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NUMERICAL INVESTIGATION OF RECIRCULATION IN THE UTSI MHD COMBUSTOR

Topical Report

By
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September 1983

Work Performed Under Contract No. AC02-79ET10815

University of Tennessee Space Institute
Tullahoma, Tennessee

U. S. DEPARTMENT OF ENERGY

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ABSTRACT

Numerical studies were carried out to investigate the gross structure of flow in cylindrical combustors. The combustor configurations studied are variations of a working design used at The University of Tennessee Space Institute to burn pulverized coal at temperatures in excess of 3000K for generation of a plasma feeding a magnetohydrodynamic channel. The numerical studies were conducted for an isothermal fluid; the main objective of the calculations was to study the effect of the oxidant injection pattern on the gross structure of recirculating flows within the combustor. The calculations illustrate the basic features of the flow in combustors of this type and suggest implications for the injection of coal and oxidizer in this type of combustor.

CONTENTS

FIGURES

TABLES

Nomenclature

C_1	Constant in governing eq. (6), 1.45
C_2	Constant in governing eq. (6), 2.00
C_3	Constant in governing eq. (7), 0.09
D	Combustor diameter, m
k	Turbulent kinetic energy, m^2/sec^2
K	Thermal Conductivity, W/sec m K
L	Combustor length, m
P	Static pressure, N/m^2
r	Radial coordinate in physical plane, m
R1	Radius of core, m
R2	Radius of outer wall, m
u	Axial Velocity, m/sec
U	Inlet bulk velocity, m/sec
v	Radial velocity, m/sec
z	Axial coordinate in physical plane, m

Greek Letters

ρ	Density, kg/m^3
μ	Dynamic viscosity, $\mu_t + \mu_l$, N sec/m^2
μ_l	Laminar viscosity, N sec/m^2
μ_t	Turbulent viscosity, N sec/m^2
ω	Vorticity, $\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r}$, 1/sec
ψ	Stream function, $\int(\rho urdr - \rho vrdz)$, kg/sec
ε	Dissipation rate of turbulent kinetic energy, W
σ_k	Prandtl number for the transport of k, 1.0
σ_ε	Prandtl number for the transport of ε, 1.3
ϕ	General dependent variable
ξ	Axial coordinate in computational plane
η	Radial coordinate in computational plane

1.0 INTRODUCTION

1.1 Recirculating Flows in Combustors

The purpose of this study is to provide conceptual understanding of the complex aerodynamic environment that exists in the primary combustor of the Coal Fired Flow Facility (CFFF). This combustor burns pulverized seeded coal and oil with preheated mixtures of oxygen and nitrogen to generate a plasma for MHD flow experiments and downstream heat recovery and pollutant control systems. The primary combustor operates based on recirculating flow mixing. It is important to understand the characteristics of recirculation zones, shown schematically in Figure 1.

Zones of recirculating fluid in combustors exist in general and, for many types of combustors, e.g. turbojet and ramjet engine combustors, the recirculation is sometimes created by physical devices called flameholders. It is desirable that the recirculation zones be generated by purely aerodynamic methods to avoid the mechanical problems associated with flameholder hardware. The usual aerodynamic mechanisms for generating recirculation zones are jet mixing and designing of the combustor with an abrupt increase in cross-sectional area. The latter method gives rise to combustors that are called sudden-expansion or "dump" combustors.

The recirculation zones act as flameholders by providing a flow regime where the flames may persist and propagate to the remainder of the reacting flow. The recirculation zones prevent flame blow-off, increase the effective residence time for thermochemical processes to occur, and, when properly oriented in the combustor, produce steady, highly efficient combustor operation. The present study examines the orientation, location, and strength (or intensity) of recirculation zones in the UTSI MHD combustor for various oxidizer injection radial locations and for various combustor geometries. The study does not attempt to simulate the effect of the pulverized coal jets on the mixing and recirculation processes. In the future these effects will be investigated, but at present, they are thought to be small.

Different aerodynamic environments affect coal injector performance; the injectors must distribute the seeded pulverized coal in an intensely turbulent flow. For a coal injector which is centrally located, the intensity of the recirculating fluid on the centerline near the coal injector is of critical importance. Too strong a recirculation will result in the pulverized coal blowing back on the injector; too weak a recirculation will result in the coal jets penetrating too far downstream and not being adequately dispersed and mixed with surrounding oxidizer. It seems reasonable to assume that for such a combustor, the fluid in a central recirculation zone near the axis will be fuel rich, partially burned products of combustion. Hence, the output chemistry of the flow leaving any such combustor will depend on the combustion occurring in the central and outer annular recirculation zones and how this flow leaves these zones to mix and pass downstream. Clues to flow field behavior are provided in a gross sense by calculated results of the kind presented herein. A preliminary study of this type was carried out for simple cylindrical combustors and presented in Reference 1.

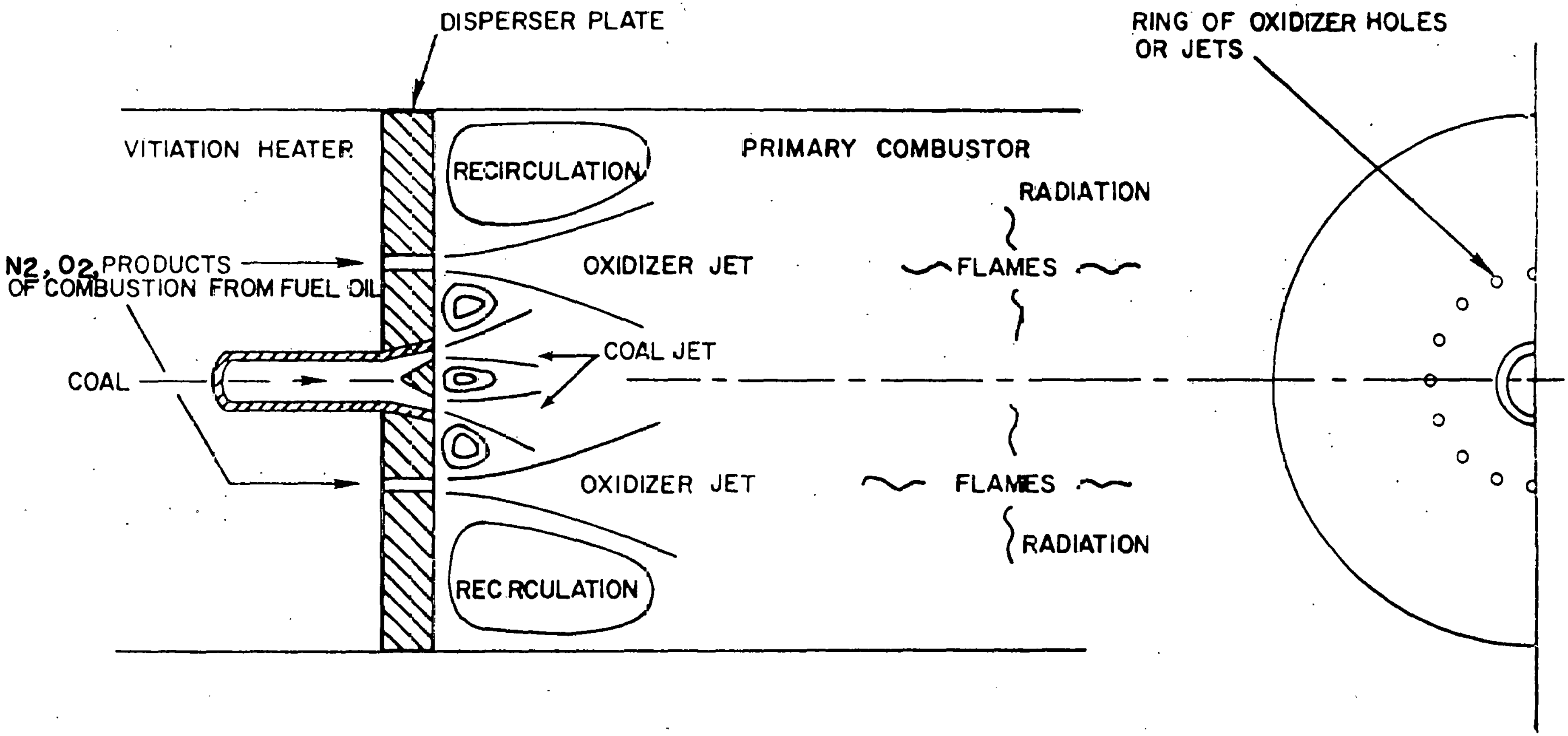


Figure 1. Schematic of Primary Combustor Flow Field Created by Disperser Plate

1.2 The UTSI MHD Combustor

The UTSI MHD combustor was generally thought to operate as shown schematically in Figure 1. Oxidizer jets enter the combustor through annular rings of holes in the upstream end of the combustor. Pulverized coal is injected into the primary combustor through a central coal injector, and enters the combustor in multiple, dense phase jets. The oxidizer jets penetrate and mix with the coal jets, and, since the oxidizer is preheated to about 1140K, the coal and oxidizer react and burn. The coal is seeded with potassium carbonate (K_2CO_3) and at the equilibrium temperature of about 3000K, the products of combustion are electrically conducting because of potassium atom species ionization.

The combustor generates plasma that is fed to a MHD channel downstream through a sonic or choked flow nozzle. For the UTSI experiments, the nozzle downstream of the choked or sonic throat continues to expand in cross-sectional area, hence, the flow is expanded to low (Mach 1.25) supersonic flow speed before entering the channel. Figure 2 shows a schematic of the UTSI upstream combustor system up to the nozzle inlet station.

