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Matrix Models of 2d Gravity

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These are introductory lectures for a general audience that give an overview of the subject of matrix models and their application to random surfaces, 2d gravity, and string theory. They are intentionally 1.5 years out of date.

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0. Canned Diatribe, Introduction, and Apologies

Following the discovery of spacetime anomaly cancellation in 1984 [1], string theory has undergone rapid development in several directions. The early hope of making direct contact with conventional particle physics phenomenology has however long since dissipated, and there is as yet no experimental program for finding even indirect manifestations of underlying string degrees of freedom in nature [2]. The question of whether string theory is “correct” in the physical sense thus remains impossible to answer for the foreseeable future. String theorists nonetheless continue to be tantalized by the richness of the theory and by its natural ability to provide a consistent microscopic underpinning for both gauge theory and gravity.

One of the prime obstacles to our understanding of string theory has been an inability to penetrate beyond its perturbative expansion. Our understanding of gauge theory is enormously enhanced by having a fundamental formulation based on the principle of local gauge invariance from which the perturbative expansion can be derived. Symmetry breaking and nonperturbative effects such as instantons admit a clean and intuitive presentation. In string theory, our lack of a fundamental formulation is compounded by our ignorance of the true

ground state of the theory. Roughly two years ago, there was some progress [3-5] towards extracting such nonperturbative information from string theory, at least in some simple contexts. The aim of these lectures is to provide the conceptual background for this work, and to describe some of its immediate consequences.

In string theory we wish to perform an integral over two dimensional geometries and a sum over two dimensional topologies,

$$Z \sim \sum_{\text{topologies}} \int \mathcal{D}g \mathcal{D}X e^{-S},$$

where the spacetime physics (in the case of the bosonic string) resides in the conformally invariant action

$$S \propto \int d^2\xi \sqrt{g} g^{ab} \partial_a X^\mu \cdot \partial_b X^\nu G_{\mu\nu}(X).$$

Here μ, ν run from $1, \dots, D$ where D is the number of spacetime dimensions, $G_{\mu\nu}(X)$ is the spacetime metric, and the integral $\mathcal{D}g$ is over worldsheet metrics. Typically we write the worldsheet metric $g_{ab} = e^\varphi \delta_{ab}$, where φ is known as the Liouville mode. Following the formulation of string theory in this form (and in particular following the appearance of [6]), there was much work to develop the quantum Liouville theory (some of which is reviewed in section 4 here), and conformal field theory itself has been characterized as “an unsuccessful attempt to solve Liouville theory” [7]. Evaluation of the partition function Z above without taking into account the integral over geometry, however, does not solve the problem of interest, and moreover does not provide a systematic basis for a perturbation series in any known parameter.

The basic idea of [3-5] relied on a discretization of the string worldsheet to provide a method of taking the continuum limit which incorporated simultaneously the contribution of 2d surfaces with any number of handles. At one fowl swoop, it was thus possible not only to integrate over all possible deformations of a given genus surface (the analog of the integral over Feynman parameters for a given loop diagram), but also to sum over all genus. This would in principle free us from the mathematically fascinating but physically irrelevant problems

of calculating conformal field theory correlation functions on surfaces of fixed genus with fixed moduli (objects which we never knew how to integrate over moduli or sum over genus anyway). The progress, however, is limited in the sense that these methods only apply currently for non-critical strings embedded in dimensions $D \leq 1$ (or critical strings embedded in $D \leq 2$), and the nonperturbative information even in this restricted context has proven incomplete. Due to familiar problems with lattice realizations of supersymmetry and chiral fermions, these methods have also resisted extension to the supersymmetric case.

The developments we shall describe here nonetheless provide at least a half-step in the correct direction, if only to organize the perturbative expansion in a most concise way. They have also prompted much useful evolution of related continuum methods. Our point of view here is that string theories embedded in $D \leq 1$ dimensions provide a simple context for testing ideas and methods of calculation. Just as we would encounter much difficulty calculating infinite-dimensional functional integrals without some prior experience with their finite-dimensional analogs [8], progress in string theory should be aided by experimentation with systems possessing a restricted number of degrees of freedom.

These notes have been confined in content essentially to the lectures actually given, in order to keep them reasonably short and accessible. (Other review references on the same general subject are [9,10]). This means that we stop well short of some of the more interesting recent developments in the field (some of which were covered by later lecturers at this school), including the application of the critical $D = 2$ dimensional models to address issues of principle such as topology change in 2d quantum gravity, and their relation as well to recent work on $D = 2$ black holes in string theory. We shall present no formal conclusions here other than to note that the subject remains in active development, and we have tried at various points in the text to draw attention to issues in need of further understanding.

1. Discretized surfaces, matrix models, and the continuum limit

1.1. Discretized surfaces

We begin by considering a “ $D = 0$ dimensional string theory”, i.e. a pure theory of surfaces with no coupling to additional “matter” degrees of freedom on the string worldsheet. This is equivalent to the propagation of strings in a non-existent embedding space. For partition function we take

$$Z = \sum_h \int \mathcal{D}g e^{-\beta A + \gamma \chi}, \quad (1.1)$$

where the sum over topologies is represented by the summation over h , the number of handles of the surface, and the action consists of couplings to the area $A = \int \sqrt{g}$, and to the Euler character $\chi = \frac{1}{4\pi} \int \sqrt{g} R = 2 - 2h$.

The integral $\int \mathcal{D}g$ over the metric on the surface in (1.1) is difficult to calculate in general. The most progress in the continuum has been made via the Liouville approach which we briefly review in section 4. If we discretize the surface, on the other hand, it turns out that (1.1) is much easier to calculate, even before removing the finite cutoff. We consider in particular a “random triangulation” of the surface [11], in which the surface is constructed from triangles, as in fig. 1. The triangles are designated to be equilateral,¹ so that there is positive (negative) curvature at vertices i where the number N_i of incident triangles is more (less) than six, and zero curvature when $N_i = 6$. The summation over all such random triangulations is thus the discrete analog to the integral $\int \mathcal{D}g$ over all possible geometries,

$$\sum_{\text{genus } h} \int \mathcal{D}g \rightarrow \sum_{\text{random triangulations}}. \quad (1.2)$$

¹ We point out that this constitutes a basic difference from the Regge calculus, in which the link lengths are geometric degrees of freedom. Here the geometry is encoded entirely into the coordination numbers of the vertices.

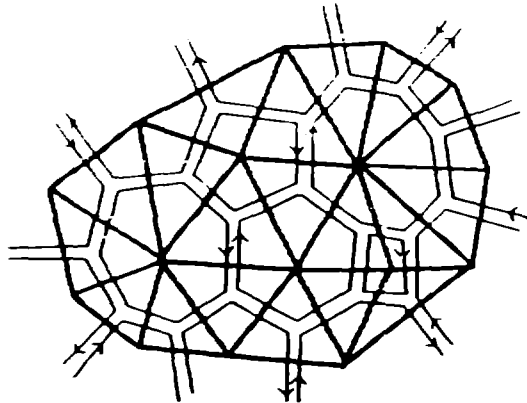


Fig. 1: A piece of a random triangulation of a surface. Each of the triangular faces is dual to a three point vertex of a quantum mechanical matrix model.

The discrete counterpart to the infinitesimal volume element \sqrt{g} is $\sigma_i = N_i/3$, so that the total area $|S| = \sum_i \sigma_i$ just counts the total number of triangles, each designated to have unit area. (The factor of $1/3$ in the definition of σ_i is because each triangle has three vertices and is counted three times.) The discrete counterpart to the Ricci scalar R at vertex i is $R_i = \pi(6 - N_i)/N_i$, so that

$$\int \sqrt{g} R \rightarrow \sum_i 2\pi(1 - N_i/6) = 2\pi(V - \frac{1}{2}F) = 2\pi(V - E + F) = 2\pi\chi.$$

Here we have used the simplicial definition which gives the Euler character χ in terms of the total number of vertices, edges, and faces V , E , and F of the triangulation (and we have used the relation $3F = 2E$ obeyed by triangulations of surfaces, since each face has three edges each of which is shared by two faces).

In the above, triangles do not play an essential role and may be replaced by any set of polygons. General random polygonifications of surfaces with appropriate fine tuning of couplings may, as we shall see, have more general critical behavior, but can in particular always reproduce the pure gravity behavior of triangulations in the continuum limit.

1.2. Matrix models

We now demonstrate how the integral over geometry in (1.1) may be performed in its discretized form as a sum over random triangulations. The trick is to use a certain matrix integral as a generating functional for random triangulations. The essential idea goes back to work [12] on the large N limit of QCD, followed by work on the saddle point approximation [13].

We first recall the (Feynman) diagrammatic expansion of the (0-dimensional) field theory integral⁴

$$\int_{-\infty}^{\infty} \frac{d\varphi}{\sqrt{2\pi}} e^{-\varphi^2/2 + \lambda\varphi^4/4!}, \quad (1.3)$$

where φ is an ordinary real number.³ In a formal perturbation series in λ , we would need to evaluate integrals such as

$$\frac{\lambda^n}{n!} \int_{\varphi} e^{-\varphi^2/2} \left(\frac{\varphi^4}{4!}\right)^n. \quad (1.4)$$

Up to overall normalization we can write

$$\int_{\varphi} e^{-\varphi^2/2} \varphi^{2k} = \frac{\partial^{2k}}{\partial J^{2k}} \int_{\varphi} e^{-\varphi^2/2 + J\varphi} \Big|_{J=0} = \frac{\partial^{2k}}{\partial J^{2k}} e^{J^2/2} \Big|_{J=0}. \quad (1.5)$$

Since $\frac{\partial}{\partial J} e^{J^2/2} = J e^{J^2/2}$, applications of $\partial/\partial J$ in the above need to be paired so that any factors of J are removed before finally setting $J = 0$. Therefore if we represent each “vertex” $\lambda\varphi^4$ diagrammatically as a point with four emerging lines (see fig. 2b), then (1.4) simply counts the number of ways to group such objects in pairs. Diagrammatically we represent the possible pairings by connecting lines between paired vertices. The connecting line is known as the propagator $\langle\varphi\varphi\rangle$ (see fig. 2a) and the diagrammatic rule we have described for connecting vertices in pairs is known in field theory as the Wick expansion.

² We apologize for this recapitulation of standard Feynman diagram technology, but prefer to keep these notes at least marginally accessible to the mathematics community.

³ The integral is understood to be defined by analytic continuation to negative λ

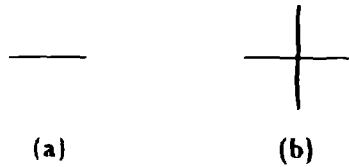


Fig. 2: (a) the scalar propagator. (b) the scalar four-point vertex.

When the number of vertices n becomes large, the allowed diagrams begin to form a mesh reminiscent of a 2-dimensional surface. Such diagrams do not yet have enough structure to specify a Riemann surface. The additional structure is given by widening the propagators to ribbons (to give so-called “thick” graphs). From the standpoint of (1.3), the required extra structure is given by replacing the scalar φ by an $N \times N$ hermitian matrix M^i_j . The analog of (1.5) is given by adding indices and traces:

$$\begin{aligned} \int_M e^{-\text{tr}M^2/2} M^{i_1 j_1} \dots M^{i_n j_n} &= \frac{\partial}{\partial J^{j_1 i_1}} \dots \frac{\partial}{\partial J^{j_n i_n}} e^{-\text{tr}M^2/2 + \text{tr}JM} \Big|_{J=0} \\ &= \frac{\partial}{\partial J^{j_1 i_1}} \dots \frac{\partial}{\partial J^{j_n i_n}} e^{\text{tr}J^2/2} \Big|_{J=0}, \end{aligned} \quad (1.6)$$

where the source J^i_j is as well now a matrix. The measure in (1.6) is the invariant $dM = \prod_i dM^i_i \prod_{i < j} d\text{Re}M^i_j d\text{Im}M^i_j$, and the normalization is such that $\int_M e^{-\text{tr}M^2/2} = 1$. To calculate a quantity such as

$$\frac{\lambda^n}{n!} \int_M e^{-\text{tr}M^2/2} (\text{tr}M^4)^n, \quad (1.7)$$

we again lay down n vertices (now of the type depicted in fig. 3b), and connect the legs with propagators (M^i_j, M^k_l) = $\delta^i_j \delta^k_l$ (fig. 3a). The presence of upper and lower matrix indices is represented in fig. 3 by the double lines⁴ and it is understood that the sense of the arrows is to be preserved when linking together vertices. The resulting diagrams are similar to those of the scalar theory, except

⁴ This is the same notation employed in the large N expansion of QCD [12].

that each external line has an associated index i , and each internal closed line corresponds to a summation over an index $j = 1, \dots, N$. The “thickened” structure is now sufficient to associate a Riemann surface to each diagram, because the closed internal loops uniquely specify locations and orientations of faces.

