tilde{\Gamma}^N\, A^{N-2}\, \psi\, . \tag{5}$$

If we assume that $\delta\tilde{S}$ is gauge invariant, then it is obvious how to generalize from the two point function: in Eq. 4, merely replace the ordinary derivative, ∂_μ, with the covariant derivative in the fundamental representation, $D^f_\mu = \partial_\mu + igA_\mu$ ($A_\mu = t^a A^a_\mu$, with t^a the color matrix for the fundamental representation). Thus

$$\delta\tilde{S} = -\frac{g^2T^2\,C_f}{2}\int\frac{d\Omega}{4\pi}\,\overline{\psi}\,\frac{\hat{K}\!\!\!/}{D^f\cdot\hat{K}}\,\psi\, . \tag{6}$$

This is equivalent to Eq. 7 of Taylor and Wong.[5] Eq. 6 generates correlation functions in momentum space, so ∂_μ represents the momentum operator. Alternately, Taylor and Wong write $\delta\tilde{S}$ in coordinate space. Introducing the vector $\hat{K}^* = (-i,\hat{k})$, the position variable congugate to $P\cdot\hat{K}$ is $x_k = x\cdot\hat{K}^*$; then the fundamental propagator $1/D^f\cdot\hat{K}$ depends nontrivially only upon x_k, equal to a step function times a path ordered exponential of $igA\cdot\hat{K}$, Eqs. 5 and 6 of Ref. 5. I'm used to momentum space, so to me the $\delta\tilde{S}$ above is more familiar.

To be honest, $\delta\tilde{S}$ should have been obvious to us in Ref. 4. In Eq. 3.6 $\delta\tilde{\Gamma}^N$ is a sum over strings of t^a's, clearly the expansion of $1/D^f\cdot\hat{K}$. What we did not notice is that the angular integrals which enter in $\delta\tilde{\Gamma}^N$, Eq. 4.5, reduce to Eq. 6. But all one needs to show this is to plug the identity of Eq. 4.6 into Eq. 4.5.

3. Gluon amplitudes

While $\delta\tilde{S}$ follows easily from the quark self energy (and gauge invariance!), the same is not true for the generating functional of hard thermal loops in purely gluonic amplitudes, δS. The derivation of δS by Taylor and Wong[5] is an exquisite example of the power of gauge invariance.

Write the complete generating functional as a sum of two terms:

$$\delta S = \frac{2T^2}{3}\left(N_c+\frac{N_f}{2}\right)\, tr\left\{\int d^4x\,\frac{g^2}{2}\,A_0^2 + \int\frac{d\Omega}{4\pi}\;W(gA\cdot\hat{K})\right\} . \tag{7}$$

I assume that there are N_f flavors of massless quarks in the fundamental representation; tr is the trace over color. The prefactor in δS, $N_c + N_f/2$, is the sum of Casimir's for gluon and quark loops.

The first term on the right hand side is just the usual static electric mass for a gluon at nonzero temperature. The second term, $W(gA\cdot\hat{K})$, is the functional to be derived. I assume W is a functional of $gA\cdot\hat{K}$: the g is obvious, for W arises from one loop diagrams, where each factor of an external A_μ brings in one power of g. That A_μ enters only as $A\cdot\hat{K}$, with an integral over $\hat{k}$, is taken from experience with hard thermal loops.[2–4]

Consider first the static limit when $p^0 = 0$ for all gauge fields. This can be studied in imaginary time, where it is well known that the only hard thermal loop is the static electric mass, with $W = 0$. This is consistent with the gauge invariance of δS, for under static gauge transformations A_0 transforms like a colored scalar field, and $tr(A_0^2)$ is gauge invariant all on its own.

Matters are very different after analytic continuation, $p^0 \to -iE$. Now the allowed gauge transformations are time dependent, and A_μ transforms as

$$\delta_\omega A_\mu \;=\; e^{-ig\omega}\left(A_\mu + \frac{1}{ig}\partial_\mu\right)e^{ig\omega}\,, \tag{8}$$

$\omega = \omega^a t^a$. For small ω,

$$\delta_\omega A_\mu \;=\; D_\mu\omega \;=\; (\partial_\mu + ig[A_\mu,\,)\,\omega\,; \tag{9}$$

D_μ is the adjoint covariant derivative.

Unlike the static case, the electric mass $tr(A_0^2)$ is clearly not invariant under Eqs. 8 and 9. For an infintesimal gauge transformation,

$$\int d^4x\,\delta_\omega\, tr\left(\frac{g^2}{2}\,A_0^2\right) \;=\; \int d^4x\, tr(-\,g^2\,\partial_0 A_0\,\omega)\,. \tag{10}$$

The commutator term in the covariant derivative drops out because $tr(A_0\,[A_0,\omega]) = -tr([A_0,A_0]\omega) = 0$; Eq. 10 follows after integrating by parts with respect to ∂_0. Next comes a trick: to bring it into a form similar to that of W, I introduce an integral over $\int d\Omega$, and use the fact that $\int d\Omega\;\vec{A}\cdot\hat{k} = 0$. Then Eq. 10 equals

$$\int d^4x \int \frac{d\Omega}{4\pi}\, tr(\,i\,g^2\,\partial_0(A\cdot\hat{K})\omega)\,. \tag{11}$$