2.0 NUMERICAL SOLUTION PROCEDURE

2.1 The Governing Equations for Isothermal Flows

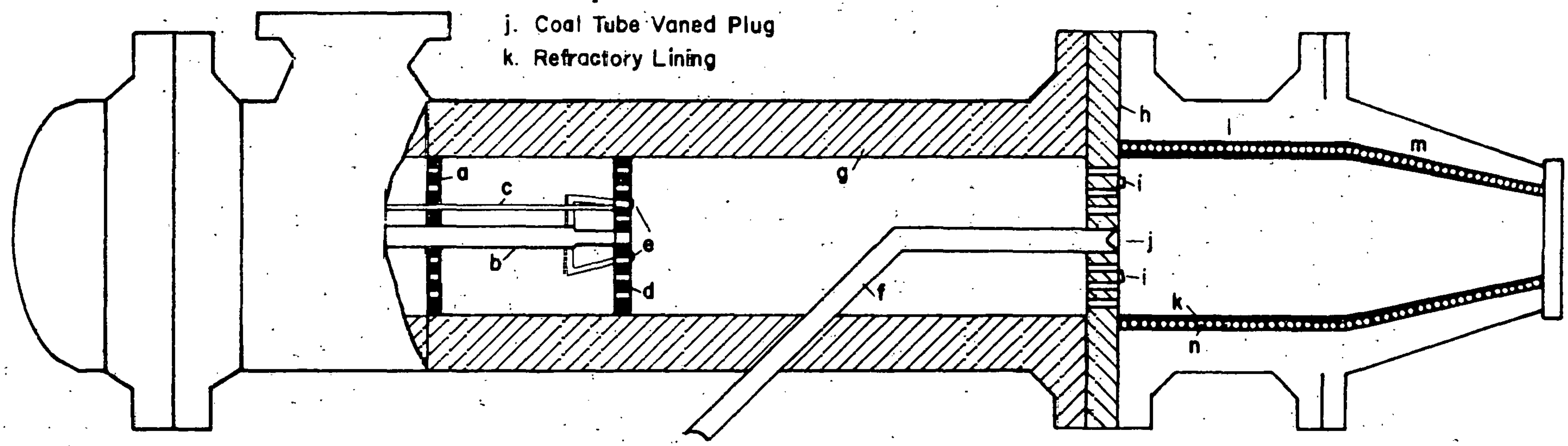
The numerical solution procedure for studying the UTSI MHD combustor flow fields is a Navier-Stokes solver. The Navier-Stokes equations are recast into a vorticity transport equation, shown in Table I, equation (1). To close the system of equations, a relationship between stream function and vorticity was derived, equation (2), of Table I. For laminar, incompressible flows, the system of equations is complete with the specification of the viscosity, equation (3), of Table I. For compressible flows, an energy equation is also required for closure through the equation of state. For either incompressible flows or compressible flows, if the pressure field is required, an additional equation can be derived from the Navier-Stokes equation, for example equation (4), of Table I, (taken from Ref. 2), to "recover" pressure. The flows that were calculated in the present study were treated as isothermal and incompressible to provide a qualitatively correct understanding of the behavior of the flows.

Finally, for turbulent flows, it is assumed that the vorticity transport equation, equation (1), was valid to describe the motion of the flow when the laminar viscosity was replaced by an effective or turbulent viscosity. A model for turbulent viscosity is thus required and two models are often used. These are a constant effective viscosity, also given by equation (3), and a two-equation model based on turbulent kinetic energy and its (locally isotropic) rate of dissipation, given by equations (5) and (6) respectively. For the two-equation model of turbulent transport, the turbulent viscosity is related to the turbulence kinetic energy and dissipation by equation (7), of Table I.

2.2 Numerical Solution System

The present study used coordinate transformations to transform the actual axisymmetric combustor geometry and the equivalent nozzle geometry into a uniform cylindrical coordinate geometry, Figure 3. Then each

a. Porous Plate Flow Straightner
b. Fuel Oil Line & Central Support Sting
c. Spark Igniter
d. Porous Plate for Mixing & Injection of Fuel Oil
e. Oil Injectors
f. Coal Tube
g. Cast Refractory
h. Disperser Plate
i. Oil Injectors
j. Coal Tube Vaned Plug
k. Refractory Lining



l. Cylindrical Section
m. Conical Section
n. Cooling Tubes

Figure 2. Schematic of Vitiation Heater and Primary Combustor Hardware as Tested

TABLE I - SYSTEM OF GOVERNING EQUATIONS

VORTICITY EQUATION

$$\frac{\partial^2\omega}{\partial z^2} + \frac{\partial^2\omega}{\partial r^2} - \left(\frac{\rho u}{\mu} - \frac{2}{\mu}\frac{\partial\mu}{\partial z}\right)\frac{\partial\omega}{\partial z} - \left(\frac{\rho v}{\mu} - \frac{2}{\mu}\frac{\partial\mu}{\partial r} - \frac{1}{r}\right)\frac{\partial\omega}{\partial r}$$

$$+\left[\frac{1}{\mu}\frac{\partial^2\mu}{\partial z^2} + \frac{1}{\mu}\frac{\partial^2\mu}{\partial r^2} + \left(\frac{\rho v}{\mu r} + \frac{1}{\mu r}\frac{\partial\mu}{\partial r} - \frac{1}{r^2}\right)\right]\omega$$

$$+ \frac{2}{\mu}\left[\frac{\partial^2\mu}{\partial z\partial r}\left(\frac{\partial v}{\partial r} - \frac{\partial u}{\partial z}\right) + \frac{\partial^2\mu}{\partial z^2}\frac{\partial u}{\partial r} - \frac{\partial^2\mu}{\partial r^2}\frac{\partial v}{\partial z} + \frac{\partial\mu}{\partial z}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r}\right)\right.$$

$$\left.- \frac{\partial\mu}{\partial r}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r}\right)\right] + \frac{1}{2\mu}\left[\frac{\partial(u^2+v^2)}{\partial z}\frac{\partial\rho}{\partial r} - \frac{\partial(u^2+v^2)}{\partial r}\frac{\partial\rho}{\partial z}\right] = 0 \quad (1)$$

STREAM FUNCTION EQUATION

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{\rho}\frac{\partial\rho}{\partial z}\frac{\partial\psi}{\partial z} - \left(\frac{1}{\rho}\frac{\partial\rho}{\partial r} + \frac{1}{r}\right)\frac{\partial\psi}{\partial r} + \rho r\omega = 0 \quad (2)$$

LAMINAR VISCOSITY

μ or $\mu_E = \mu_{ref}$ (constant effective viscosity) (3)

PRESSURE EQUATION

$$\frac{\partial^2 P}{\partial z^2} + \frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} - w^2 + u\left(\frac{\partial\omega}{\partial r} + \frac{\omega}{r}\right) - v\left(\frac{\partial\omega}{\partial z}\right)$$

$$+\left[\frac{\partial^2}{\partial z^2}\left(\frac{u^2+v^2}{2}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{u^2+v^2}{2}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{u^2+v^2}{2}\right)\right] = 0 \quad (4)$$

TURBULENT KINETIC ENERGY EQUATION

$$\frac{\partial^2 k}{\partial z^2} + \frac{\partial^2 k}{\partial r^2} + \left(\frac{1}{\mu}\frac{\partial\mu}{\partial z} - \frac{\rho u\,\sigma_k}{\mu}\right)\frac{\partial k}{\partial z} + \left(\frac{1}{r} + \frac{1}{\mu}\frac{\partial\mu}{\partial r} - \frac{\rho v\,\sigma_k}{\mu}\right)\frac{\partial k}{\partial r}$$

$$- \frac{\rho\varepsilon\sigma_k}{\mu} + \sigma_k\left\{2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{v}{r}\right)^2\right] + \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right)^2\right\} = 0 \quad (5)$$

TURBULENT KINETIC ENERGY DISSIPATION EQUATION

$$\frac{\partial^2\varepsilon}{\partial z^2} + \frac{\partial^2\varepsilon}{\partial r^2} + \left(\frac{1}{\mu}\frac{\partial\mu}{\partial z} - \frac{\rho u\,\sigma_\varepsilon}{\mu}\right)\frac{\partial\varepsilon}{\partial z} + \left(\frac{1}{r} + \frac{1}{\mu}\frac{\partial\mu}{\partial r} - \frac{\rho v\,\sigma_\varepsilon}{\mu}\right)\frac{\partial\varepsilon}{\partial r}$$

$$- \frac{C_2\rho\varepsilon^2\sigma_\varepsilon}{\mu k} + \frac{C_1\varepsilon\,\sigma_\varepsilon}{k}\left\{2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{v}{r}\right)^2\right] + \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right)^2\right\} = 0 \quad (6)$$

TURBULENT VISCOSITY FOR TWO-EQUATION MODEL

$\mu_t = C_3\rho k^2/\varepsilon,$ (7)

governing equation was transformed using chain-rule differentiation from a physical coordinate (z,r) system into a transformed coordinate (ξ,η) system. Each equation was then transformed into the general form given in equation (8) for use in the actual

$$a_{1\phi}\frac{\partial^2\phi}{\partial\xi^2} + a_{2\phi}\frac{\partial^2\phi}{\partial\eta^2} - b_{1\phi}\frac{\partial\phi}{\partial\xi} - b_{2\phi}\frac{\partial\phi}{\partial\eta} + d_\phi = 0 \qquad (8)$$

finite difference numerical solution method. The numerical solution method is a point-by-point, under/over relaxation method that has been described in References 2 - 6. The details of the application and operation of this solution procedure are not discussed in this paper.

The boundary conditions for the present study are shown in Figure 4. When using the two-equation turbulence model, the turbulent kinetic energy (k) and dissipation (ε) had to be evaluated and assigned at the walls during the course of the solution. The usual procedure of assuming nearly equilibrium conditions (turbulence production = turbulence dissipation) at the walls led to numerical instability if the equations were not relaxed carefully. This resulted in excessive computing times since so many cases were being studied. So to make qualitative estimates of the nature of the recirculating flows, a constant effective viscosity model was used in the majority of the calculations. Essentially, this "laminarizes" the computed flows, which is an acceptable approximation for the purposes of the present study. A more thorough theoretical study has recently been completed, and will be presented in the future, that identifies the effects of turbulence modelling on the predicted internal flow fields of such configurations, Reference 7.

2.3 Parameters of the Calculation

The primary parameters varied in the present study were the combustor length to diameter ratio and the outlet conical contraction angle, for fixed oxidizer inlet conditions. Then, the combustor geometry was fixed and the oxidizer inlet location was varied radially, compared to a base or reference case. In all cases the input mass flux and velocity were held constant. The governing set of variables for each case therefore, were ρ, u, and μ_E, and the geometry of the combustor.

3.0 CALCULATED RESULTS

3.1 Fixed Inlet Conditions for Variable Combustor Geometry

Figures 5, 6, and 7 show, for fixed inlet conditions and approximately equal L/D, the effect of conical contraction angle on the internal recirculation patterns. The contraction angle has a relatively minor effect on the structure of the recirculation zones in the main combustor for this L/D ratio combustor. The outlet velocity profiles are significantly affected, however. The outlet velocity profiles are for incompressible flow and thus are not accurate in terms of magnitude. Nevertheless, the profile shapes can be compared. The profiles shown in Figures 6 and 7 still exhibit a wake-like character which suggests that fluid from the inner and outer recirculation regions may not be well mixed when leaving the combustor. Based on these limited comparisons, it is inferred from a purely aerodynamic viewpoint, that steep (greater than 43°) contractions should not be used for combustors with L/D approximately of the order of unity.