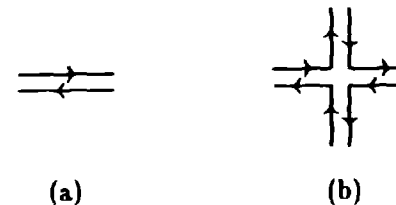


Fig. 3: (a) the hermitian matrix propagator. (b) the hermitian matrix four-point vertex.

To make contact with the random triangulations discussed earlier, we consider the diagrammatic expansion of the matrix integral

$$e^Z = \int dM e^{-\frac{1}{2}\text{tr}M^2 + \frac{g}{N}\text{tr}M^3} \quad (1.8)$$

(with M an $N \times N$ hermitian matrix, and the integral again understood to be defined by analytic continuation in the coupling g .) The term of order g^n in a power series expansion counts the number of diagrams constructed with n 3pt vertices. The dual to such a diagram (in which each face, edge, and vertex is associated respectively to a dual vertex, edge, and face) is identically a random triangulation inscribed on some orientable Riemann surface (fig. 1). We see that the matrix integral (1.8) automatically generates all such random triangulations.⁵ Since each triangle has unit area, the area of the surface is just n . We can thus make formal identification with (1.1) by setting $g = e^{-\beta}$. Actually the matrix integral generates both connected and disconnected

⁵ Had we used real symmetric matrices rather than the hermitian matrices M , the two indices would be indistinguishable and there would be no arrows in the propagators and vertices of fig. 3. Such orientationless vertices and propagators generate an ensemble of both orientable and non-orientable surfaces.

surfaces, so we have written e^Z on the left hand side of (1.8). As familiar from field theory, the exponential of the connected diagrams generates all diagrams, so Z as defined above represents contributions only from connected surfaces. We see that the free energy from the matrix model point of view is actually the partition function Z from the 2d gravity point of view.

There is additional information contained in N , the size of the matrix. If we change variables $M \rightarrow M\sqrt{N}$ in (1.8), the matrix action becomes $N \text{tr}(-\frac{1}{2}\text{tr}M^2 + g\text{tr}M^3)$, with an overall factor of N .⁶ This normalization makes it easy to count the power of N associated to any diagram. Each vertex contributes a factor of N , each propagator (edge) contributes a factor of N^{-1} (because the propagator is the inverse of the quadratic term), and each closed loop (face) contributes a factor of N due to the associated index summation. Thus each diagram has an overall factor

$$N^{V-E+F} = N^\chi = N^{2-2h}, \quad (1.9)$$

where χ is the Euler character of the surface associated to the diagram. We observe that the value $N = e^\gamma$ makes contact with the coupling γ in (1.1). In conclusion, if we take $g = e^{-\beta}$ and $N = e^\gamma$, we can formally identify the continuum limit of the partition function Z in (1.8) with the Z defined in (1.1). The metric for the discretized formulation is not smooth, but one can imagine how an effective metric on larger scales could arise after averaging over local irregularities. In the next subsection, we shall see explicitly how this works.

(Actually (1.8) automatically calculates (1.1) with the measure factor in (1.2) corrected to $\sum_S \frac{1}{|G(S)|}$, where $|G(S)|$ is the order of the (discrete) group of symmetries of the triangulation S . This is familiar from field theory where diagrams with symmetry result in an incomplete cancellation of $1/n!$'s such as in (1.4) and (1.7). The symmetry group $G(S)$ is the discrete analog of the isometry group of a continuum manifold.)

⁶ Note that N remains distinguished from the coupling g in the model, since it enters as well into the traces via the $N \times N$ size of the matrix.

The graphical expansion of (1.8) enumerates graphs as shown in fig. 1, where the triangular faces that constitute the random triangulation are dual to the 3-point vertices. Had we instead used 4-point vertices as in fig. 3b, then the dual surface would have square faces (a ‘‘random squarification’’ of the surface), and higher point vertices ($g_k/N^{k/2-1}$) $\text{tr}M^k$ in the matrix model would result in more general ‘‘random polygonifications’’ of surfaces. (The powers of N associated with the couplings are chosen so that the rescaling $M \rightarrow M\sqrt{N}$ results in an overall factor of N multiplying the action, so that the argument leading to (1.9) remains valid, and the power of N continues to measure the Euler character of a surface constructed from arbitrary polygons.) The different possibilities for generating vertices constitute additional degrees of freedom that can be realized as the coupling of 2d gravity to different varieties of matter in the continuum limit.

1.3. The continuum limit

From (1.9), it follows that we may expand Z in powers of N ,

$$Z(g) = N^2 Z_0(g) + Z_1(g) + N^{-2} Z_2(g) + \dots = \sum N^{2-2h} Z_h(g), \quad (1.10)$$

where Z_h gives the contribution from surfaces of genus h . In the conventional large N limit, we take $N \rightarrow \infty$ and only Z_0 , the planar surface (genus zero) contribution, survives. Z_0 itself may be expanded in a perturbation series in the coupling g , and for large order n behaves as (see [14] for a review)

$$Z_0(g) \sim \sum_n n^{\gamma-3} (g/g_c)^n \sim (g_c - g)^{2-\gamma}. \quad (1.11)$$

These series thus have the property that they diverge as g approaches some critical coupling g_c . We can extract the continuum limit of these surfaces by tuning $g \rightarrow g_c$. This is because the expectation value of the area of a surface is given by

$$\langle A \rangle = \langle n \rangle = \frac{\partial}{\partial y} \ln Z_0(g) \sim \frac{1}{g - g_c}$$

(recall that the area is proportional to the number of vertices n , which appears as the power of the coupling in the factor g^n associated to each graph). As

$\gamma \rightarrow \gamma_c$, we see that $A \rightarrow \infty$ so that we may rescale the area of the individual triangles to zero, thus giving a continuum surface with finite area. Intuitively, by tuning the coupling to the point where the perturbation series diverges the integral becomes dominated by diagrams with infinite numbers of vertices, and this is precisely what we need to define continuum surfaces.

There is no direct proof as yet that this procedure for defining continuum surfaces is “correct”, i.e. that it coincides with the continuum definition (1.1). We are able, however, to compare properties of the partition function and correlation functions calculated by matrix model methods with those (few) properties that can be calculated directly in the continuum (for a review, see [15]). This gives implicit confirmation that the matrix model approach is sensible and gives reason to believe other results derivable by matrix model techniques (e.g. for higher genus) that are not obtainable at all by continuum methods.

One of the properties of these models derivable via the continuum Liouville approach is a “critical exponent” γ , defined in terms of the area dependence of the partition function for surfaces of fixed large area A as

$$Z(A) \sim A^{(\gamma-2)/2-1}. \quad (1.12)$$

To anticipate some relevant results, we recall that the unitary discrete series of conformal field theories is labelled by an integer $m \geq 2$ and has central charge $D = 1 - 6/m(m+1)$ (for a review, see e.g. [16]), where the central charge is normalized such that $D = 1$ corresponds to a single free boson. If we couple conformal field theories with these fractional values of D to 2d gravity, the continuum Liouville theory prediction for the exponent γ is (see section 4)

$$\gamma = \frac{1}{12} (D - 1 - \sqrt{(D-1)(D-25)}) = -\frac{1}{m}. \quad (1.13)$$

The case $m = 2$, for example, corresponds to $D = 0$ and hence $\gamma = -\frac{1}{2}$ for pure gravity. The next case $m = 3$ corresponds to $D = 1/2$, i.e. to a $1/2$ -boson or fermion. This is the conformal field theory of the critical Ising model, and we learn from (1.13) that the Ising model coupled to 2d gravity has $\gamma = -\frac{1}{3}$. Notice that (1.13) ceases to be sensible for $D > 1$. This is the first indication of a “barrier” at $D = 1$ which will reappear in various guises in what follows.

In section 2 we shall present the solution to the matrix model formulation of the problem, and the value of the exponent γ provides a coarse means of determining which specific continuum model results from taking the continuum limit of a particular matrix model. Indeed the coincidence of γ and other scaling exponents (to be defined in section 4) calculated from the two points of view were originally the only evidence that the continuum limit of matrix models was a suitable definition for the continuum problem of interest. In the past year, the simplicity of matrix model results for correlation functions has spurred a rapid evolution of continuum Liouville technology so that as well many correlation functions can be computed in both approaches and are found to coincide.⁷

1.4. The double scaling limit

Thus far we have discussed the naive $N \rightarrow \infty$ limit which retains only planar surfaces. It turns out that the successive coefficient functions $Z_h(g)$

⁷ By way of very superficial overview: following the confirmation that the matrix model approach reproduced the scaling results of [17], some 3-point couplings for order parameters at genus zero were calculated in [18] from the standpoint of ADE face models on fluctuating lattices. The connection to KdV (reviewed in section 3 here) was made in [19], and then general correlations of order parameters (not yet known in the continuum) were calculated in [20]. The first step in the calculation of continuum correlators was provided in [21], where the free field formulation by zero mode integration of the Liouville field was established. This was employed in [22] together with a necessary analytic continuation of the scaling parameter to calculate some continuum correlation functions: the incorporation of the Liouville mode was shown to cancel the ghastly assemblage of Γ -functions familiar from the conformal field theory result and reproduce the relatively simple matrix model result. Additional genus zero correlation functions for $D \leq 1$ were then computed in [23]. The genus one partition function for the AD series was calculated via KdV methods in [20], and was confirmed from the continuum Liouville approach in [24]. For $D = 1$, the matrix model approach of [25,26] was used in [27] (also [28,29]) to calculate a variety of correlation functions. These were also calculated in the collective field approach [30] where up to 6-point amplitudes were derived, and found to be in agreement with the Liouville results of [23].

in (1.10) as well diverge at the same critical value of the coupling $g = g_c$ (this should not be surprising since the divergence of the perturbation series is a local phenomenon and should not depend on global properties such as the effective genus of a diagram). As we shall see in the next section, for the higher genus contributions (1.11) is generalized to

$$Z_h(g) \sim \sum_n n^{(\gamma-2)h/2-1} (g/g_c)^n \sim (g_c - g)^{(2-\gamma)h/2}. \quad (1.14)$$

We see that the contributions from higher genus are enhanced as $g \rightarrow g_c$. This suggests that if we take the limits $N \rightarrow \infty$ and $g \rightarrow g_c$ not independently, but together in a correlated manner, we may compensate the large N high genus suppression with a $g \rightarrow g_c$ enhancement. This would result in a coherent contribution from all genus surfaces [3-5].

To see how this works explicitly, we write the leading singular piece of the $Z_h(g)$ as

$$Z_h(g) \sim f_h (g - g_c)^{(2-\gamma)h/2}.$$

Then in terms of

$$\kappa^{-1} \equiv N(g - g_c)^{(2-\gamma)/2}, \quad (1.15)$$

the expansion (1.10) can be rewritten⁸

$$Z = \kappa^{-2} f_0 + f_1 + \kappa^2 f_2 + \dots = \sum_h \kappa^{2h-2} f_h. \quad (1.16)$$

The desired result is thus obtained by taking the limits $N \rightarrow \infty$, $g \rightarrow g_c$ while holding fixed the "renormalized" string coupling κ of (1.15). This is known as the "double scaling limit".

⁸ Strictly speaking the first two terms here have additional non-universal pieces that need to be subtracted off.