If the generating functional δS is gauge invariant, the gauge transformation of the electric mass in Eq. 11 must be cancelled by that of W. For small ω,

$$\delta_\omega\, W(gA\cdot\hat{K}) \;=\; \int d^4x\,\frac{\delta W}{\delta(A\cdot\hat{K})}\,\delta_\omega\left(A\cdot\hat{K}\right) \;=\; \int d^4x\,\frac{\delta W}{\delta(A\cdot\hat{K})}\,D\cdot\hat{K}\,\omega\,. \tag{12}$$

Thus δS is gauge invariant if

$$\frac{\delta W}{\delta(A\cdot\hat{K})} = \int d^4y\,(-ig^2)\partial_0(A\cdot\hat{K})\,\frac{1}{D\cdot\hat{K}}\,. \tag{13}$$

Because W is a functional of $gA\cdot\hat{K}$, Eq. 13 can be used to compute the derivative of W with respect to g:

$$\frac{dW}{dg} = \int d^4x\,\frac{1}{g}\,\frac{\delta W}{\delta(A\cdot\hat{K})}\,\frac{d(gA\cdot\hat{K})}{dg} = \int d^4x \int d^4y\,(-ig)\,\partial_0(A\cdot\hat{K})\,\frac{1}{D\cdot\hat{K}}\,A\cdot\hat{K}\,. \tag{14}$$

This is equivalent to Eq. 22 of Taylor and Wong.[5] Their form is given by writing the adjoint propagator, $1/D\cdot\hat{K}$, in coordinate space.

Taylor and Wong express W as the integral of dW/dg with respect to g. I make the trivial observation that since g only enters dW/dg through the combination $g/D\cdot\hat{K} = g/(\partial + ig[A,)\cdot\hat{K}$, at least formally it is possible to integrate Eq. 14 with respect with g. The final result for W is:

$$W = \int d^4x \int d^4y\;\partial_0(A\hat{K})\frac{1}{[A\cdot\hat{K},}\left\{-g+\;\partial\cdot\hat{K}\,log\left(\frac{1}{\partial\cdot\hat{K}}D\cdot\hat{K}\right)\frac{1}{i[A\cdot\hat{K},}\right\}A\hat{K}\,. \tag{15}$$

In this expression there is strict ordering of the operators: *e.g.*, the expansion of the logarithm is as a power series in $(1/\partial\cdot\hat{K})\,ig[A\cdot\hat{K},$.

Eq. 15 can be checked in two limits. First, since it starts with a time derivative, $\partial_0(A\cdot\hat{K})$, W vanishes if all momenta are static, which is consistent with the euclidean analysis discussed above. Secondly, we can consider the analogue of Eqs. 14 and 15 in the abelian theory. For hot QED the hard thermal loops in the N–photon functions vanish for all $N > 2$.[4] To see this, follow Ref. 5 and write $D\cdot\hat{K} = \Omega^{-1}\partial\cdot\hat{K}\Omega$, where $\Omega(x_k) = exp(ie\int_{-\infty}^{x_k} A(x'_k)\cdot\hat{K}\,dx'_k)$, $x_k = x\cdot\hat{K}^*$. With path ordering $D\cdot\hat{K}$ can be written in this form for either abelian or nonabelian fields. For nonabelian fields Ω doesn't commute with $A\cdot\hat{K}$, and this representation of $D\cdot\hat{K}$ does not reduce Eqs. 14 or 15. For abelian fields, however, Ω can be commuted through. Thus Eq. 14, which starts out with the two–point function, stops there. This is less obvious with Eq. 15: because of the factors of $1/[A\cdot\hat{K},$, the term quadratic in $A\cdot\hat{K}$ does not vanish, but all higher powers do.

The effective actions in Eqs. 6, 7, and 15 agree with the diagrammatic analysis of Ref. 4. Each is proportional to the Casimir of the virtual field in the loop: the quark action $\delta\tilde{S}$ to C_f, the gluon action δS to $N_c + N_f/2$. The expansion of $\delta\tilde{S}$ and δS in powers of $A\cdot\hat{K}$ gives a color structure typical of a tree amplitude. For the quark action the expansion of the fundamental covariant derivative produces a string of t^a's, Eq. 3.6. For the gluon action the expansion of the adjoint covariant derivative in powers of $ig[A\cdot\hat{K}$, produces a series of nested commutators which reduce to a string of structure constants, Eqs. 2.17 and 2.20. As discussed at the end of Sec. 2 it is easy to see that the angular integrals generated by the expansion of Eq. 6 agrees with Ref. 4. This is not at all obvious for the gluon action in Eqs. 7 and 15. This is best shown by checking that the two– and three–point functions agree. Given that, the Ward identities satisfied by hard thermal loops imply that all higher point functions concur (in the spirit of Eq. 2.14).

I conclude with speculation. The gluon action δS is reminiscent of the Wess–Zumino action in two space–time dimensions: as W is added to $tr(A_0^2)$ to make δS gauge invariant, so in the Wess–Zumino action is a complicated functional added to $tr(A_\mu^2)$ to make the sum gauge invariant. Perhaps this implies that there is something topological about δS, and is why it is proportional to a rational number, $N_c + N_f/2$.

4. References

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