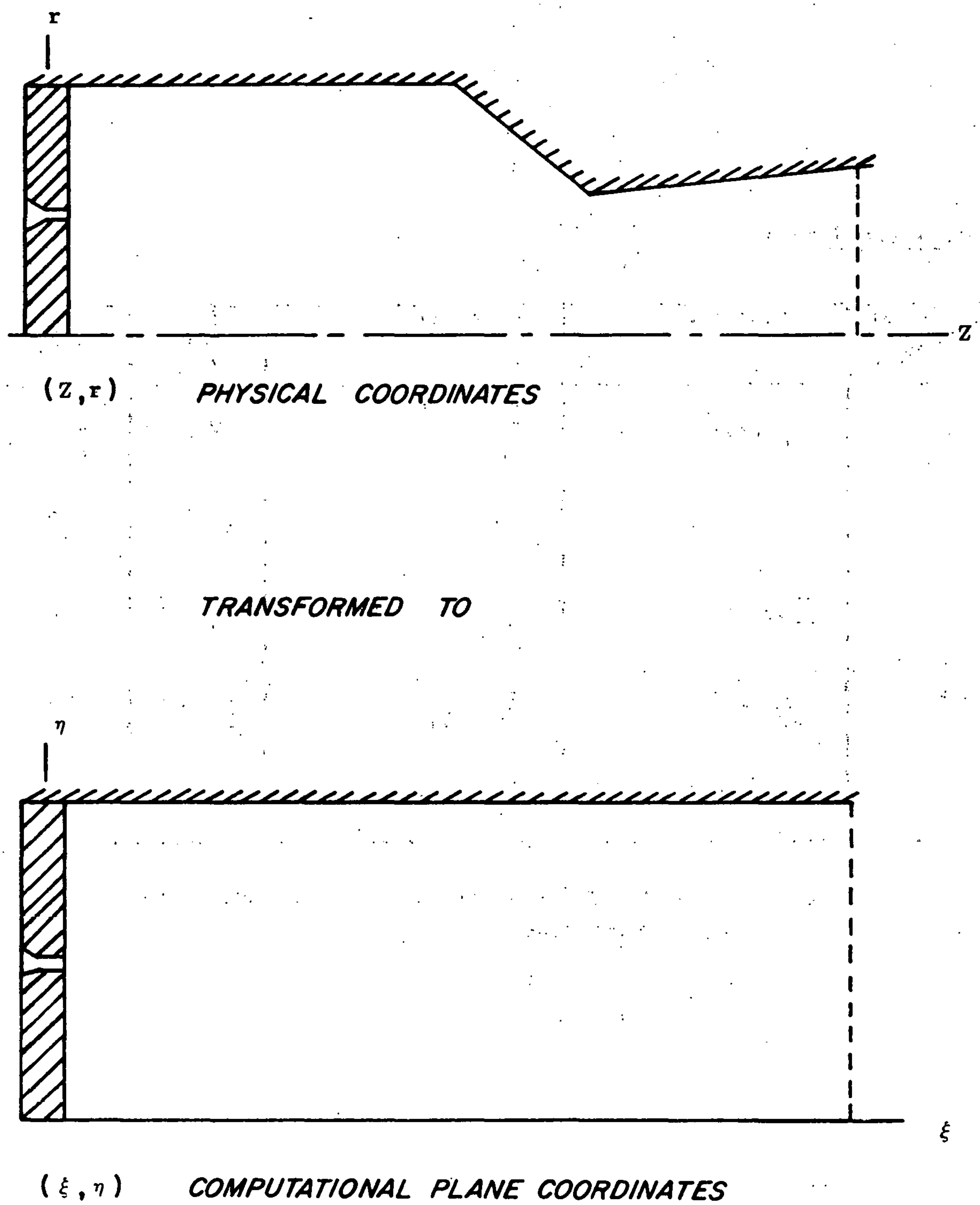


Figure 3. Coordinate Transformation used to Transform Physical Geometry to Uniform Rectangular Computational Geometry

Boundary Conditions

Region / Variable	Inlet Plane 1	Outer Wall 2	Exit Plane 3	AXIS OF SYMMETRY 4
u	$v_o\ f(r)$	0	$\frac{\partial u}{\partial z} = 0$	$\frac{\partial u}{\partial r} = 0$
v	0	0	$\frac{\partial v}{\partial z} = 0$	0
ψ	$\int_o^r \rho u r dr$	$\int_o^{R2} \rho u r dr$	$\frac{\partial \psi}{\partial z} = 0$	0
ω	$\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r}$	$\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r}$	$\frac{\partial \omega}{\partial z} = 0$	0