2. All genus partition functions

2.1. Orthogonal polynomials

In order to justify the claims made at the end of the previous section, we introduce some formalism to solve the matrix models. We begin by rewriting the partition function (1.8) in the form

$$e^Z = \int dM e^{-\text{tr}V(M)} = \int \prod_{i=1}^N d\lambda_i \Delta^2(\lambda) e^{-\sum_i V(\lambda_i)}, \quad (2.1)$$

where we now allow a general polynomial potential $V(M)$. In (2.1), the λ_i 's are the N eigenvalues of the hermitian matrix M , and

$$\Delta(\lambda) = \prod_{i<j} (\lambda_j - \lambda_i) \quad (2.2)$$

is the Vandermonde determinant.⁹ Due to antisymmetry in interchange of any two eigenvalues, (2.2) can be written $\Delta(\lambda) = \det \lambda_i^{j-1}$ (where the normalization is determined by comparing leading terms). In the case $N = 3$ for example we have

$$(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) = \det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}.$$

⁹ (2.1) may be derived via the usual Fadeev-Popov method: Let U_0 be the unitary matrix such that $M = U_0^\dagger \Lambda U_0$, where Λ is a diagonal matrix with eigenvalues λ_i . The right hand side of (2.1) follows by substituting the definition $1 = \int dU \delta(U M U^\dagger - \Lambda) \Delta^2(\lambda)$ (where $\int dU \equiv 1$). We first perform the integration over M , and then U decouples due to the cyclic invariance of the trace so the integration over U is trivial, leaving only the integral over the eigenvalues λ_i of Λ . To determine $\Delta(\lambda)$, we note that only the infinitesimal neighborhood $U = (1 + T)U_0$ contributes to the U integration, so that

$$1 = \int dU \delta(U M U^\dagger - \Lambda) \Delta^2(\lambda) = \int dT \delta([T, \Lambda]) \Delta^2(\lambda).$$

Now $[T, \Lambda]_{ij} = T_{ij}(\lambda_j - \lambda_i)$, so (2.2) follows (up to a sign) since we integrate over both real and imaginary parts of the off diagonal T_{ij} 's.

The now standard method for solving (2.1) makes use of an infinite set of polynomials $P_n(\lambda)$, orthogonal with respect to the measure

$$\int_{-\infty}^{\infty} d\lambda e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{nm} . \quad (2.3)$$

The P_n 's are known as orthogonal polynomials and are functions of a single real variable λ . Their normalization is given by having leading term $P_n(\lambda) = \lambda^n + \dots$ hence the constant h_n on the r.h.s. of (2.3). Due to the relation

$$\Delta(\lambda) = \det \lambda_i^{j-1} = \det P_{j-1}(\lambda_i) \quad (2.4)$$

(recall that arbitrary polynomials may be built up by adding linear combinations of preceding columns, a procedure that leaves the determinant unchanged), the polynomials P_n can be employed to solve (2.1). We substitute the determinant $\det P_{j-1}(\lambda_i) = \sum (-1)^P \prod_k P_{i_k-1}(\lambda_k)$ for each of the $\Delta(\lambda)$'s in (2.1). The integrals over individual λ_i 's factorize, and due to orthogonality the only contributions are from terms with all $P_i(\lambda_j)$'s paired. There are $N!$ such terms so (2.1) reduces to

$$e^Z = N! \prod_{i=0}^{N-1} h_i = N! h_0^N \prod_{k=1}^{N-1} f_k^{N-k} , \quad (2.5)$$

where we have defined $f_k \equiv h_k/h_{k-1}$.

In the naive large N limit (the planar limit), the rescaled index k/N becomes a continuous variable ξ that runs from 0 to 1, and f_k/N becomes a continuous function $f(\xi)$. In this limit, the partition function (up to an irrelevant additive constant) reduces to a simple one-dimensional integral:

$$\frac{1}{N^2} Z = \frac{1}{N} \sum_k (1 - k/N) \ln f_k \sim \int_0^1 d\xi (1 - \xi) \ln f(\xi) . \quad (2.6)$$

To derive the functional form for $f(\xi)$, we assume for simplicity that the potential $V(\lambda)$ in (2.3) is even. Since the P_i 's form a complete set of basis vectors in the space of polynomials, it is clear that $\lambda P_n(\lambda)$ must be expressible as a linear combination of lower P_i 's, $\lambda P_n(\lambda) = \sum_{i=0}^{n+1} a_i P_i(\lambda)$ (with

$a_i = h_i^{-1} \int e^{-V} \lambda P_n P_i$). In fact, the orthogonal polynomials satisfy the simple recursion relation,

$$\lambda P_n = P_{n+1} + r_n P_{n-1} , \quad (2.7)$$

with r_n a scalar coefficient independent of λ . This is because any term proportional to P_n in the above vanishes due to the assumption that the potential is even, $\int e^{-V} \lambda P_n P_n = 0$. Terms proportional to P_i for $i < n-1$ also vanish since $\int e^{-V} P_n \lambda P_i = 0$ (recall λP_i is a polynomial of order at most P_{i+1} which is orthogonal to P_n for $i+1 < n$).

By considering the quantity $P_n \lambda P_{n-1}$ with λ paired alternately with the preceding or succeeding polynomial, we derive

$$\int e^{-V} P_n \lambda P_{n-1} = r_n h_{n-1} = h_n .$$

This shows that the ratio $f_n = h_n/h_{n-1}$ for this simple case¹⁰ is identically the coefficient defined by (2.7), $f_n = r_n$. Similarly if we pair the λ in $P'_n \lambda P_n$ before and afterwards, integration by parts gives

$$n h_n = \int e^{-V} P'_n \lambda P_n = \int e^{-V} P'_n r_n P_{n-1} = r_n \int e^{-V} V' P_n P_{n-1} . \quad (2.8)$$

This is the key relation that will allow us to determine r_n .

2.2. The genus zero partition function

Our intent now is to find an expression for $f_n = r_n$ and substitute into (2.6) to calculate a partition function. For definiteness, we take as example the potential

$$V(\lambda) = \frac{1}{2g} \left(\lambda^2 + \frac{\lambda^4}{N} + b \frac{\lambda^6}{N^2} \right) , \quad (2.9)$$

with derivative $gV'(\lambda) = \lambda + 2 \frac{\lambda^3}{N} + 3b \frac{\lambda^5}{N^2} .$

The right hand side of (2.8) involves terms of the form $\int e^{-V} \lambda^{2p-1} P_n P_{n-1}$. According to (2.7), these may be visualized as "walks" of $2p-1$ steps ($p-1$

¹⁰ In other models, e.g. multimatrix models, $f_n = h_n/h_{n-1}$ has a more complicated dependence on recursion coefficients.

steps up and p steps down) starting at n and ending at $n - 1$, where each step down from m to $m - 1$ receives a factor of r_m and each step up receives a factor of unity. The total number of such walks is given by $\binom{2p-1}{p}$, and each results in a final factor of h_{n-1} (from the integral $\int e^{-V} P_{n-1} P_{n-1}$) which combines with the r_n to cancel the h_n on the left hand side of (2.8). For the potential (2.9), (2.8) thus gives

$$gn = r_n + \frac{2}{N} r_n (r_{n+1} + r_n + r_{n-1}) + \frac{36}{N^2} (10 \text{ rrr terms}) . \quad (2.10)$$

(The 10 rrr terms start with $r_n(r_n^2 + r_{n+1}^2 + r_{n-1}^2 + \dots)$ and may be found e.g. in [31].)

As mentioned before (2.6), in the large N limit the index n becomes a continuous variable ξ , and we have $r_n/N \rightarrow r(\xi)$ and $r_{n\pm 1}/N \rightarrow r(\xi \pm \varepsilon)$, where $\varepsilon \equiv 1/N$. To leading order in $1/N$, (2.10) reduces to

$$\begin{aligned} g\xi = r + 6r^2 + 30br^3 = W(r) \\ = g_c + \frac{1}{2} W''|_{r=r_c} (r(\xi) - r_c)^2 + \dots \end{aligned} \quad (2.11)$$

In the second line, we have expanded $W(r)$ for r near a critical point r_c at which $W'|_{r=r_c} = 0$ (which always exists without any fine tuning of the parameter b), and $g_c \equiv W(r_c)$. We see from (2.11) that

$$r - r_c \sim (g_c - g\xi)^{1/2} .$$

(For a general potential $V(\lambda) = \frac{1}{2g} \sum_p a_p \lambda^{2p}$ in (2.9), we would have $W(r) = \sum_p a_p \frac{(2p-1)!}{(p-1)!g} r^{2p-1}$.)

To make contact with the 2d gravity ideas of the preceding section, let us suppose more generally that the leading singular behavior of $f(\xi)$ ($= r(\xi)$) for large N is

$$f(\xi) - f_c \sim (g_c - g\xi)^{-\gamma} \quad (2.12)$$

for g near some g_c (and ξ near 1). (We shall see that γ in the above coincides with the critical exponent γ defined in (1.12).) The behavior of (2.6) for g near

g_c is then

$$\begin{aligned} \frac{1}{N^2} Z &\sim \int_0^1 d\xi (1-\xi)(g_c - g\xi)^{-\gamma} \sim \int_0^1 d\xi (1-\xi)(g_c - g\xi)^{-\gamma+1} \Big|_0^1 + \int_0^1 d\xi (g_c - g\xi)^{-\gamma+1} \\ &\sim (g_c - g)^{-\gamma+2} \sim \sum_n n^{\gamma-3} (g/g_c)^n . \end{aligned} \quad (2.13)$$

Comparison with (1.12) shows that the large area (large n) behavior identifies the exponent γ in (2.12) with the critical exponent defined earlier. We also note that the second derivative of Z with respect to $r = g_c - g$ has leading singular behavior

$$Z'' \sim (g_c - g)^{-\gamma} \sim f(1) . \quad (2.14)$$

From (2.12) and (2.13) we see that the behavior in (2.11) implies a critical exponent $\gamma = -1/2$. From (1.13), we see that this corresponds to the case $D = 0$, i.e. to pure gravity. It is natural that pure gravity should be present for a generic potential. With fine tuning of the parameter b in (2.9), we can achieve a higher order critical point, with $W'|_{r=r_c} = W''|_{r=r_c} = 0$, and hence the r.h.s. of (2.11) would instead begin with an $(r - r_c)^3$ term. By the same argument starting from (2.12), this would result in a critical exponent $\gamma = -1/3$. With a general potential $V(M)$ in (2.1), we have enough parameters to achieve an m^{th} order critical point [32] at which the first $m - 1$ derivatives of $W(r)$ vanish at $r = r_c$. The behavior is then $r - r_c \sim (g_c - g\xi)^{1/m}$ with associated critical exponent $\gamma = -1/m$. As anticipated at the end of subsection 1.2, we see that more general polynomial matrix interactions provide the necessary degrees of freedom to result in matter coupled to 2d gravity in the continuum limit.

2.3. The all genus partition function

We now search for another solution to (2.10) and its generalizations that describes the contribution of all genus surfaces to the partition function (2.6). We shall retain higher order terms in $1/N$ in (2.10) so that e.g. (2.11) instead reads

$$\begin{aligned} g\xi = W(r) + 2r(\xi)(r(\xi + \varepsilon) + r(\xi - \varepsilon) - 2r(\xi)) \\ = g_c + \frac{1}{2} W''|_{r=r_c} (r(\xi) - r_c)^2 + 2r(\xi)(r(\xi + \varepsilon) + r(\xi - \varepsilon) - 2r(\xi)) + \dots \end{aligned} \quad (2.15)$$

As suggested at the end of section 1, we shall simultaneously let $N \rightarrow \infty$ and $g \rightarrow g_c$ in a particular way. Since $g - g_c$ has dimension $[\text{length}]^2$, it is convenient to introduce a parameter a with dimension length and let $g - g_c = \kappa^{-4/5} a^2$, with $a \rightarrow 0$. Our ansatz for a coherent large N limit will be to take $\varepsilon \equiv 1/N = a^{5/2}$ so that the quantity $\kappa^{-1} = (g - g_c)^{5/4} N$ remains finite as $g \rightarrow g_c$ and $N \rightarrow \infty$.

Moreover since the integral (2.6) is dominated by ξ near 1 in this limit, it is convenient to change variables from ξ to z , defined by $g_c - g\xi = a^2 z$. Our scaling ansatz in this region is $r(\xi) = r_c + a u(z)$. If we substitute these definitions into (2.11), the leading terms are of order a^2 and result in the relation $u^2 \sim z$. To include the higher derivative terms, we note that

$$r(\xi + \varepsilon) + r(\xi - \varepsilon) - 2r(\xi) \sim \varepsilon^2 \frac{\partial^2 r}{\partial \xi^2} = a \frac{\partial^2}{\partial z^2} a u(z) \sim a^2 u'' ,$$

where we have used $\varepsilon(\partial/\partial\xi) = -ga^{1/2}(\partial/\partial z)$ (which follows from the above change of variables from ξ to z). Substituting into (2.15), the vanishing of the coefficient of a^2 implies the differential equation

$$z = u^2 - \frac{1}{3} u'' \quad (2.16)$$

(after a suitable rescaling of u and z). In (2.14), we saw that the the second derivative of the partition function (the “specific heat”) has leading singular behavior given by $f(\xi)$ with $\xi = 1$, and thus by $u(z)$ for $z = (g - g_c)/a^2 = \kappa^{-4/5}$. The solution to (2.16) characterizes the behavior of the partition function of pure gravity to all orders in the genus expansion. (Notice that the leading term is $u \sim z^{1/2}$ so after two integrations the leading term in Z is $z^{5/2} = \kappa^{-2}$, consistent with (1.16).)

Eq. (2.16) is known in the mathematical literature as the Painlevé I equation. The perturbative solution in powers of $z^{-5/2} = \kappa^2$ takes the form $u = z^{1/2}(1 - \sum_{k=1} u_k z^{-5k/2})$, where the u_k are all positive.¹¹ For large k , the

¹¹ The first term, i.e. the contribution from the sphere, is dominated by a regular part which has opposite sign. This is removed by taking an additional derivative of u , giving a series all of whose terms have the same sign — negative in the conventions of (2.16). The other solution, with leading term $-z^{1/2}$, has an expansion with alternating sign which is presumably Borel summable, but not physically relevant.

u_k go asymptotically as $(2k)!$, so the solution for $u(z)$ is not Borel summable (for a review of these issues in the context of 2d gravity, see e.g. [33]). Our arguments in section 1 show only that the matrix model results should agree with 2d gravity order by order in perturbation theory. How to insure that we are studying nonperturbative gravity as opposed to nonperturbative matrix models is still an open question. Some of the constraints that the solution to (2.16) should satisfy are reviewed in [34]. In particular it is known that real solutions to (2.16) cannot satisfy the Schwinger-Dyson (loop) equations for the theory.

In the case of the next higher multicritical point, with δ in (2.11) adjusted so that $W' = W'' = 0$ at $r = r_c$, we have $W(r) \sim g_c + \frac{1}{6} W''''|_{r=r_c} (r - r_c)^3 + \dots$ and critical exponent $\gamma = -1/3$. In general, we take $g - g_c = \kappa^{2/(\gamma-2)} a^2$, and $\varepsilon = 1/N = a^{2-\gamma}$ so that the combinatoric $(g - g_c)^{1-\gamma/2} N = \kappa^{-1}$ is fixed in the limit $a \rightarrow 0$. The value $\xi = 1$ now corresponds to $z = \kappa^{2/(\gamma-2)}$, so the string coupling $\kappa^2 = z^{\gamma-2}$. The general scaling ansatz is $r(\xi) = r_c + a^{-2\gamma} u(z)$, and the change of variables from ξ to z gives $\varepsilon(\partial/\partial\xi) = -ga^{-\gamma}(\partial/\partial z)$.

For the case $\gamma = -1/3$, this means in particular that $r(\xi) = r_c + a^{2/3} u(z)$, $\kappa^2 = z^{-7/3}$, and $\varepsilon(\partial/\partial\xi) = -ga^{1/3} \frac{\partial}{\partial z}$. Substituting into the large N limit of (2.10) gives (again after suitable rescaling of u and z)

$$z = u^3 - uu'' - \frac{1}{2}(u')^2 + c u'''' , \quad (2.17)$$

with $\alpha = \frac{1}{10}$. The solution to (2.17) takes the form $u = z^{1/3}(1 + \sum_k u_k z^{-7k/3})$. It turns out that the coefficients u_k in the perturbative expansion of the solution to (2.17) are positive definite only for $\alpha < \frac{1}{12}$, so the 3th order multicritical point does not describe a unitary theory of matter coupled to gravity. Although from (1.13) we see that the critical exponent $\gamma = -1/3$ coincides with that predicted for the (unitary) Ising model coupled to gravity, it turns out [31,35] that (2.17) with $\alpha = \frac{1}{10}$ instead describes the conformal field theory of the Yang Lee edge singularity (a critical point obtained by coupling the Ising model to a particular value of imaginary magnetic field) coupled to gravity. The specific heat of the conventional critical Ising model coupled to gravity turns out (see the next section here) to be as well determined by the differential equation (2.17), but instead with $\alpha = \frac{2}{27}$.

For the general m^{th} order critical point of the potential $W(r)$, we have seen that the associated model of matter coupled to gravity has critical exponent $\gamma = -1/m$. With scaling ansatz $r(\xi) = r_c + a^{2/m}u(z)$, we find leading behavior $u \sim z^{1/m}$ (and $Z \sim z^{2+1/m} = \kappa^{-2}$ as expected). The differential equation that results from substituting the double scaling behaviors given before (2.17) into the generalized version of (2.10) turns out to be the m^{th} member of the KdV hierarchy of differential equations (of which Painlevé I results for $m = 2$). In the next section, we shall provide some marginal insight into why this structure emerges.

In the nomenclature of [36], so-called “minimal conformal field theories” (those with a finite number of primary fields) are specified by a pair of relatively prime integers (p, q) . (The unitary discrete series is the subset specified by $(p, q) = (m + 1, m)$.) After coupling to gravity, these have critical exponent $\gamma = -2/(p + q - 1)$. In general, the m^{th} order multicritical point of the one-matrix model turns out to describe the $(2m - 1, 2)$ model (in general non-unitary) coupled to gravity, so its critical exponent $\gamma = -1/m$ happens to coincide with that of the m^{th} member of the unitary discrete series coupled to gravity. The remaining (p, q) models coupled to gravity can be realized in terms of multi-matrix models (to be defined in the next section).

3. KdV equations and other models

3.1. KdV equations

We now wish to describe superficially why the KdV hierarchy of differential equations plays a role in 2d gravity. To this end it is convenient to switch from the basis of orthogonal polynomials P_n employed in the previous section to a basis of orthonormal polynomials $\Pi_n(\lambda) = P_n(\lambda)/\sqrt{h_n}$ that satisfy

$$\int_{-\infty}^{\infty} d\lambda e^{-V} \Pi_n \Pi_m = \delta_{nm} . \quad (3.1)$$

In terms of the Π_n , eq. (2.7) becomes

$$\begin{aligned} \lambda \Pi_n &= \sqrt{\frac{h_{n+1}}{h_n}} \Pi_n + r_n \sqrt{\frac{h_{n-1}}{h_n}} \Pi_{n-1} = \sqrt{r_{n+1}} \Pi_{n+1} + \sqrt{r_n} \Pi_{n-1} \\ &= Q_{nm} \Pi_m . \end{aligned}$$

In matrix notation, we write this as $\lambda \Pi = Q \Pi$, where the matrix Q has components

$$Q_{nm} = \sqrt{r_m} \delta_{m,n+1} + \sqrt{r_n} \delta_{m+1,n} . \quad (3.2)$$

Due to the orthonormality property (3.1), we see that $\int e^{-V} \lambda \Pi_n \Pi_m = Q_{nm} = Q_{mn}$, and Q is a symmetric matrix. In the continuum limit, Q will therefore become a hermitian operator.

To see how this works explicitly [19,37], we substitute the scaling ansatz $r(\xi) = r_c + a^{2/m}u(z)$ for the m^{th} multicritical model into (3.2),

$$Q \rightarrow (r_c + a^{2/m}u(z))^{1/2} e^{\varepsilon \frac{\partial}{\partial \xi}} + e^{-\varepsilon \frac{\partial}{\partial \xi}} (r_c + a^{2/m}u(z))^{1/2} .$$

With the substitution $\varepsilon \frac{\partial}{\partial \xi} \rightarrow -ga^{1/m} \frac{\partial}{\partial z}$, we find the leading terms

$$Q = 2r_c^{1/2} + \frac{a^{2/m}}{\sqrt{r_c}} (u + r_c \kappa^2 \partial_z^2) , \quad (3.3)$$

of which the first is a non-universal constant and the second is a hermitian 2nd order differential operator.

The other matrix that naturally arises is defined by differentiation,

$$\frac{\partial}{\partial \lambda} \Pi_n = A_{nm} \Pi_m , \quad (3.4)$$

and automatically satisfies $[A, Q] = 1$. The matrix A does not have any particular symmetry or antisymmetry properties so it is convenient to correct it to a matrix P that satisfies the same commutator as A . From our definitions, it follows that

$$0 = \int \frac{\partial}{\partial \lambda} (\Pi_n \Pi_m e^{-V}) \Rightarrow A + A^T = V'(Q) ,$$

where we have differentiated term by term and used $\int e^{-V} \lambda^l \Pi_n \Pi_m = (Q^l)_{nm}$. The matrix $P \equiv A - \frac{1}{2}V'(Q) = \frac{1}{2}(A - A^T)$ is therefore anti-symmetric and satisfies

$$[P, Q] = 1 . \quad (3.5)$$

To determine the order of the differential operator Q in the continuum limit, let us assume for example that the potential V is of order 2ℓ , i.e. $V = \sum_{k=0}^{\ell} a_k \lambda^{2k}$. For $m > n$, the integral $A_{mn} = \int e^{-V} \Pi_n \frac{\partial}{\partial \lambda} \Pi_m = \int e^{-V} V' \Pi_n \Pi_m$ may be nonvanishing for $m - n \leq 2\ell - 1$. That means that $P_{mn} \neq 0$ for $|m - n| \leq 2\ell - 1$, and thus has enough parameters to result in a $(2\ell - 1)^{\text{th}}$ order differential operator in the continuum. The single condition $W' = 0$ results in P tuned to a 3^{rd} order operator, and the $\ell - 1$ conditions $W' = \dots = W^{(\ell-1)} = 0$ allow P to be realized as a $(2\ell - 1)^{\text{th}}$ order differential operator. In (3.3), we see that the universal part of Q after suitable rescaling takes the form $Q = d^2 - \mathbf{u}$. For the simple critical point $W' = 0$, the continuum limit of P is the antihermitian operator $P = d^3 - \frac{3}{4}\{\mathbf{u}, d\}$, and the commutator

$$1 = [P, Q] = 4R_2' = \left(\frac{3}{4}\mathbf{u}^2 - \frac{1}{4}\mathbf{u}'' \right)' \quad (3.6)$$

is easily integrated with respect to z to give an equation equivalent to (2.16), the string equation for pure gravity. In (3.6), the notation R_2 is conventional for the first member of the ordinary KdV hierarchy. The emergence of the KdV hierarchy in this context is due to the natural occurrence of the fundamental commutator relation (3.5), which also occurs in the Lax representation of the KdV equations. (The topological gravity approach has as well been shown at length to be equivalent to KdV, for a review see [38].)

In general the differential equations

$$[P, Q] = 1 \quad (3.7)$$

that follow from (3.5) may be determined directly in the continuum. Given an operator Q , the differential operator P that can satisfy this commutator is constructed as a "fractional power" of the operator Q .

Before showing how this construction works, we first expand slightly the class of models from single matrix to multi-matrix models. The free energy of

a particular $(q - 1)$ -matrix model, generalizing (2.1), may be written [39]

$$\begin{aligned} Z &= \ln \int \prod_{i=1}^{q-1} dM_i e^{-\text{tr}(\sum_{i=1}^{q-1} V_i(M_i) - \sum_{i=1}^{q-2} c_i M_i M_{i+1})} \\ &= \ln \int \prod_{\substack{i=1, q-1 \\ \alpha=1, \dots, N}} d\lambda_i^{(\alpha)} \Delta(\lambda_i) e^{-\sum_{i,\alpha} V_i(\lambda_i^{(\alpha)}) + \sum_{i,\alpha} c_i \lambda_i^{(\alpha)} \lambda_{i+1}^{(\alpha)}} \Delta(\lambda_{q-1}), \end{aligned} \quad (3.8)$$

where the M_i (for $i = 1, \dots, q - 1$) are $N \times N$ hermitian matrices, the $\lambda_i^{(\alpha)}$ ($\alpha = 1, \dots, N$) are their eigenvalues, and $\Delta(\lambda_i) = \prod_{\alpha < \beta} (\lambda_i^{(\alpha)} - \lambda_i^{(\beta)})$ is the Vandermonde determinant. The result in the second line of (3.8) depends on having c_i 's that couple matrices along a line (with no closed loops so that the integrations over the relative angular variables in the M_i 's can be performed.) Via a diagrammatic expansion, the matrix integrals in (3.8) can be interpreted to generate a sum over discretized surfaces, where the different matrices M_i represent $q - 1$ different matter states that can exist at the vertices. The quantity Z in (3.8) thereby admits an interpretation as the partition function of 2d gravity coupled to matter.