Figure 4. Boundary Conditions for the Present Study

Nondimensional Units

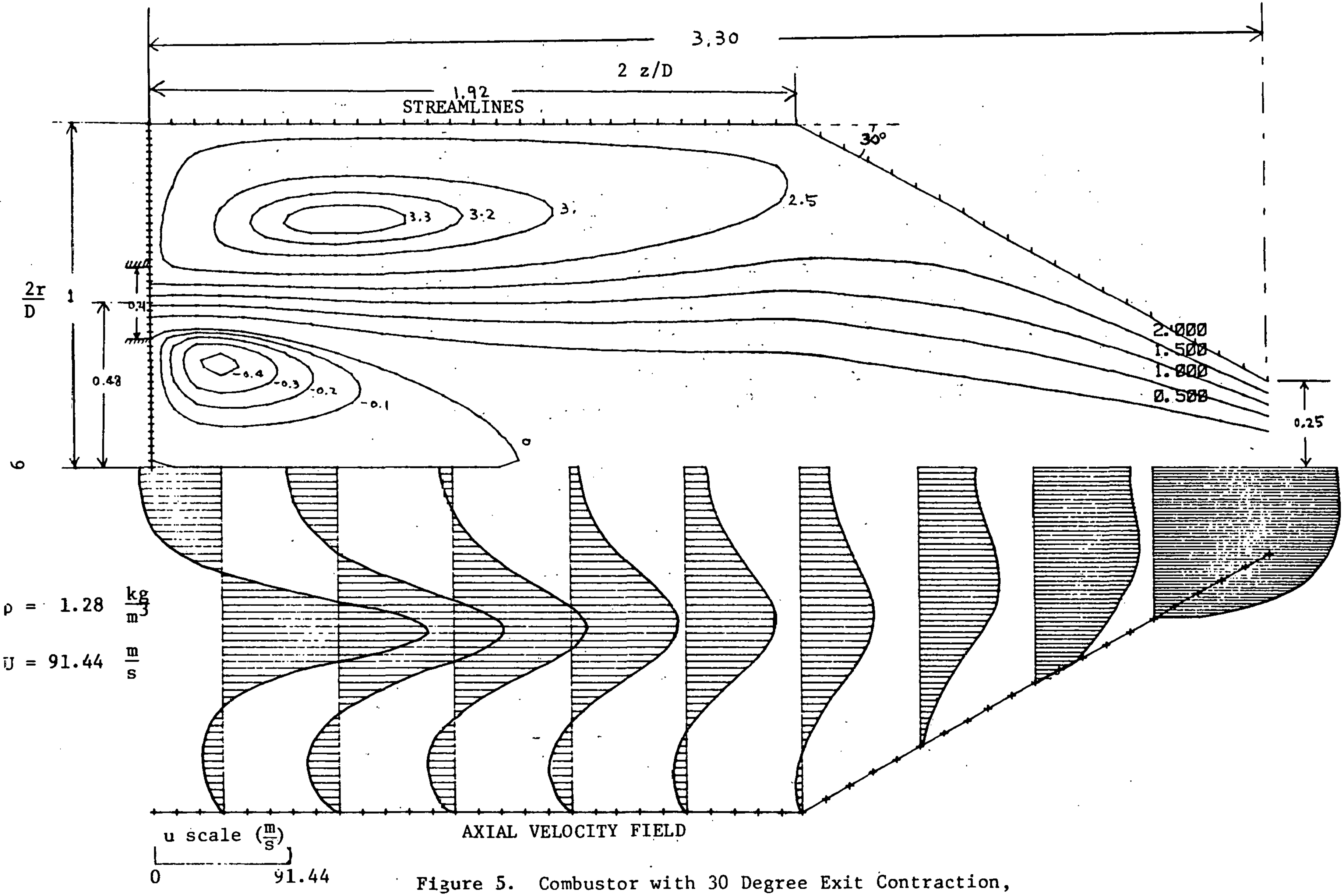


Figure 5. Combustor with 30 Degree Exit Contraction, L/D = 0.96

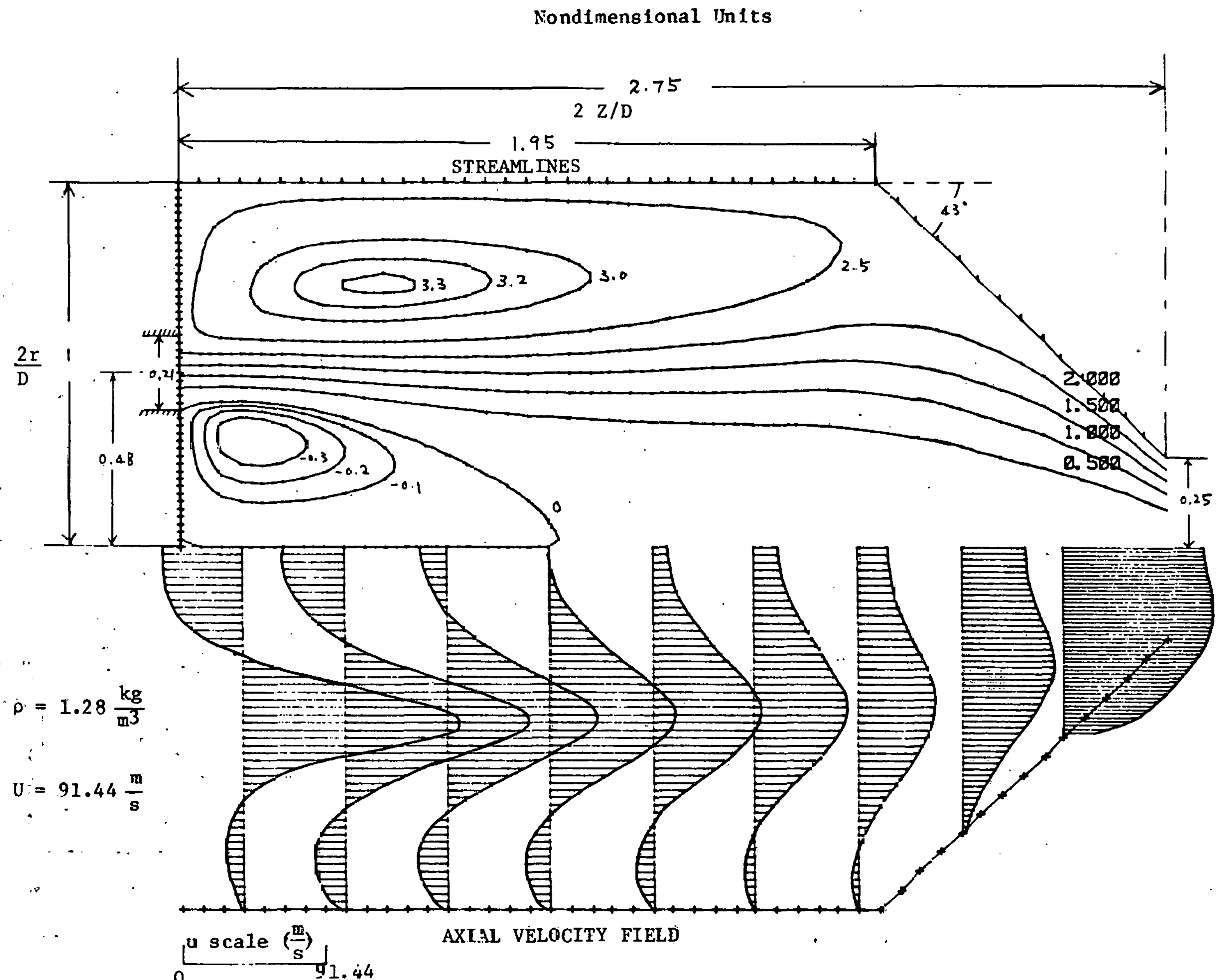


Figure 6. Combustor with 43 Degree Exit Contraction, L/D = 0.975

Nondimensional Units

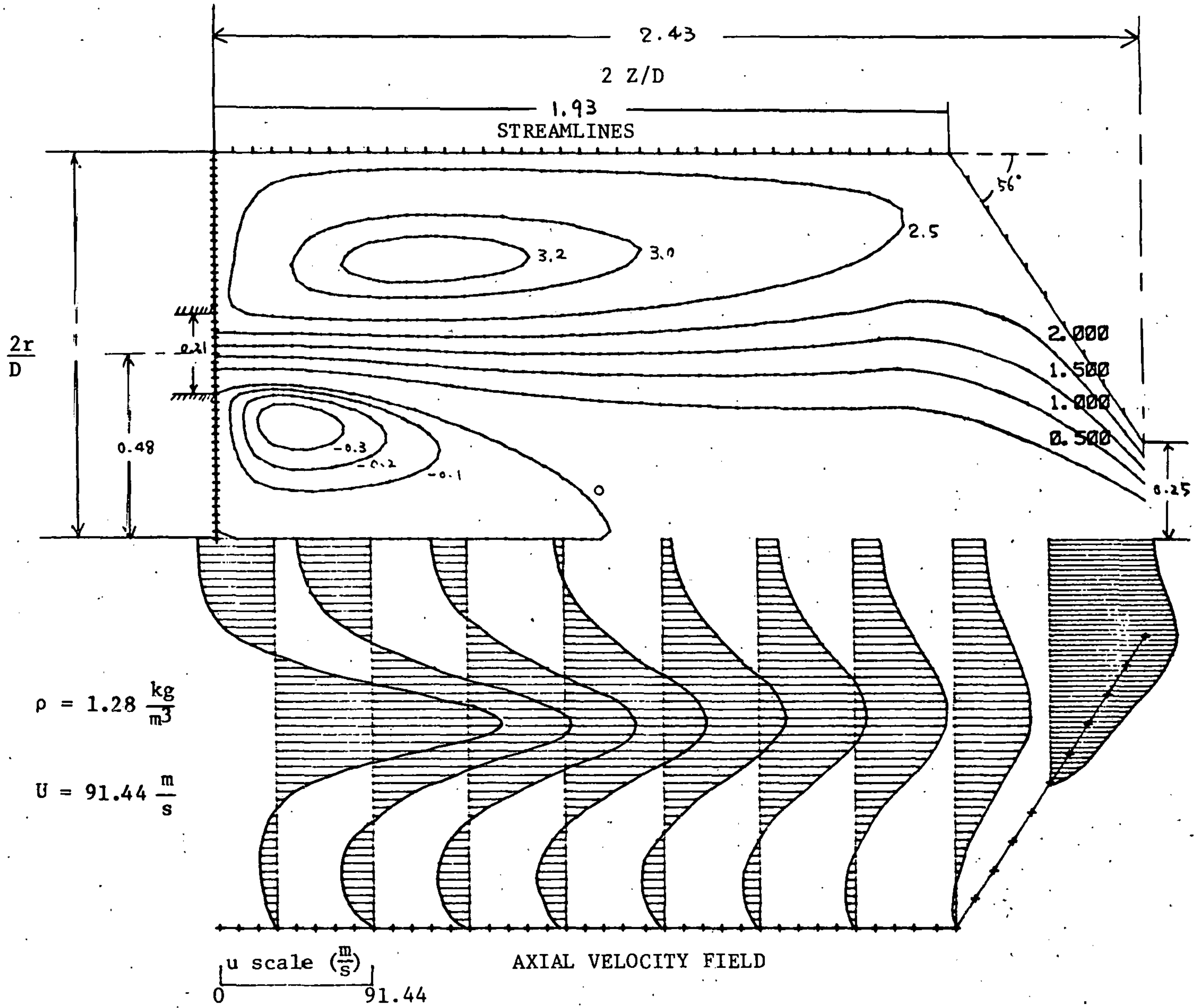


Figure 7. Combustor with 56 Degree Exit Contraction, L/D = 0.965

Figures 8, 9 and 10 show, for fixed inlet conditions and contraction angle, the effect of varying L/D on the recirculation zone structure. A comparison of the figures shows that the structure of the inner or central recirculation zone and the velocity of the backflow, is relatively unaffected by the L/D ratios evaluated. On the other hand, the outer recirculation zone is greatly affected by the length of the combustor. The outer recirculation region can extend into the conical section of a short combustor. The velocities of the backflowing fluid do not seem to be greatly affected in either recirculation zone. The calculations suggest that if reacting flows were contained by these combustors of different length, the flow chemistry and the uniformity of the temperature and concentration profiles at the combustor exit would be very sensitive to combustor length or L/D. For this contraction angle (43°), it takes a relatively long combustor, L/D greater than 1.725, for the wake-like character of the velocity profiles to begin to dissipate, again suggesting that steep contraction angles should be avoided in short combustors, and that a combination of L/D and contraction angle can be found to optimize the combustor performance.

3.2 Fixed Combustor Geometry and Variable Inlet Conditions

The calculations, presented in Fig 11 and 12, together with the results of Fig 6, show the effect of varying radial location of the oxidizer injection ports on the structure of the recirculating flows inside combustors of fixed geometry. For these calculations, the total mass flow, the injection velocity, the L/D ratio, and the contraction angle of the conical outlet have been fixed. Because the flow is treated two-dimensionally, the rings of oxidizer injection holes were replaced by annular slots. The axisymmetric affect of radius of the annular slots requires that the slot widths be decreased for increasing radial location, and vice versa, for constant input stream flow area. The figures show that the internal structure of the flow field inside the combustor is very dependent on the radial location of the oxidizer injection ports. The three figures document the conclusion that for fixed input mass flow, the radial location of the oxidizer jets is the single most important control parameter for establishing the size of the inner and outer recirculation zones. The calculations can be used to argue that favorable locations for the oxidizer holes will probably be at the midspan of the inlet plane or at slightly greater radial locations. Such configurations result in recirculation zones of approximately equal size in the outer annular region and in the central core region into which the coal is injected. Considering the previous results, for a combustor of L/D between 1.0 and 2.0, and a 30° conical contracting combustor outlet, the oxidizer injection pattern should be near optimum for performance based on generating a uniform temperature and composition plasma when located at the midpoint between the centerline and the outer wall. This is about the best or limit of utility for a cold flow analytical study in terms of what it can provide to describe optimum combustor geometry.

3.3 General Combustor Configuration

In the previous calculations, the nozzle of the combustor was treated as a contracting conical section. To demonstrate the generality of the program to treat nozzles of arbitrary geometry, a calculation was made for a combustor with L/D ≈ 1.4, oxidizer injector location near the half-radius and a 45° contracting conical nozzle with an extension. The program was