Following [39], we can introduce operators Q_i and P_i that represent the insertions of λ_i and $d/d\lambda_i$, respectively in the integral (3.8). These operators necessarily satisfy $[P_i, Q_i] = 1$. In the $N \rightarrow \infty$ limit, we have seen (following [19]) that P and Q become differential operators of finite order, say p, q respectively (where we assume $p > q$), and these continue to satisfy (3.7). In the continuum limit of the matrix problem (i.e. the "double" scaling limit, which here means couplings in (3.8) tuned to critical values), Q becomes a differential operator of the form

$$Q = d^q + \{v_{q-2}(z), d^{q-2}\} + \dots + 2v_0(z), \quad (3.9)$$

where $d = d/dz$. (By a change of basis of the form $Q \rightarrow f^{-1}(z)Qf(z)$, the coefficient of d^{q-1} may always be set to zero.) The continuum scaling limit of the multi-matrix models is thus abstracted to the mathematical problem of finding solutions to (3.7).

The differential equations (3.7) may be constructed as follows. For p, q relatively prime, a p^{th} order differential operator that can satisfy (3.7) is constructed as a fractional power of the operator Q of (3.9). Formally, a q^{th} root may be represented within an algebra of formal pseudo-differential operators (see, e.g. [40]) as

$$Q^{1/q} = d + \sum_{i=1}^{\infty} \{c_i, d^{-i}\}, \quad (3.10)$$

where d^{-1} is defined to satisfy $d^{-1}f = \sum_{j=0}^{\infty} (-1)^j f^{(j)} d^{-j-1}$. The differential equations describing the (p, q) minimal model coupled to 2d gravity are given by

$$[Q_+^{p/q}, Q] = 1, \quad (3.11)$$

where $P = Q_+^{p/q}$ indicates the part of $Q^{p/q}$ with only non-negative powers of d , and is a p^{th} order differential operator.

To illustrate the procedure we reproduce now the results for the one-matrix models, which can be used to generate (p, q) of the form $(2l-1, 2)$. From (3.3), these models are obtained by taking Q to be the hermitian operator

$$Q = K \equiv d^2 - u(z). \quad (3.12)$$

The formal expansion of $Q^{l-1/2} = K^{l-1/2}$ (an anti-hermitian operator) in powers of d is given by

$$K^{l-1/2} = d^{2l-1} - \frac{2l-1}{4} \{u, d^{2l-3}\} + \dots \quad (3.13)$$

(where only symmetrized odd powers of d appear in this case). We now decompose $K^{l-1/2} = K_+^{l-1/2} + K_-^{l-1/2}$, where $K_+^{l-1/2} = d^{2l-1} + \dots$ contains only non-negative powers of d , and the remainder $K_-^{l-1/2}$ has the expansion

$$K_-^{l-1/2} = \sum_{i=1}^{\infty} \{c_{2i-1}, d^{-2i-1}\} = \{R_l, d^{-1}\} + O(d^{-3}) + \dots \quad (3.14)$$

Here we have identified $R_l \equiv c_1$ as the first term in the expansion of $K_-^{l-1/2}$. For $K^{1/2}$, for example, we find $K_+^{1/2} = d$ and $R_1 = -u/4$.

The prescription (3.11) with $p = 2l-1$ corresponds here to calculating the commutator $[K_+^{l-1/2}, K]$. Since K commutes with $K^{l-1/2}$, we have

$$[K_+^{l-1/2}, K] = [K, K_-^{l-1/2}]. \quad (3.15)$$

But since K begins at d^2 , and since from the l.h.s. above the commutator can have only positive powers of d , only the leading (d^{-1}) term from the r.h.s. can contribute, which results in

$$[K_+^{l-1/2}, K] = \text{leading piece of } [K, 2R_l d^{-1}] = 4R_l'. \quad (3.16)$$

After integration, the equation $[K_+^{l-1/2}, K] = 1$ thus takes the simple form

$$c R_l[u] = z, \quad (3.17)$$

where the constant c may be fixed by suitable rescaling of z and u (enabled by the property that all terms in R_l have fixed grade, namely $2l$).

The quantities R_l in (3.14) are easily seen to satisfy a simple recursion relation. From $K^{l+1/2} = K K^{l-1/2} = K^{l-1/2} K$, we find

$$K_+^{l+1/2} = \frac{1}{2} \left(K_+^{l-1/2} K + K K_+^{l-1/2} \right) + \{R_l, d\}.$$

Commuting both sides with K and using (3.16), simple algebra gives [41]

$$R_{l+1}' = \frac{1}{4} R_l'' - u R_l' - \frac{1}{2} u' R_l. \quad (3.18)$$

While this recursion formula only determines R_l' , by demanding that the R_l ($l \neq 0$) vanish at $u = 0$, we obtain

$$\begin{aligned} R_0 &= \frac{1}{2}, & R_1 &= -\frac{1}{4}u, & R_2 &= \frac{3}{16}u^2 - \frac{1}{16}u'', \\ R_3 &= -\frac{5}{32}u^3 + \frac{5}{32}(uu'' + \frac{1}{2}u'^2) - \frac{1}{64}u^{(4)}. \end{aligned} \quad (3.19)$$

We summarize as well the first few $K_+^{l-1/2}$,

$$\begin{aligned} K_+^{1/2} &= d, & K_+^{3/2} &= d^3 - \frac{3}{4}\{u, d\}, \\ K_+^{5/2} &= d^5 - \frac{5}{4}\{u, d^3\} + \frac{5}{16}\{(3u^2 + u''), d\}. \end{aligned} \quad (3.20)$$

After rescaling, we recognize R_3 in (3.19) as eq. (2.17) with $\alpha = \frac{1}{10}$, i.e. the equation for the (2,5) model. In general, the equations determined by (3.7) for general p, q characterize the partition function of the (p, q) minimal model (mentioned at the end of section 2) coupled to gravity. To realize these equations in the continuum limit turns out [42,43] to require only a two-matrix model of the type (3.8). The argument given after (3.5) for the one-matrix case is easily generalized to the recursion relations for the two-matrix case and shows that for high enough order potentials, there are enough couplings to tune the matrices P and Q to become p^{th} and q^{th} order differential operators. In subsection 3.2, we shall show how to realize a $D = 1$ theory coupled to gravity in terms of a two-matrix model. In [44], it is argued that one can as well realize a wide variety of $D < 1$ theories by means of a one-matrix model coupled to an external potential.

3.2. Other models

As a specific example of a two-matrix model, we consider

$$e^Z = \int dU dV e^{-\text{tr}(U^2 + V^2 - 2cUV + \frac{g}{N}(e^H U^4 + e^{-H} V^4))} \quad (3.21)$$

where U and V are hermitian $N \times N$ matrices and H is a constant. In the diagrammatic expansion of the right hand side, we now have two different quartic vertices of the type depicted in fig. 3b, corresponding to insertions of U^4 and V^4 . The propagator is determined by the inverse of the quadratic term,

$$\begin{pmatrix} 1 & -c \\ -c & 1 \end{pmatrix}^{-1} = \frac{1}{1-c^2} \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}.$$

We see that double lines connecting vertices of the same type (either generated by U^4 or V^4) receive a factor of $1/(1-c^2)$, while those connecting U^4 vertices to V^4 vertices receive a factor of $c/(1-c^2)$.

This is identically the structure necessary to realize the Ising model on a random lattice. Recall that the Ising model is defined to have a spin $\sigma = \pm 1$ at each site of a lattice, with an interaction $\sigma_i \sigma_j$ between nearest neighbor sites (ij) . This interaction takes one value for equal spins and another value

for unequal spins. Up to an overall additive constant to the free energy, the diagrammatic expansion of (3.21) results in the 2d partition function

$$Z = \sum_{\text{lattices}} \sum_{\text{spin configurations}} e^{\beta \sum_{(ij)} \sigma_i \sigma_j + H \sum_i \sigma_i}$$

where H is the magnetic field. The weights for equal and unequal neighboring spins are $e^{\pm J}$, so fixing the ratio $e^{2\beta} = 1/c$ relates the parameter c in (3.21) to the temperature β . It turns out that the Ising model is much easier to solve summed over random lattices than on a regular lattice, and in particular is solvable even in the presence of a magnetic field. This is because there is much more symmetry after coupling to gravity, since the complicating details of any particular lattice (e.g. square) are effectively integrated out.

We briefly outline the method for solving (3.21) (see [45,31,35] for more details). By methods similar to those used to derive (2.1), we can write (3.21) in terms of the eigenvalues x_i and y_i of U and V ,

$$e^Z = \int \prod_i dx_i dy_i e^{-W(x_i, y_i)} \Delta(x) \Delta(y),$$

where $W(x_i, y_i) \equiv x_i^2 + y_i^2 - 2c x_i y_i + \frac{g}{N}(e^H x_i^4 + e^{-H} y_i^4)$. The polynomials we define for this problem are orthogonal with respect to the bilocal measure

$$\int dx dy e^{-W(x, y)} P_n(x) Q_m(y) = h_n \delta_{nm}$$

(where $P_n \neq Q_n$ for $H \neq 0$). The result for the partition function is identical to (2.5),

$$e^Z \propto \prod_i h_i \propto \prod_i f_i^{N-1},$$

and the recursion relations for this case generalize (2.7),

$$x P_n(x) = P_{n+1} + r_n P_{n-1} + s_n P_{n-3},$$

$$y Q_m(y) = Q_{m+1} + q_m Q_{m-1} + t_m Q_{m-3}.$$

We still have $f_n \equiv h_n/h_{n-1}$, and f_n can be determined in terms of the above recursion coefficients (although the formulae are more complicated than in the

one-matrix case). After we substitute the scaling ansätze described in subsection 2.3, the formula for the scaling part of f is derived via straightforward algebra. The result is that the specific heat $\alpha \propto Z''$ is given by (2.17) with $\alpha = \frac{2}{27}$.

Other conventional statistical mechanical models can be formulated on random lattices and solved in the continuum limit. The ADE face models (with $D < 1$), for example, have been considered in [18]. One way of formulating $D = 1$ is to generalize (3.8) to an infinite line of matrices. In dual form, this is equivalent to strings propagating on a circle of finite radius (see e.g. [25,46]). Another formulation involves letting the index i specifying the matrix M_i become a continuous index $t \in (-\infty, \infty)$. In this limit we trade off matrix quantum mechanics for a field theory of matrices theory $M(t)$. This is a problem that was originally solved in [13], and was used to analyze 2d gravity at genus zero in [15] and was then applied to higher genus starting in [25,26]. A connection to Liouville theory was pointed out in [47], and carried further by the free fermion and collective field formulations of [3].

Yet another means of formulating 2d gravity coupled to $D = 1$ matter is via the 8-vertex model, which renormalizes at criticality (the 6-vertex model) onto a single boson at finite radius.¹² Since this has not been treated in the literature, we give a quick description of the formulation. The simplest vertex models are those for which the degrees of freedom are (two-state) arrows that live on links, and are defined on lattices which have four links meeting at each vertex. Each possible arrow configuration at a vertex is given a statistical weight, and the partition function is given by summing over all arrow configurations, with each assigned an overall weight equal to the product of the statistical weights over the vertices. In the 8-vertex model, the vertices are restricted to the set of eight with an even number of arrows both incoming and outgoing. In the 6-vertex model, the source and sink (all four arrows outgoing or incoming) are excluded.

¹² For an overview geared towards string/particle physicists, see e.g. [48]. On a regular lattice, the radius r of the boson (in conventions in which $r = 1/\sqrt{2}$ is the self-dual point) and the conventional weights a, b, c of the 6-vertex model are related by $\cos \frac{1}{2} \pi r^2 = (a^2 + b^2 - c^2)/2ab$

which leaves the four distinct rotated versions of fig. 4a, and the two distinct rotated versions of fig. 4b.

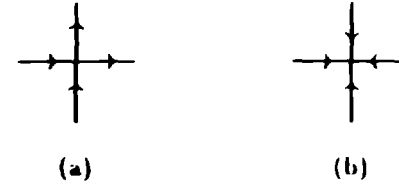


Fig. 4: (a) vertex with weight a . (b) vertex with weight c .

The coupling to gravity is given by summing over random lattices that maintain four links at each vertex, but can have arbitrary polygonal faces. It is simple to write down a matrix model that generates 6-vertex configurations on random lattices. Rather than a hermitian matrix, we employ an arbitrary complex $N \times N$ matrix $\varphi = A + iB$, where A and B are hermitian. The propagator $\langle \varphi^l \varphi \rangle$ now has an overall orientation, which we identify by an arrow on the propagator. (In what follows we suppress the underlying double-lined notation of fig. 3.) The graphs of interest are generated by the matrix integral

$$\int_{\varphi} e^{\text{tr}(-\frac{1}{2}\varphi^l \varphi + a\varphi^2 \varphi^{l2} + c(\varphi^l \varphi)^2)} \quad (3.22)$$

where the vertices shown in figs. 4a and 4b are assigned weights a and c respectively.¹³

The model has not yet been solved in this formulation except at the analog of the Kosterlitz-Thouless point, $a = c$. At that point we can use the identity

$$\text{tr}[\varphi^2 \varphi^{l2} + (\varphi^l \varphi)^2] = \frac{1}{8} [(\varphi + \varphi^l)^2 - (\varphi - \varphi^l)^2]^2 = 2 \text{tr}(A^2 + B^2)^2$$

¹³ On a regular square lattice, the four rotated versions of fig. 4a are further subdivided into two mirror reflected pairs, which can be given different weights a and b . On a random lattice such distinctions are academic, a property embodied by the cyclicity of the trace in (3.22), and we automatically generate the so-called F-model with $a = b$

to rewrite the action in terms of the hermitian matrices A, B . By introducing an additional integration matrix M , we can reduce the action to terms quadratic in A and B .

$$e^{-2\pi\text{tr}(A^2 + B^2)} = \int_M e^{\text{tr}(-\frac{1}{2}M^2 + 2\sqrt{r}(A^2 + B^2)M)}$$

In this form, the model reduces to a standard transcription of the $O(n)$ model for $n = 2$. (For general $O(n)$, $A^2 + B^2$ is replaced in the above by $\sum_{i=1}^n A_i^2$.) This is reasonable since $S(O(2))$ is just the circle S^1 normalized to a particular radius. The genus zero solution (due to M. Gaudin) is reproduced in [29].

4. Quick tour of Liouville theory

For completeness, we give here a brief overview¹⁴ of how the continuum results we have used here are calculated. As previously mentioned, the coincidence of these results with those of the matrix model approach originally served to give post-facto verification of both methods. This section may be considered as an appendix to the preceding three.

4.1 String susceptibility

We consider the continuum partition function

$$Z = \int \frac{Dg DX}{\text{Vol}(\text{Diff})} e^{-S_M(X;g) - \frac{\mu_0}{2\pi} \int d^2\xi \sqrt{g}} \quad (4.1)$$

¹⁴ In preparing this section, I may have shamelessly plagiarized some material from a similar section in [9]. Historically, after the work of [6] some of the results here were derived in [49], where the conformal quantization of Liouville theory was studied (but general correlation functions were not calculated). The quantum Liouville theory was also studied in [50]. More recently, the calculation of critical exponents in lightcone gauge was carried out in [17] (using $SL(2, \mathbb{R})$ current algebra). The results were subsequently redetived in conformal gauge in [51], which is the approach we follow here since it applies also to higher genus. Reviews of Liouville theory may be found in [52].

where S_M is some conformally invariant action for matter fields coupled to a two dimensional surface Σ with metric g , μ_0 is a bare cosmological constant, and we have symbolically divided the measure by the “volume” of the diffeomorphism group (which acts as a local symmetry) of Σ . For the free bosonic string, we take $S_M = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{g} g^{ab} \partial_a \vec{X} \cdot \partial_b \vec{X}$ where the $\vec{X}(\xi)$ specify the embedding of Σ into flat D -dimensional spacetime.

To define (4.1), we need to specify the measures for the integrations over X and g (see, e.g. [53]). The measure DX is determined by requiring that $\int Dg \delta X e^{-\|\delta X\|_g^2} = 1$, where the norm in the gaussian functional integral is given by $\|\delta X\|_g^2 = \int d^2\xi \sqrt{g} \delta \vec{X} \cdot \delta \vec{X}$. Similarly, the measure Dg is determined by normalizing $\int Dg \delta g e^{-\frac{1}{2}\|\delta g\|_g^2} = 1$, where $\|\delta g\|_g^2 = \int d^2\xi \sqrt{g} (g^{ac}g^{bd} + 2g^{ab}g^{cd}) \delta g_{ab} \delta g_{cd}$, and δg represents a metric fluctuation at some point g_{ij} in the space of metrics on a genus h surface.

The measures DX and Dg are invariant under the group of diffeomorphisms of the surface, but not necessarily under conformal transformations $g_{ab} \rightarrow e^\sigma g_{ab}$. Indeed due to the metric dependence in the norm $\|\delta X\|_g^2$, it turns out that

$$D_{e^\sigma g} X = e^{\frac{D}{40\pi} S_L(\sigma)} D_g X, \quad (4.2)$$

where

$$S_L(\sigma) = \int d^2\xi \sqrt{g} \left(\frac{1}{2} g^{ab} \partial_a \sigma \partial_b \sigma + R\sigma + \mu e^\sigma \right) \quad (4.3)$$

is known as the Liouville action. (This result may be derived diagrammatically, via the Fujikawa method, or via an index theorem; for a review see [54].)

The metric measure Dg as well has an anomalous variation under conformal transformations. To express it in a form analogous to (4.2), we first need to recall some basic facts about the domain of integration. The space of metrics on a compact topological surface Σ modulo diffeomorphisms and Weyl transformations is a finite dimensional compact space \mathcal{M}_g , known as moduli space. (It is 0-dimensional for genus $h = 0$; 2-dimensional for $h = 2$; and $(6h - 6)$ -dimensional for $h \geq 2$). If for each point $\tau \in \mathcal{M}_g$, we choose a representative metric $\hat{g}_{i,j}$, then the orbits generated by the diffeomorphism and Weyl groups

acting on \hat{g}_τ , generate the full space of metrics on Σ . Thus given the slice $\hat{g}(\tau)$, any metric can be represented in the form

$$f^* g = e^\nu \hat{g}(\tau) .$$

where f^* represents the action of the diffeomorphism $f : \Sigma \rightarrow \Sigma$.

Since the integrand of (4.1) is diffeomorphism invariant, the functional integral would be infinite unless we formally divide out by the volume of orbit of the diffeomorphism group. This is accomplished by gauge fixing to the slice $\hat{g}(\tau)$; the Jacobian that enters can be represented in terms of Fadeev-Popov ghosts, as familiar from the analogous procedure in gauge theory. We parametrize an infinitesimal change in the metric as

$$\delta g_{z\bar{z}} = \nabla_z \xi_{\bar{z}} , \quad \delta g_{\bar{z}z} = \nabla_{\bar{z}} \xi_z$$

(where for convenience we employ complex coordinates, and recall that the components $g_{z\bar{z}} = g^{z\bar{z}}$ are parametrized by e^ν). The measure Dg at $\hat{g}(\tau)$ splits into an integration $[d\tau]$ over moduli, an integration $D\varphi$ over the conformal factor, and an integration $D\xi D\bar{\xi}$ over diffeomorphisms. The change of integration variables $D\delta g_{z\bar{z}} D\delta g_{\bar{z}z} = (\det \nabla_z \det \nabla_{\bar{z}}) D\xi D\bar{\xi}$ introduces the Jacobian $\det \nabla_z \det \nabla_{\bar{z}}$ for the change from δg to ξ . The determinants in turn can be represented as

$$\begin{aligned} \det \nabla_z \det \nabla_{\bar{z}} &= \int D\bar{b} Dc D\bar{b} D\bar{c} e^{-\int d^2\xi \sqrt{g} b_{z\bar{z}} \nabla_z c^z - \int d^2\xi \sqrt{g} b_{\bar{z}z} \nabla_{\bar{z}} \bar{c}^{\bar{z}}} \\ &\equiv \int D(\text{gh}) e^{-S_{\text{gh}}(b, c, \bar{b}, \bar{c})} . \end{aligned} \quad (4.4)$$

where $D(\text{gh}) \equiv D\bar{b} Dc D\bar{b} D\bar{c}$ is an abbreviation for the measures associated to the ghosts b, c, \bar{b}, \bar{c} , $b_{z\bar{z}}$ is a holomorphic quadratic differential, and c^z ($\bar{c}^{\bar{z}}$) is a holomorphic (anti-holomorphic) vector.

Finally, the ghost measure $D(\text{gh})$ is not invariant under the conformal transformation $g \rightarrow e^\nu g$, instead we have [6.53,54]

$$D_{e^\nu g}(\text{gh}) = e^{-\frac{26}{12\pi} S_L(\sigma, g)} D_g(\text{gh}) . \quad (4.5)$$

where S_L is again the Liouville action (4.3). (In units in which the contribution of a single scalar field to the conformal anomaly is $c = 1$, and hence $c = 1/2$ for a single Majorana-Weyl fermion, the conformal anomaly due to a spin j reparametrization ghost is given by $c = (-1)^F 2(1+6j)(j-1)$). The contribution from a spin $j = 2$ reparametrization ghost is thus $c = -26$.)

We have thus far succeeded to reexpress the partition function (4.1) as

$$Z = \int [d\tau] D_\theta \varphi D_\theta(\text{gh}) D_\theta X e^{-S_M - S_{\text{gh}} - \frac{26}{24} \int d^2\xi \sqrt{g}}$$

Choosing a metric slice $g = e^\nu \hat{g}$ gives

$$D_{e^\nu \hat{g}} \varphi D_{e^\nu \hat{g}}(\text{gh}) D_{e^\nu \hat{g}} X = J(\varphi, \hat{g}) D_\theta \varphi D_\theta(\text{gh}) D_\theta X ,$$

where the Jacobian $J(\varphi, \hat{g})$ is easily calculated for the matter and ghost sectors ((4.2) and (4.5)) but not for the Liouville mode φ . The functional integral over φ is complicated by the implicit metric dependence in the norm

$$\|\delta\varphi\|_g^2 = \int d^2\xi \sqrt{g} (\delta\varphi)^2 = \int d^2\xi \sqrt{\hat{g}} e^\nu (\delta\varphi)^2 ,$$

since only if the e^ν factor were absent above would the $D_\theta \varphi$ measure reduce to that of a free field. In [51], it is simply assumed¹⁵ that the overall Jacobian $J(\varphi, \hat{g})$ takes the form of an exponential of a local Liouville-like action $\int d^2\xi \sqrt{\hat{g}} (\bar{a} \hat{g}^{ab} \partial_a \varphi \partial_b \varphi + \bar{b} \hat{R} \varphi + \mu e^{\hat{c}\varphi})$, where \bar{a} , \bar{b} , and \bar{c} are constants that will be determined by requiring overall conformal invariance (\bar{c} is inserted in anticipation of rescaling of φ). With this assumption, the partition function (4.1) takes the form

$$\begin{aligned} Z &= \int [d\tau] D_\theta \varphi D_\theta(\text{gh}) D_\theta X e^{-S_M(X, \hat{g}) - S_{\text{gh}}(b, c, \bar{b}, \bar{c}; \hat{g})} \\ &\quad \cdot e^{-\int d^2\xi \sqrt{\hat{g}} (\bar{a} \hat{g}^{ab} \partial_a \varphi \partial_b \varphi + \bar{b} \hat{R} \varphi + \mu e^{\hat{c}\varphi})} \end{aligned} \quad (4.6)$$

where the φ measure is now that of a free field.