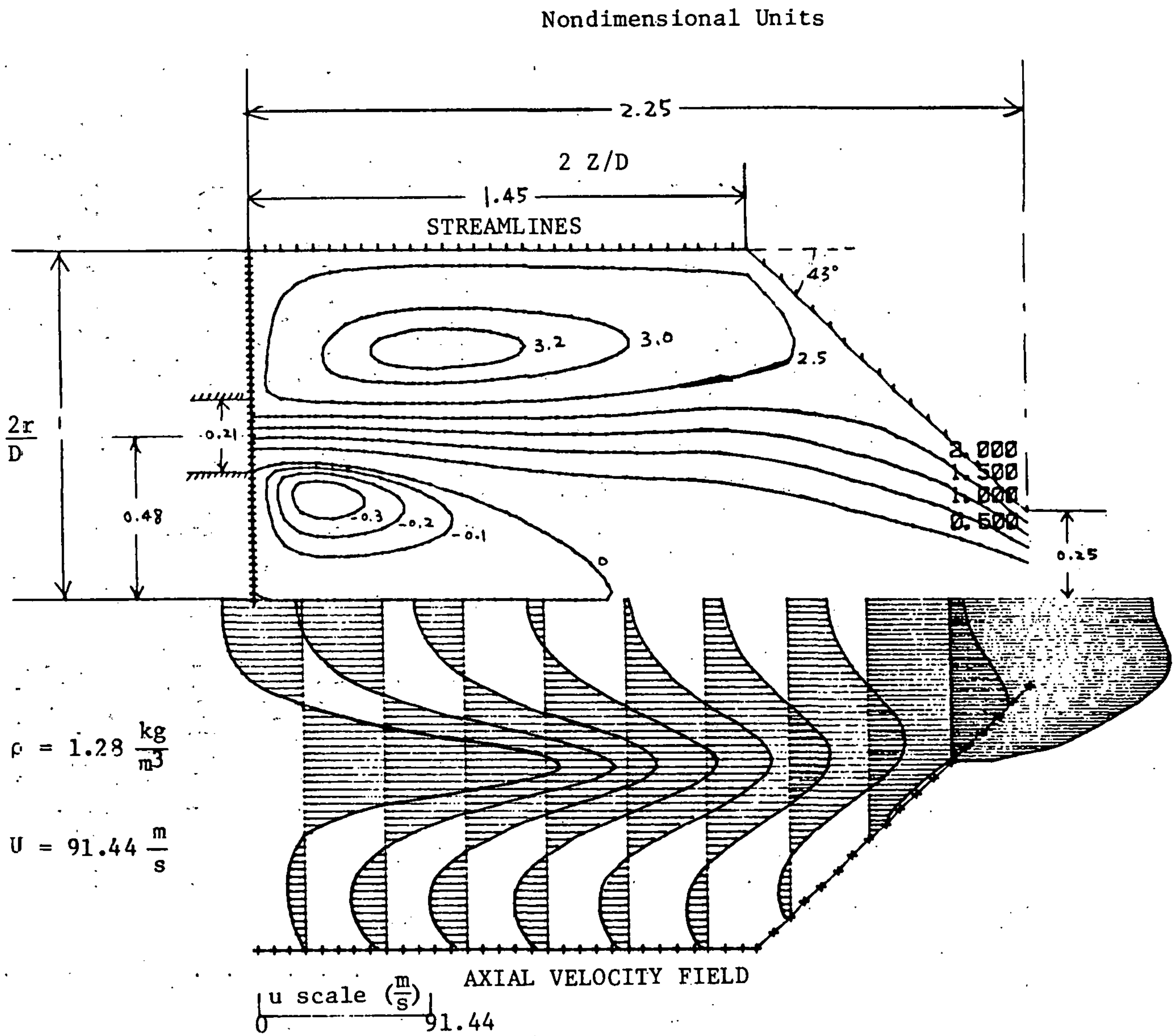


Figure 8. Combustor with Shortened Length Cylindrical Section, L/D = 0.725

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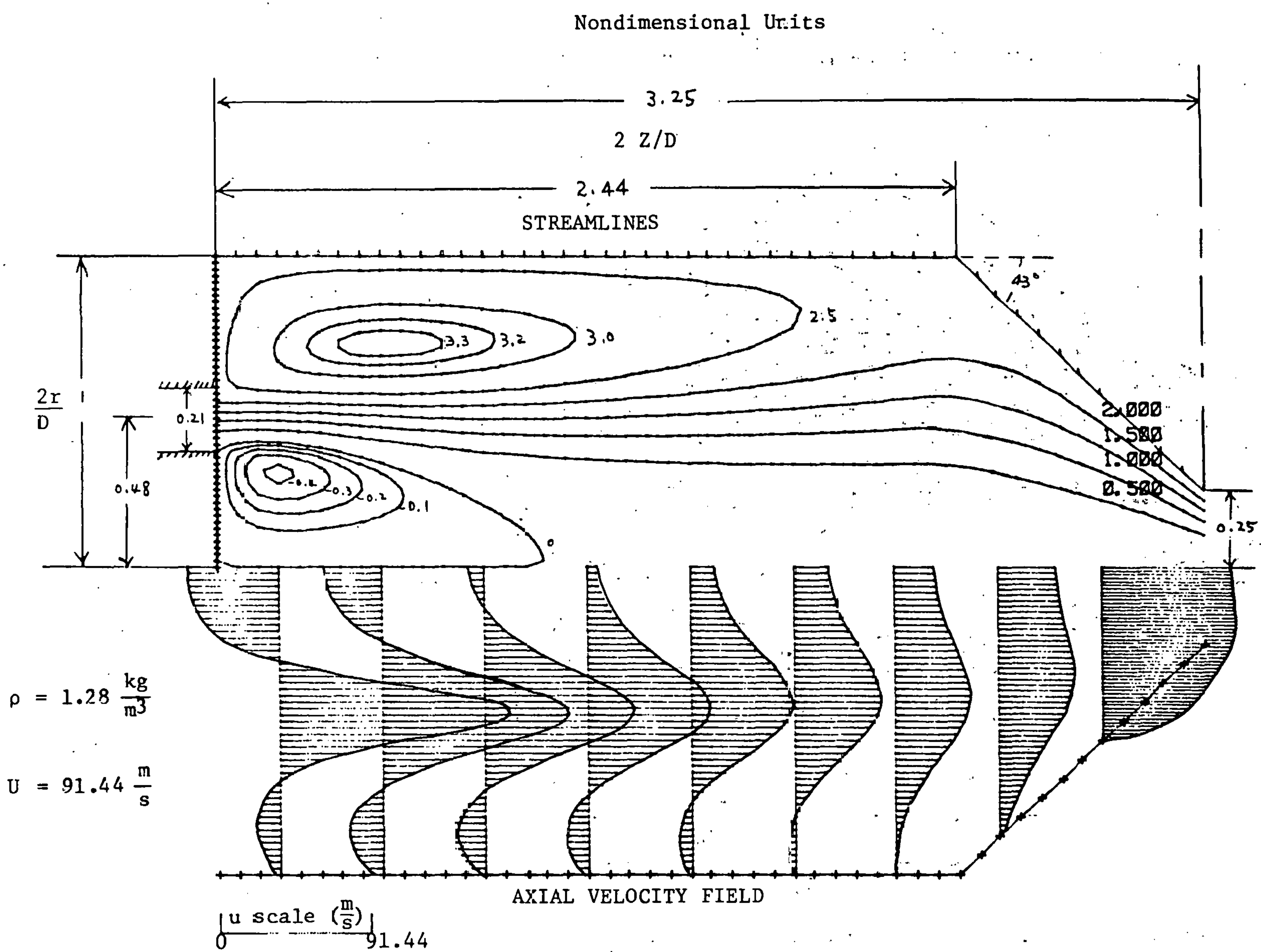

Figure 9. Combustor with Extended Length Cylindrical Section, L/D = 1.22

Nondimensional Units

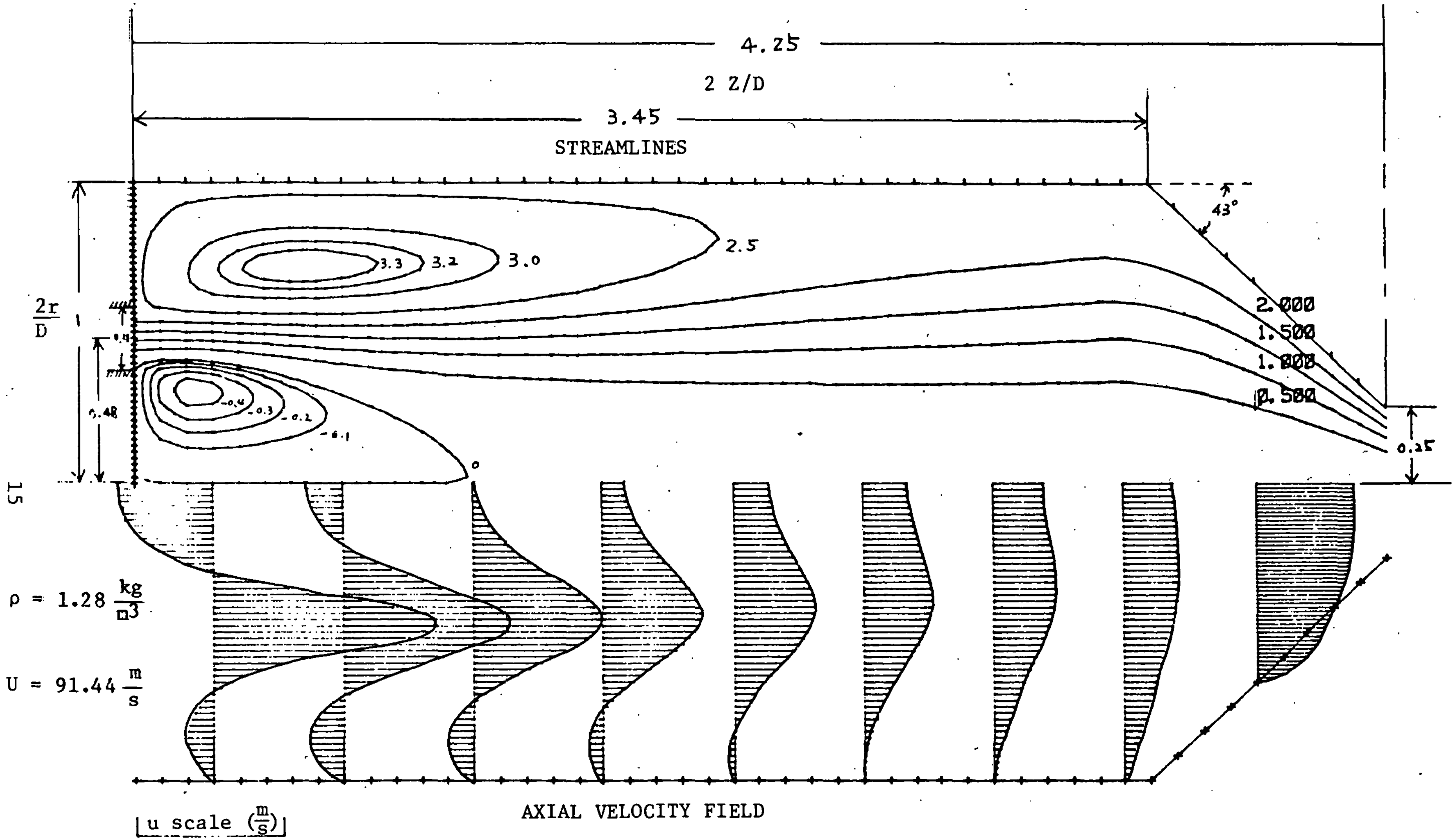

Figure 10. Combustor with Extended Length Cylindrical Section, L/D = 1.725

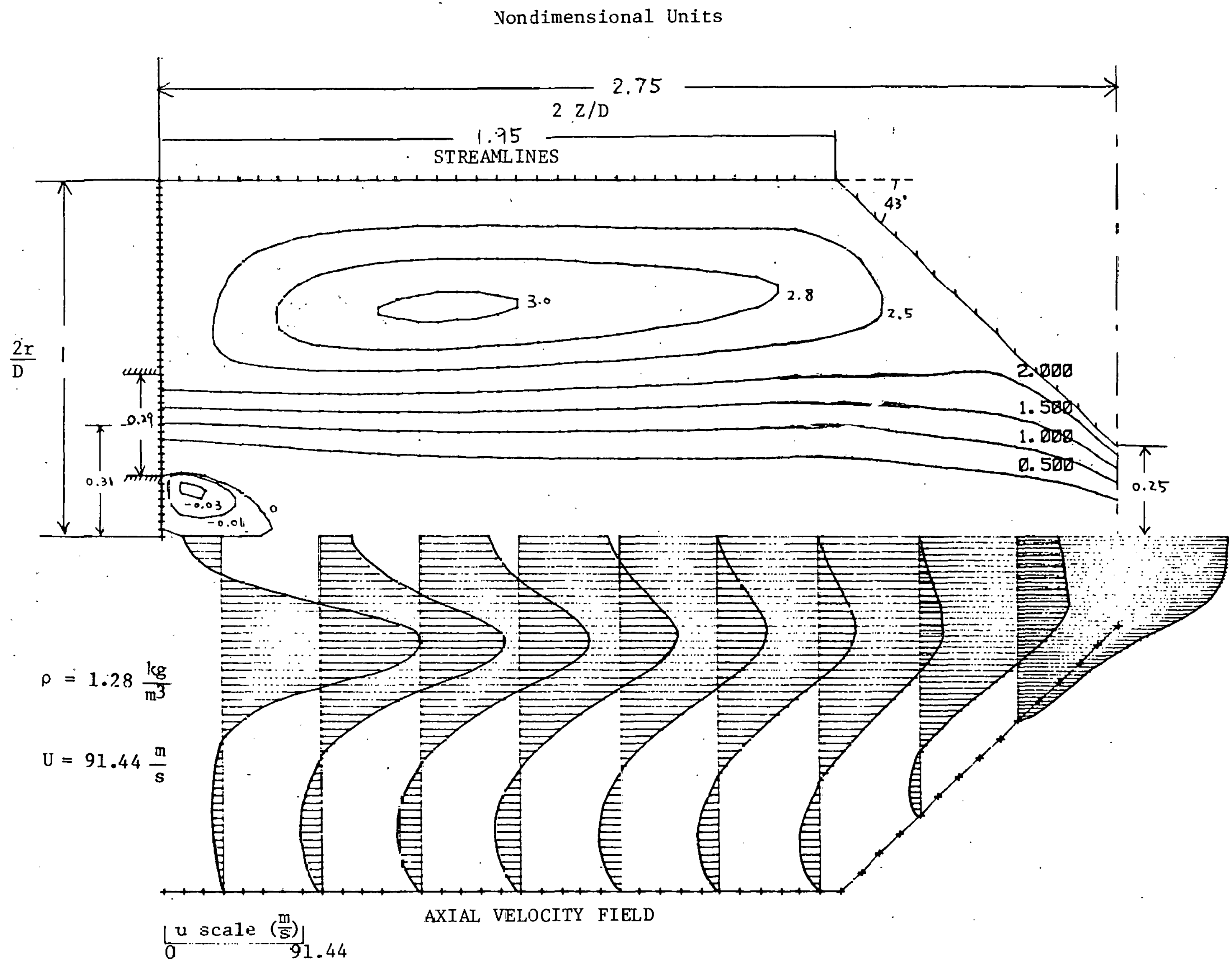

Figure 11. Combustor with Jet Radial Location Moved Toward Centerline, L/D = 0.975

Nondimensional Units

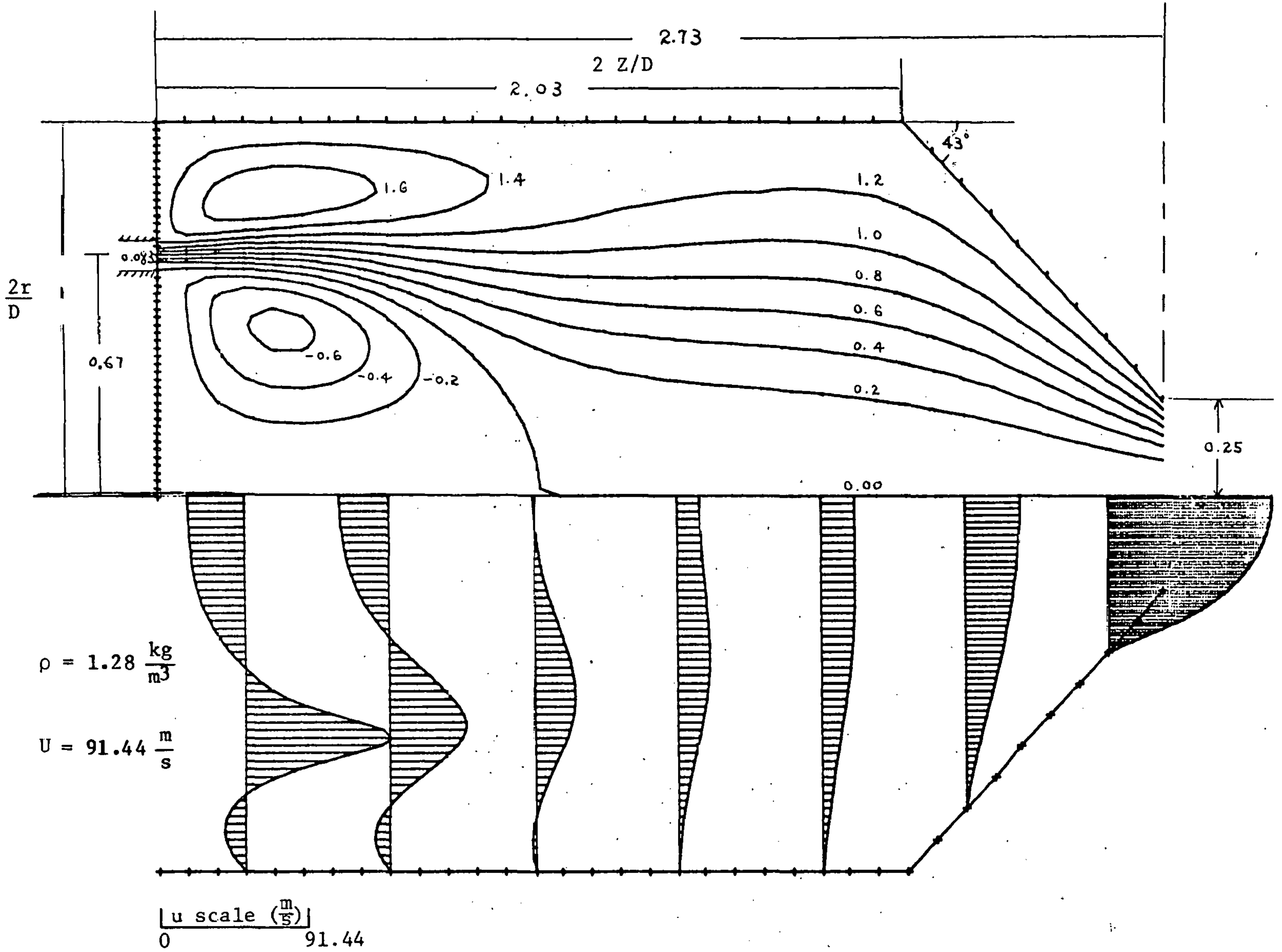

Figure 12. Combustor with Jet Radial Location Moved Farther Out From Centerline, L/D = 0.975

able to easily compute the flow in this configuration and the results are displayed in Fig 13. Thus, in principle, the program can be used to compute the flow from combustor inlet plane to the beginning of a real nozzle throat regime. Fig 13 demonstrates the power of the coordinate sketching technique for handling arbitrary axisymmetric nozzle geometries. It must be pointed out that the effects of compressibility were not incorporated in the present model formulation which was based on constant density.

4.0 SUMMARY

The results of the calculation for an isothermal analytical combustor model showed that the radial location of the oxidant injector had the single greatest effect in establishing the characteristics of the recirculation zones inside the combustor. A radial location, for a single row of holes, about half-way between the axis and the wall may be about the optimum jet position for generating an outer annular and central recirculation zone of about the same size and strength. The effect of the angle of the conical contracting nozzle inlet was secondary on the internal recirculation zone structure, but did affect the degree of homogeneity of the flow leaving the combustor and entering the nozzle. A contraction angle greater than about 45° should be avoided based on the results of the present calculations if the combustor length is short. The effect of varying the combustor L/D ratio on the inner recirculation zone structure was small, but, for the radial location of the injector studied, the effect of L/D on the outer recirculation zones was significant, hence, short (L/D $\leq$ 1.0) combustors should be avoided if possible. There is, of course, a trade-off: shorter lengths for fixed diameters means lower heat loss and hence higher thermal efficiencies for the combustor, while on the other hand, short combustors may not provide for sufficient mixing, hence, lower aerodynamic performance and combustion efficiency. Thus, in the present study, an isothermal flow analysis indicates that a combustor of L/D ~ 1.25, conical nozzle inlet angle of 30°, and a radial location for oxidizer injection at about the half-radius position should lead to good combustor performance from a purely aerodynamic viewpoint. To verify this result, the effects of combustion chemistry on the flame structure in actual combustors must be studied.