¹⁵ Some attempts to justify this assumption may be found in [55]

The path integral (4.6) was defined to be reparametrization invariant, and should depend only on $e^{\tilde{g}} = g$ (up to diffeomorphism), not on the specific slice \tilde{g} . Due to diffeomorphism invariance, (4.6) should thus be invariant under the infinitesimal transformation

$$\delta\tilde{g} = \varepsilon(\xi)\tilde{g} \quad , \quad \delta\varphi = -\varepsilon(\xi) \quad , \quad (4.7)$$

and we can use the known conformal anomalies (4.2) and (4.5) for φ , X , and the ghosts to determine the constants \bar{a} , \bar{b} , \bar{c} . Substituting the variations (4.7) in (4.6), we find terms of the form

$$\left(\frac{D-26+1}{48\pi} + \bar{b}\right) \int d^2\xi \sqrt{\tilde{g}} \tilde{R} \varepsilon \quad \text{and} \quad (2\bar{a} - \bar{b}) \int d^2\xi \sqrt{\tilde{g}} \varepsilon \square \varphi \quad ,$$

where the $D-26$ on the left is the contribution from the matter and ghost measures $D_{\tilde{g}}X$ and $D_{\tilde{g}}(gh)$, and the additional 1 comes from the $D_{\tilde{g}}\varphi$ measure. Invariance under (4.7) thus determines

$$\bar{b} = \frac{25-D}{48\pi} \quad , \quad \bar{a} = \frac{1}{2}\bar{b} \quad . \quad (4.8)$$

Substituting the values of \bar{a} , \bar{b} into the Liouville action in (4.6) gives

$$\frac{1}{8\pi} \int d^2\xi \sqrt{\tilde{g}} \left(\frac{25-D}{12} \tilde{g}^{ab} \partial_a \varphi \partial_b \varphi + \frac{25-D}{6} \tilde{R} \varphi \right) \quad . \quad (4.9)$$

To obtain a conventionally normalized kinetic term $\frac{1}{8\pi} \int (\partial\varphi)^2$, we rescale $\varphi \rightarrow \sqrt{\frac{12}{25-D}} \tilde{\varphi}$. (This normalization leads to the leading short distance expansion $\varphi(z)\varphi(w) \sim -\log(z-w)$.) In terms of the rescaled $\tilde{\varphi}$, we write the Liouville action as

$$\frac{1}{8\pi} \int d^2\xi \sqrt{\tilde{g}} \left(\tilde{g}^{ab} \partial_a \tilde{\varphi} \partial_b \tilde{\varphi} + Q \tilde{R} \tilde{\varphi} \right) \quad , \quad (4.10)$$

where

$$Q \equiv \sqrt{\frac{25-D}{3}} \quad . \quad (4.11)$$

The energy-momentum tensor $T = -\frac{1}{2}\partial\tilde{\varphi}\partial\tilde{\varphi} + \frac{Q}{2}\partial^2\tilde{\varphi}$ derived from (4.10) has leading short distance expansion $T(z)T(w) \sim \frac{1}{2}c_{\text{Liouville}}/(z-w)^4 + \dots$, where $c_{\text{Liouville}} = 1+3Q^2$. Note that if we substitute (4.11) into $c_{\text{Liouville}}$ and add an

additional $c = D-26$ from the matter and ghost sectors, we find that the total conformal anomaly vanishes (consistent with the required overall conformal invariance).

It remains to determine the coefficient \bar{c} in (4.6). We have since rescaled φ , so we write instead $e^{\alpha\varphi}$ and determine α by the requirement that the physical metric be $g = \tilde{g}e^{\alpha\varphi}$. Geometrically, this means that the area of the surface is represented by $\int d^2\xi \sqrt{\tilde{g}} e^{\alpha\varphi}$. α is thereby determined by the requirement that $e^{\alpha\varphi}$ behave as a (1,1) conformal field (so that the combination $d^2\xi e^{\alpha\varphi}$ is conformally invariant). For the energy-momentum tensor mentioned after (4.11), the conformal weight¹⁶ of $e^{\alpha\varphi}$ is

$$h(e^{\alpha\varphi}) = \bar{h}(e^{\alpha\varphi}) = -\frac{1}{2}\alpha(\alpha - Q) \quad . \quad (4.12)$$

Requiring that $h(e^{\alpha\varphi}) = \bar{h}(e^{\alpha\varphi}) = 1$ determines that $Q = 2/\alpha + \alpha$. Using (4.11) and solving for α then gives¹⁷

$$\alpha = \frac{1}{\sqrt{12}} (\sqrt{25-D} - \sqrt{1-D}) \quad . \quad (4.13)$$

For spacetime embedding dimension $d \leq 1$, we find from (4.11) and (4.13) that Q and α are both real (with $\gamma \leq Q/2$). The $D \leq 1$ domain is thus where the Liouville theory is well-defined and most easily interpreted. For $D \geq 25$, on the other hand, both α and Q are imaginary. To define a real physical metric $g = e^{\alpha\varphi}\tilde{g}$, we need to Wick rotate $\varphi \rightarrow -i\varphi$. (This changes the sign of the kinetic term for φ . Precisely at $D = 25$ we can interpret $X^0 = -i\varphi$ as a free time coordinate. In other words, for a string naively embedded in 25 flat euclidean dimensions, the Liouville mode turns out to

¹⁶ Recall that h is given by the leading term in the operator product expansion $T(z)e^{\alpha\varphi}(w) \sim h e^{\alpha\varphi}/(z-w)^2 + \dots$. Recall also that for a conventional energy-momentum tensor $T = -\frac{1}{2}\partial\varphi\partial\varphi$, the conformal weight of $e^{\alpha\varphi}$ is $h = \bar{h} = \alpha^2/2$.

¹⁷ The choice of root for α is determined by making contact with the classical limit of the Liouville action. Note that the effective coupling in (4.9) goes as $(25-D)^{-1}$ so the classical limit is given by $D \rightarrow \infty$. In this limit the above choice of root has the classical $\alpha \rightarrow 0$ behavior.

provide automatically a single timelike dimension, dynamically realizing a string embedded in 26 dimensional minkowski spacetime. Finally, in the regime $1 < D < 25$, α is complex, and Q is imaginary. Sadly, it is not yet known how to make sense of the Liouville approach for the regime of most physical interest. We mention as well that so-called non-critical strings (i.e. whose conformal anomaly is compensated by a Liouville mode) in D dimensions can always be reinterpreted as critical strings in $D + 1$ dimensions, where the Liouville mode provides the additional (interacting) dimension. (The converse, however, is not true since it is not always possible to gauge-fix a critical string and artificially disentangle the Liouville mode.)

It remains to extract the string susceptibility γ of (1.13) in this formalism. We write the partition function for fixed area A as

$$Z(A) = \int D\varphi DX e^{-S} \delta\left(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\varphi} - A\right), \quad (4.14)$$

where for convenience we now group the ghost determinant and integration over moduli into DX . We define a string susceptibility γ as in (1.14) by

$$Z(A) \sim A^{(\gamma-2)\lambda/2-1}, \quad A \rightarrow \infty,$$

and determine γ by a simple scaling argument. (Note that for genus zero, we have $Z(A) \sim A^{\gamma-3}$ as in (1.11).) Under the shift $\varphi \rightarrow \varphi + \rho/\alpha$ for ρ constant, the measure in (4.14) does not change. The change in the action (4.10) comes from the term

$$\frac{Q}{8\pi} \int d^2\xi \sqrt{\hat{g}} \hat{R} \varphi \rightarrow \frac{Q}{8\pi} \int d^2\xi \sqrt{\hat{g}} \hat{R} \varphi + \frac{Q}{8\pi} \frac{\rho}{\alpha} \int d^2\xi \sqrt{\hat{g}} \hat{R}.$$

Substituting in (4.14) and using the Gauss-Bonnet formula $\frac{1}{4\pi} \int d^2\xi \sqrt{\hat{g}} \hat{R} = \chi$ together with the identity $\delta(\lambda x) = \delta(x)/|\lambda|$ gives $Z(A) = e^{-Q\rho\chi/2\alpha-\rho} Z(e^{-\rho/\alpha}A)$.

We may now choose $e^{-\rho/\alpha} = A$, which results in

$$Z(A) = A^{-Q\chi/2\alpha-1} Z(1) = A^{(\gamma-2)\lambda/2-1} Z(1),$$

and we confirm from (4.11) and (4.13) that

$$\gamma = 2 - \frac{Q}{\alpha} = \frac{1}{12} (D-1 - \sqrt{(D-25)(D-1)}). \quad (4.15)$$

This result reproduces (1.13), which we used to compare with the result of the matrix model calculation (recall that $\gamma = -1/m$ for $D = 1 - 6/m(m+1)$).

4.2. Dressed operators / dimensions of fields

Now we wish to determine the effective dimension of fields after coupling to gravity. Suppose that Φ_0 is some spinless primary field in a conformal field theory with conformal weight $h_0 = h(\Phi_0) = \bar{h}(\Phi_0)$ before coupling to gravity. The gravitational “dressing” can be viewed as a form of wave function renormalization that allows Φ_0 to couple to gravity. The dressed operator $\Phi = e^{\beta\varphi}\Phi_0$ is required to have dimension (1,1) so that it can be integrated over the surface Σ without breaking conformal invariance. (This is the same argument used prior to (4.13) to determine α). Recalling the formula (4.12) for the conformal weight of $e^{\beta\varphi}$, we find that β is determined by the condition

$$h_0 - \frac{1}{2}\beta(\beta - Q) = 1. \quad (4.16)$$

We may now associate a critical exponent h to the behavior of the one-point function of Φ at fixed area A ,

$$F_\Phi(A) \equiv \frac{1}{Z(A)} \int D\varphi DX e^{-S} \delta\left(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\varphi} - A\right) \int d^2\xi \sqrt{\hat{g}} e^{\beta\varphi} \Phi_0 \sim A^{1-h}. \quad (4.17)$$

This definition conforms to the standard convention that $h < 1$ corresponds to a relevant operator, $h = 1$ to a marginal operator, and $h > 1$ to an irrelevant operator (and in particular that relevant operators tend to dominate in the infrared, i.e. large area, limit).

To determine h , we employ the same scaling argument that led to (4.15). We shift $\varphi \rightarrow \varphi + \rho/\alpha$ with $e^{-\rho/\alpha} = A$ on the right hand side of (4.17), to find

$$F_\Phi(A) = \frac{A^{-Q\chi/2\alpha-1+\beta/\alpha}}{A^{-Q\chi/2\alpha-1}} F_\Phi(1) = A^{\beta/\alpha} F_\Phi(1),$$

where the additional factor of $e^{\beta\rho/\alpha} = A^{\beta/\alpha}$ comes from the $e^{\beta\varphi}$ gravitational dressing of Φ_0 . The gravitational scaling dimension h defined in (4.17) thus satisfies

$$h = 1 - \beta/\alpha. \quad (4.18)$$

Appealing to the semiclassical argument employed before (4.13), we solve (4.16) for β with the branch

$$\beta = \frac{1}{2}Q - \sqrt{\frac{1}{4}Q^2 - 2 + 2h_0} = \frac{1}{\sqrt{12}}(\sqrt{25 - D} - \sqrt{1 - D + 24h_0})$$

(for which $\beta \leq Q/2$, and $\beta \rightarrow 0$ as $D \rightarrow -\infty$). Finally we substitute the above result for β and the value (4.13) for α into (4.18), and find¹⁸

$$h = \frac{\sqrt{1 - D + 24h_0} - \sqrt{1 - D}}{\sqrt{25 - D} - \sqrt{1 - D}}. \quad (4.19)$$

As an example, we apply these results to the minimal models [36] mentioned at the end of section 2. These have a set of operators labelled by two integers p, q (satisfying $1 \leq r \leq q-1$, $1 \leq s \leq p-1$). Coupled to gravity, these operators turn out to have dressed conformal weights

$$h_{r,s} = \frac{p+q - |pr - qs|}{p+q-1} \quad 1 \leq r \leq q-1, \quad 1 \leq s \leq p-1, \quad (4.20)$$

in agreement with the weights determined from the (p, q) formalism discussed in section 3 for the generalized KdV hierarchy (see e.g. [19,56]).