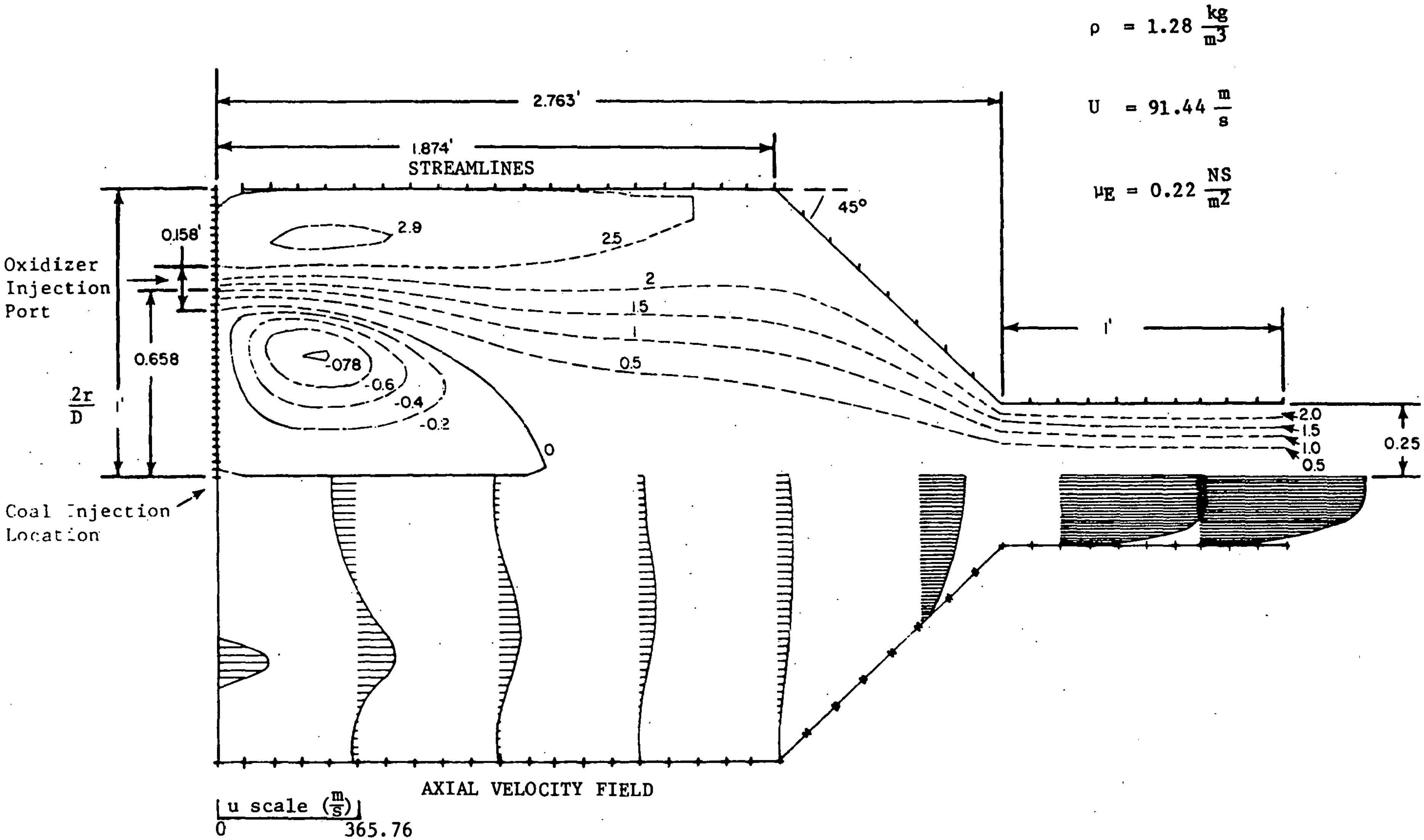

Figure 13. Theoretical Calculation of Flow Field Inside the UTSI Design Pulverized Coal Combustor
Single Oxidizer Injector Ring

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APPENDIX A

Coefficients in the Standard Form* for the Governing Equations for the Transformed Coordinate System

*[Equation 8]

GOVERNING EQUATION FOR VORTICITY (ω)

Vorticity is defined, for two-dimensional, axisymmetric flow as

$$\omega = \frac{\partial v}{\partial z} - \frac{\partial u}{\partial r} \tag{9}$$

and the transport equation for vorticity, derived from the Navier-Stokes equation is

$$\frac{\partial^2\omega}{\partial z^2} + \frac{\partial^2\omega}{\partial r^2} - \left(\frac{\rho u}{\mu} - \frac{2}{\mu}\frac{\partial \mu}{\partial z}\right)\frac{\partial \omega}{\partial z} - \left(\frac{\rho v}{\mu} - \frac{2}{\mu}\frac{\partial \mu}{\partial r}\frac{\omega}{r} - \frac{\partial \omega}{\partial r}\right) +$$

$$\left[\frac{1}{\mu}\frac{\partial^2\mu}{\partial z^2} + \frac{1}{\mu}\frac{\partial^2\mu}{\partial r^2} + \left(\frac{\rho v}{\mu r} + \frac{1}{\mu r}\frac{\partial \mu}{\partial r} - \frac{1}{r^2}\right)\right]\omega +$$

$$\frac{2}{\mu}\left[\frac{\partial^2\mu}{\partial z\partial r}\left(\frac{\partial v}{\partial r} - \frac{\partial u}{\partial z}\right) + \frac{\partial^2\mu}{\partial z^2}\frac{\partial u}{\partial r} - \frac{\partial^2\mu}{\partial r^2}\frac{\partial v}{\partial z} + \frac{\partial \mu}{\partial z}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r}\right) - \right.$$

$$\left.\frac{\partial \mu}{\partial r}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r}\right)\right] + \frac{1}{2\mu}\left[\frac{\partial(u^2 + v^2)}{\partial z}\frac{\partial \rho}{\partial r} - \frac{\partial(u^2 + v^2)}{\partial r}\frac{\partial \rho}{\partial z}\right] = 0 \tag{10}$$

This equation is transformed to stretched coordinates (ξ,η) as

$$z2\frac{\partial^2\omega}{\partial \xi^2} + z3\frac{\partial \omega}{\partial \xi} + 2z1\ z4\frac{\partial^2\omega}{\partial \xi\partial \eta} + z5\frac{\partial^2\omega}{\partial \eta^2} + z6\frac{\partial \omega}{\partial \eta} +$$

$$z8\frac{\partial^2\omega}{\partial \xi^2} + z9\frac{\partial \omega}{\partial \xi} + 2z7\ z10\frac{\partial^2\omega}{\partial \xi\partial \eta} + z11\frac{\partial^2\omega}{\partial \eta^2} + z12\frac{\partial \omega}{\partial \eta} -$$

$$\frac{1}{\mu}\left[\rho u - 2\left(z1\frac{\partial \mu}{\partial \xi} + z4\frac{\partial \mu}{\partial \eta}\right)\right]\left(z1\frac{\partial \omega}{\partial \xi} + z4\frac{\partial \omega}{\partial \eta}\right) -$$

$$\frac{1}{\mu}\left[\rho v - 2\left(z7\frac{\partial \mu}{\partial \xi} + z10\frac{\partial \mu}{\partial \eta}\right) - \frac{\mu}{r}\right]\left(z7\frac{\partial \omega}{\partial \xi} + z10\frac{\partial \omega}{\partial \eta}\right) +$$

$$\frac{1}{\mu}\left\{\left(z2\frac{\partial^2\mu}{\partial \xi^2} + z3\frac{\partial \mu}{\partial \xi} + 2z1\ z4\frac{\partial^2\mu}{\partial \xi\partial \eta} + z5\frac{\partial^2\mu}{\partial \eta^2} + z6\frac{\partial \mu}{\partial \eta}\right) +\right.$$

$$\left(z8\frac{\partial^2\mu}{\partial \xi^2} + z9\frac{\partial \mu}{\partial \xi} + 2z7\ z10\frac{\partial^2\mu}{\partial \xi\partial \eta} + z11\frac{\partial^2\mu}{\partial \eta^2} + z12\frac{\partial \mu}{\partial \eta}\right) +$$

$$\left.\left[\frac{\rho v}{r} + \frac{1}{r}\left(z7\frac{\partial \mu}{\partial \xi} + z10\frac{\partial \mu}{\partial \eta}\right) - \frac{\mu}{r^2}\right]\right\}\omega +$$

$$\frac{2}{\mu}\left\{(z13\,\frac{\partial\mu}{\partial\xi} + z1\ z7\,\frac{\partial^2\mu}{\partial\xi^2} + z4\ z7\,\frac{\partial^2\mu}{\partial\xi\partial\eta} + z14\,\frac{\partial\mu}{\partial\eta} + z4\ z10\,\frac{\partial^2\mu}{\partial\eta^2} + z1\ z10\,\frac{\partial^2\mu}{\partial\xi\partial\eta})\right.$$

$$(z7\,\frac{\partial v}{\partial\xi} + z10\,\frac{\partial v}{\partial\eta} - z1\,\frac{\partial u}{\partial\xi} - z4\,\frac{\partial u}{\partial\eta}) + (z2\,\frac{\partial^2\mu}{\partial\xi^2} + z3\,\frac{\partial\mu}{\partial\xi} + 2z1\ z4\,\frac{\partial^2\mu}{\partial\xi\partial\eta}\ +$$

$$z5\,\frac{\partial^2\mu}{\partial\eta^2} + z6\,\frac{\partial\mu}{\partial\eta})\ (z7\,\frac{\partial u}{\partial\xi} + z10\,\frac{\partial u}{\partial\eta}) - (z8\,\frac{\partial^2\mu}{\partial\xi^2} + z9\,\frac{\partial\mu}{\partial\xi} + 2z7\ z10\,\frac{\partial^2\mu}{\partial\xi\partial\eta}\ +$$

$$z11\,\frac{\partial^2\mu}{\partial\eta^2} + z12\,\frac{\partial\mu}{\partial\eta})(z1\,\frac{\partial v}{\partial\xi} + z4\,\frac{\partial v}{\partial\eta}) + (z1\,\frac{\partial\mu}{\partial\xi} + z4\,\frac{\partial\mu}{\partial\eta})$$

$$[(z13\,\frac{\partial u}{\partial\xi} + z1\ z7\,\frac{\partial^2 u}{\partial\xi^2} + z4\ z7\,\frac{\partial^2 u}{\partial\xi\partial\eta} + z14\,\frac{\partial u}{\partial\eta} + z4\ z10\,\frac{\partial^2 u}{\partial\eta^2} + z1\ z10\,\frac{\partial^2 u}{\partial\xi\partial\eta})\ +$$

$$(z8\,\frac{\partial^2 v}{\partial\xi^2} + z9\,\frac{\partial v}{\partial\xi} + 2z7\ z10\,\frac{\partial^2 v}{\partial\xi\partial\eta} + z11\,\frac{\partial^2 v}{\partial\eta^2} + z12\,\frac{\partial v}{\partial\eta}) - \frac{v}{r^2} + \frac{1}{r}\,(z7\,\frac{\partial v}{\partial\xi} + z10\,\frac{\partial v}{\partial\eta})]\ -$$

$$(z7\,\frac{\partial\mu}{\partial\xi} + z10\,\frac{\partial\mu}{\partial\eta})[(z2\,\frac{\partial^2 u}{\partial\xi^2} + z3\,\frac{\partial u}{\partial\xi} + 2z1\ z4\,\frac{\partial^2 u}{\partial\xi\partial\eta} + z5\,\frac{\partial^2 u}{\partial\eta^2} + z6\,\frac{\partial u}{\partial\eta})\ +$$

$$(z13\,\frac{\partial v}{\partial\xi} + z1\ z7\,\frac{\partial^2 v}{\partial\xi^2} + z4\ z7\,\frac{\partial^2 v}{\partial\xi\partial\eta} + z14\,\frac{\partial v}{\partial\eta} + z4\ z10\,\frac{\partial^2 v}{\partial\eta^2} + z1\ z10\,\frac{\partial^2 v}{\partial\xi\partial\eta})\ +$$