More explicitly, we consider the first member of the unitary discrete series, i.e. the $D = 1/2$ Ising model, which has $(p, q) = (4, 3)$. Before coupling to gravity, critical exponents ν, α, β can be defined in terms of the divergences of correlation length $\xi \sim t^{-\nu}$, specific heat $C \sim t^{-\alpha}$, and magnetization $m \sim t^\beta$ with respect to the deviation $t = (T - T_c)/T_c$ from the critical temperature T_c . In terms of the conformal weights of the energy and spin operators h_e and h_s , these exponents satisfy $\nu = \frac{1}{2(1-h_e)}$, $\alpha = 2(1-\nu)$, $\beta = (2-\alpha)h_s$. According to (4.20), the coupling to gravity induces the shifts $h_e = \frac{1}{2} \rightarrow \frac{2}{3}$, $h_s = \frac{1}{16} \rightarrow \frac{1}{6}$, which implies corresponding shifts in ν, α , and β .

¹⁸ We can also substitute $\beta = \alpha(1-h)$ from (4.18) into (4.16) and use $-\frac{1}{2}\alpha(\alpha-Q) = 1$ (from before (4.13)) to rederive the result $h - h_0 = h(1-h)\alpha^2/2$ for the difference between the "dressed weight" h and the bare weight h_0 [17].

References

- [1] M. B. Green and J. Schwarz (remember them?), Phys. Lett. 149B (1984) 117.
- [2] M. B. Green, private communication (1987).
- [3] M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635.
- [4] E. Brézin and V. Kazakov, Phys. Lett. B236 (1990) 144.
- [5] D. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 127; Nucl. Phys. B340 (1990) 333.
- [6] A. M. Polyakov, Phys. Lett. 103B (1981) 207, 211.
- [7] A. M. Polyakov, lecture at Northeastern Univ., spring 1990.
- [8] S. H. Shenker, private communication (1989).
- [9] L. Alvarez-Gaumé, "Random surfaces, statistical mechanics, and string theory", Lausanne lectures, winter 1990.
- [10] A. Bilal, "2d gravity from matrix models," John Hopkins Lectures, CERN TH5867/90; V. Kazakov, "Bosonic strings and string field theories in one-dimensional target space," LPTENS 90/30, to appear someday in *Random surfaces and quantum gravity*, proceedings of 1990 Cargèse workshop, edited by O. Alvarez, E. Marinari, and P. Windey (publication scandalously delayed indefinitely due to the middle editor's inability to provide an analytic index to the publisher); D. Gross, lectures at Jerusalem winter conference (winter 90/91), and lectures at Les Houches (1991) are probably edifying, but repeated requests for them go unanswered; I. Klebanov, "String Theory in Two Dimensions", Trieste lectures, spring 1991, Princeton preprint PUPT-1271 (hep-th@xxx/9108019); D. Kutasov, "Some Properties of (Non) Critical Strings", Trieste lectures, spring 1991, Princeton preprint PUPT-1277 (hep-th@xxx/9110041); J. Mañes and Y. Lozano, Introduction to Nonperturbative 2d quantum gravity, Barcelona preprint UB-ECM-PF3/91.
- [11] F. David, Nucl. Phys. B257[FS14] (1985) 45, 543; J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B257[FS14] (1985) 433; J. Fröhlich, in: *Lecture Notes in Physics*, Vol. 216, ed. L. Garrido (Springer, Berlin, 1985); V. A. Kazakov, I. K. Kostov and A. A. Migdal, Phys. Lett. 157B (1985)

- 295: D. Boulatov, V. A. Kazakov, I. K. Kostov and A. A. Migdal, Phys. Lett. B174 (1986) 87; Nucl. Phys. B275[FS17] (1986) 641.
- [12] G. 't Hooft, Nucl. Phys. B72 (1974) 461.
- [13] E. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, Comm. Math. Phys. 59 (1978) 35.
- [14] D. Bessis, C. Itzykson, and J.-B. Zuber, Adv. Appl. Math. 1 (1980) 109.
- [15] V. A. Kazakov and A. A. Migdal, Nucl. Phys. B311 (1988) 171.
- [16] P. Ginsparg, "Applied conformal field theory" Les Houches Session XLIV, 1988, *Fuchs, Strings, and Critical Phenomena*, ed. by E. Brézin and J. Zinn-Justin (1989).
- [17] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.
- [18] I. K. Kostov, Nucl. Phys. B326 (1989) 583.
- [19] M. R. Douglas, Phys. Lett. B238 (1990) 176.
- [20] P. Di Francesco and D. Kutasov, Nucl. Phys. B342 (1990) 589; and Princeton preprint PUPT-1206 (1990) to appear in proceedings of Cargèse workshop (1990).
- [21] A. Gupta, S. Trivedi and M. Wise, Nucl. Phys. B340 (1990) 475.
- [22] M. Goulian and M. Li, Phys. Rev. Lett. 66 (1991) 2051.
- [23] P. Di Francesco and D. Kutasov, Phys. Lett. 261B (1991) 385; P. Di Francesco and D. Kutasov, "World sheet and space time physics in two dimensional (super) string theory," Princeton preprint PUPT-1276 (hep-th@xxx/9109005).
- [24] M. Bershadsky and I. Klebanov, Phys. Rev. Lett. 65 (1990) 3088.
- [25] E. Brézin, V. A. Kazakov, and A. B. Zamolodchikov, Nucl. Phys. B338 (1990) 673; G. Parisi, Phys. Lett. B238 (1990) 209, 213; Europhys. Lett. 11 (1990) 595; D. J. Gross and N. Miljkovic, Phys. Lett. B238 (1990) 217.
- [26] P. Ginsparg and J. Zinn-Justin, Phys. Lett. B240 (1990) 333.
- [27] G. Moore, "Double scaled field theory at $c = 1$ ", Rutgers preprint RU-91-12, to appear in Nucl. Phys. B; G. Moore and N. Seiberg, "From loops to fields in 2-d quantum gravity", Rutgers preprint RU-91-29 (1991), to appear in Int. J. Mod. Phys.
- [28] D. Gross and I. Klebanov, Nucl. Phys. B359 (1991) 3; D. Gross and I. Klebanov, Nucl. Phys. B352 (1991) 671; D. Gross, I. Klebanov, and M. Newman, Nucl. Phys. B350 (1991) 621.
- [29] I. Kostov, "Strings embedded in Dynkin Diagrams", SACLAY-SPHT-90-133 (1990), to appear in proceedings of Cargèse Workshop (1990); Phys. Lett. B266 (1991) 42.
- [30] S. Das and A. Jevicki, Mod. Phys. Lett. A5 (1990) 1639; A. Sengupta and S. Wadia, Int. J. Mod. Phys. A6 (1991) 1961; G. Mandal, A. Sengupta, and S. Wadia, Mod. Phys. Lett. A6 (1991) 1465; K. Demeterfi, A. Jevicki, and J.P. Rodrigues, Nucl. Phys. B362 (1991) 173, and "Scattering amplitudes and loop corrections in collective string field theory. 2", Brown preprint 803 (1991); J. Polchinski, Nucl. Phys. B362 (1991) 125.
- [31] C. Crnković, P. Ginsparg, and G. Moore, Phys. Lett. B237 (1990) 196.
- [32] V. Kazakov, Mod. Phys. Lett. A4 (1989) 2125.
- [33] P. Ginsparg and J. Zinn-Justin, "Action principle and large order behavior of non-perturbative gravity", LA-UR-90-3687/SPhT/90-140 (1990), to appear in *Random surfaces and quantum gravity*, proceedings of 1990 Cargèse workshop, edited by O. Alvarez, E. Marinari, and P. Windley; P. Ginsparg and J. Zinn-Justin, Phys. Lett. B255 (1991) 189.
- [34] F. David, "Non-perturbative effects in 2D gravity and matrix models," Saclay-SPHT-90-178, to appear in proceedings of Cargèse workshop (1990).
- [35] E. Brézin, M. Douglas, V. Kazakov, and S. Shenker, Phys. Lett. B237 (1990) 43; D. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 717
- [36] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B241 (1984) 333.
- [37] T. Banks, M. Douglas, N. Seiberg, and S. Shenker, Phys. Lett. B238 (1990) 279.
- [38] R. Dijkgraaf, H. Verlinde, and E. Verlinde, "Notes on topological string theory and 2D quantum gravity", Princeton preprint PUPT-1217, to appear in proceedings of Cargèse workshop (1990).
- [39] M. L. Mehta, Comm. Math. Phys. 79 (1981) 327; S. Chadha, G. Mahoux and M. L. Mehta, J. Phys. A14 (1981) 579; C. Itzykson and J.B. Zuber, J. Math. Phys. 21 (1980) 411.
- [40] V. G. Drinfel'd and V. V. Sokolov, Jour. Sov. Math. (1985) 1975; G. Segal and G. Wilson, Pub. Math. I.H.E.S. 61 (1985), 5.
- [41] I. M. Gel'fand and L. A. Dikii, Russian Math. Surveys 30:5 (1975) 77; I. M. Gel'fand and L. A. Dikii, Funct. Anal. Appl. 10 (1976) 259.

- [42] M. Douglas. "The two-matrix model", to appear in proceedings of 1990 Cargèse workshop.
- [43] T. Tada, Phys. Lett. B259 (1991) 442.
- [44] S.Kharchev, A.Marshakov, A.Mironov, A.Morozov, A.Zabrodin, "Unification of All String Models with $c < 1$ ". Lebedev preprint FIAN/TD-9/91 (hep-th/9111037).
- [45] V. Kazakov, Phys. Lett. 119A (1986) 140;
D. Boulatov and V. Kazakov, Phys. Lett. 186B (1987) 379.
- [46] D.J. Gross and I. Klebanov, Nucl. Phys. B344 (1990) 475; Nucl. Phys. B354 (1991) 459.
- [47] J. Polchinski, Nucl. Phys. B346 (1990) 253.
- [48] P. Ginsparg, "Some statistical mechanical models and conformal field theories," lectures given at Trieste spring school, 1989, published in M. Green and A. Strominger, eds., *Superstrings '89*, World Scientific 1990.
- [49] T. Curtright and C. E. Thorn, Phys. Rev. Lett. 48 (1982) 1309;
E. Braaten, T. Curtright and C. B. Thorn, Phys. Lett. 118B (1982) 115, Ann. Phys. 147 (1983) 365;
E. Braaten, T. Curtright, G. Ghandour and C. B. Thorn, Phys. Rev. Lett. 51 (1983) 19, Ann. Phys. 153 (1984) 147.
- [50] J.L. Gervais and A. Neveu, Nucl. Phys. B199 (1982) 59; B209 (1982) 125; B224 (1983) 329; B238 (1984) 125,396; Phys. Lett. 151B (1985) 271.
- [51] F. David, Mod. Phys. Lett. A3 (1988) 1651;
J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509.
- [52] N. Seiberg, "Notes on quantum Liouville theory and quantum gravity, in Cargèse", to appear in *Random surfaces and quantum gravity*, proceedings of 1990 Cargèse workshop, edited by O. Alvarez, E. Marinari, and P. Windey;
J. Polchinski, "Remarks on the Liouville field theory, UT Austin preprint UTTC 19-90 (1990), presented at Strings '90 conference, College Stn, TX;
E. D'Hoker, "Continuum approaches to 2-D gravity, UCLA/91/TEP/41, review talk at Stonybrook Strings and Symmetries conference, May 1991.
- [53] D. Friedan, Les Houches lectures summer 1982, in *Recent Advances in Field Theory and Statistical Physics*, J.-B. Zuber and R. Stora eds, (North Holland, 1984).
- [54] O. Alvarez, in *Unified String Theories*, M. Green and D. Gross, eds., (World Scientific, Singapore, 1986).
- [55] N. E. Mavromatos and J. L. Miramontes, Mod.Phys.Lett. A4 (1989) 1847;
E. d'Hoker and P. S. Kurzepa, Mod.Phys.Lett. A5 (1990) 1411.
- [56] P. Ginsparg, M. Goulian, M. R. Plesser, and J. Zinn-Justin, Nucl. Phys. B342 (1990) 539