$$\left.\frac{1}{r}\,(z1\,\frac{\partial v}{\partial\xi} + z4\,\frac{\partial v}{\partial\eta})]\right\}\ +$$

$$\frac{1}{\mu}\left\{[u(z1\,\frac{\partial u}{\partial\xi} + z4\,\frac{\partial u}{\partial\eta}) + v(z1\,\frac{\partial v}{\partial\xi} + z4\,\frac{\partial v}{\partial\eta})](z7\,\frac{\partial\rho}{\partial\xi} + z10\,\frac{\partial\rho}{\partial\eta})\ -\right.$$

$$\left.[u(z7\,\frac{\partial u}{\partial\xi} + z10\,\frac{\partial u}{\partial\eta}) + v(z7\,\frac{\partial v}{\partial\xi} + z10\,\frac{\partial v}{\partial\eta})]\ (z1\,\frac{\partial\rho}{\partial\xi} + z4\,\frac{\partial\rho}{\partial\eta})\right\} = 0. \qquad (11)$$

or

$$(z2 + z8)\,\frac{\partial^2\omega}{\partial\xi^2} + (z5 + z11)\,\frac{\partial^2\omega}{\partial\eta^2} - \frac{1}{\mu}\left\{z1[\rho u - 2(z1\,\frac{\partial\mu}{\partial\xi} + z4\,\frac{\partial\mu}{\partial\eta})]\ +\right.$$

$$\left.z7[\rho v - 2(z7\,\frac{\partial\mu}{\partial\xi} + z10\,\frac{\partial\mu}{\partial\eta}) - \frac{\mu}{r}] - \mu(z3 + z9)\right\}\,\frac{\partial\omega}{\partial\xi} - \frac{1}{\mu}\left\{z4[\rho u - 2(z1\,\frac{\partial\mu}{\partial\xi} + z4\,\frac{\partial\mu}{\partial\eta})]\ +\right.$$

$$\left.z10[\rho v - 2\,(z7\,\frac{\partial\mu}{\partial\xi} + z10\,\frac{\partial\mu}{\partial\eta}) - \frac{\mu}{r}] - \mu(z6 + z12)\right\}\,\frac{\partial\omega}{\partial\eta}\ +\ d\omega. \qquad (12)$$

where

$$d\omega = \Bigg\{\Bigg\{ 2(z1\ z4 + z7\ z10)\frac{\partial^2\omega}{\partial\xi\partial\eta} + \frac{\omega}{\mu}\Big[(z2 + z8)\frac{\partial^2\mu}{\partial\xi^2} + (z5 + z11)\frac{\partial^2\mu}{\partial\eta^2} + (z3 + z9)\frac{\partial\mu}{\partial\xi} +$$

$$\frac{\partial\mu}{\partial\xi} + (z6 + z12)\frac{\partial\mu}{\partial\eta} + 2(z1\ z4 + z7\ z10)\frac{\partial^2\mu}{\partial\xi\partial\eta} + \frac{1}{r}\left(\rho v + z7\frac{\partial\mu}{\partial\xi} + z10\frac{\partial\mu}{\partial\eta} - \frac{\mu}{r}\right)\Big] +$$

$$\frac{2}{\mu}\Big\{\left(z13\frac{\partial\mu}{\partial\xi} + z1\ z7\frac{\partial^2\mu}{\partial\xi^2} + z4\ z7\frac{\partial^2\mu}{\partial\xi\partial\eta} + z14\frac{\partial\mu}{\partial\eta} + z4\ z10\frac{\partial^2\mu}{\partial\eta^2} + z1\ z10\frac{\partial^2\mu}{\partial\xi\partial\eta}\right)$$

$$\left(z7\frac{\partial v}{\partial\xi} + z10\frac{\partial v}{\partial\eta} - z1\frac{\partial u}{\partial\xi} - z4\frac{\partial u}{\partial\eta}\right) + \left(z2\frac{\partial^2\mu}{\partial\xi^2} + z3\frac{\partial\mu}{\partial\xi} + 2z1\ z4\frac{\partial^2\mu}{\partial\xi\partial\eta} +\right.$$

$$\left. z5\frac{\partial^2\mu}{\partial\eta^2} + z6\frac{\partial\mu}{\partial\eta}\right)\left(z7\frac{\partial u}{\partial\xi} + z10\frac{\partial u}{\partial\eta}\right) - \left(z8\frac{\partial^2\mu}{\partial\xi^2} + z9\frac{\partial\mu}{\partial\xi} + 2z7\ z10\frac{\partial^2\mu}{\partial\xi\partial\eta} +\right.$$

$$\left. z11\frac{\partial^2\mu}{\partial\eta^2} + z12\frac{\partial\mu}{\partial\eta}\right)\left(z1\frac{\partial v}{\partial\xi} + z4\frac{\partial v}{\partial\eta}\right) + \left(z1\frac{\partial\mu}{\partial\xi} + z4\frac{\partial\mu}{\partial\eta}\right)$$

$$\Big[\left(z13\frac{\partial u}{\partial\xi} + z1\ z7\frac{\partial^2 u}{\partial\xi^2} + z4\ z7\frac{\partial^2 u}{\partial\xi\partial\eta} + z14\frac{\partial u}{\partial\eta} + z4\ z10\frac{\partial^2 u}{\partial\eta^2} + z1\ z10\frac{\partial^2 u}{\partial\xi\partial\eta}\right) +$$

$$\left(z8\frac{\partial^2 v}{\partial\xi^2} + z9\frac{\partial v}{\partial\xi} + 2z7\ z10\frac{\partial^2 v}{\partial\xi\partial\eta} + z11\frac{\partial^2 v}{\partial\eta^2} + z12\frac{\partial v}{\partial\eta}\right) + \frac{1}{r}\left(z7\frac{\partial v}{\partial\xi} + z10\frac{\partial v}{\partial\eta} - \frac{v}{r}\right)\Big] -$$

$$\left(z7\frac{\partial\mu}{\partial\xi} + z10\frac{\partial\mu}{\partial\eta}\right)\Big[\left(z2\frac{\partial^2 u}{\partial\xi^2} + z3\frac{\partial u}{\partial\xi} + 2z1\ z4\frac{\partial^2 u}{\partial\xi\partial\eta} + z5\frac{\partial^2 u}{\partial\eta^2} + z6\frac{\partial u}{\partial\eta}\right) +$$

$$\left(z13\frac{\partial v}{\partial\xi} + z1\ z7\frac{\partial^2 v}{\partial\xi^2} + z4\ z7\frac{\partial^2 v}{\partial\xi\partial\eta} + z14\frac{\partial v}{\partial\eta} + z4\ z10\frac{\partial^2 v}{\partial\eta^2} + z1\ z10\frac{\partial^2 v}{\partial\xi\partial\eta}\right) +$$

$$\frac{1}{r}\left(z1\frac{\partial v}{\partial\xi} + z4\frac{\partial v}{\partial\eta}\right)\Big]\Big\} +$$

$$\frac{1}{\mu}\Big\{\Big[u\left(z1\frac{\partial u}{\partial\xi} + z4\frac{\partial u}{\partial\eta}\right) + v\left(z1\frac{\partial v}{\partial\xi} + z4\frac{\partial v}{\partial\eta}\right)\Big]\left(z7\frac{\partial\rho}{\partial\xi} + z10\frac{\partial\rho}{\partial\eta}\right) -$$

$$\Big[u\left(z7\frac{\partial u}{\partial\xi} + z10\frac{\partial u}{\partial\eta}\right) + v\left(z7\frac{\partial v}{\partial\xi} + z10\frac{\partial v}{\partial\eta}\right)\Big]\left(z1\frac{\partial\rho}{\partial\xi} + z4\frac{\partial\rho}{\partial\eta}\right)\Big\}\Bigg\}$$

The standard coefficients are defined by

$$a_{1\omega} = z2 + z8$$

$$a_{2\omega} = z5 + z11$$

$$b_{1\omega} = \frac{1}{\mu}\left\{z1[\rho u - 2(z1\frac{\partial\mu}{\partial\xi} + z4\frac{\partial\mu}{\partial\eta})] + \right.$$

$$\left. z7[\rho v - 2(z7\frac{\partial\mu}{\partial\xi} + z10\frac{\partial\mu}{\partial\eta})]\right\} - (z3 + z9 + \frac{z7}{r})$$

$$b_{2\omega} = \frac{1}{\mu}\left\{z4[\rho u - 2(z1\frac{\partial\mu}{\partial\xi} + z4\frac{\partial\mu}{\partial\eta})] + \right.$$

$$\left. z10[\rho v - 2(z7\frac{\partial\mu}{\partial\xi} + z10\frac{\partial\mu}{\partial\eta})]\right\} - (z6 + z12 + \frac{z10}{r})$$

$d\omega = \left\{\left\{ \quad \right\}\right\}$ from page A-4.

Governing equation for stream function (ψ)

For two dimensional flows, in axisymmetic coordinates, the steam function is defined as

$$\frac{\partial \psi}{\partial r} = \rho u r; \quad \frac{\partial \psi}{\partial z} = - \rho v r \tag{13}$$

or

$$\psi = \int (\rho u r dr - \rho v r dz) \tag{14}$$

By substituting derivatives of ψ for u, v in the definition of vorticity, one gets

$$\omega = \frac{\partial v}{\partial z} - \frac{\partial u}{\partial r} = - \frac{\partial}{\partial z} \left(\frac{1}{\rho r} \frac{\partial \psi}{\partial z}\right) - \frac{\partial}{\partial r} \left(\frac{1}{\rho r} \frac{\partial \psi}{\partial r}\right) \tag{15}$$

or

$$\frac{\partial}{\partial z} \left(\frac{1}{\rho r} \frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial r} \left(\frac{1}{\rho r} \frac{\partial \psi}{\partial r}\right) + \omega = 0 \tag{16}$$

or

$$\frac{1}{\rho r} \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{\rho^2 r} \frac{\partial \rho}{\partial z} \frac{\partial \psi}{\partial z} + \frac{1}{\rho r} \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{\rho^2 r} \frac{\partial \rho}{\partial r} \frac{\partial \psi}{\partial r} - \frac{1}{\rho r^2} \frac{\partial \psi}{\partial r} + \omega = 0 \tag{17}$$

or, finally, an equation relating ψ and ω as

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{\rho} \frac{\partial \rho}{\partial z} \frac{\partial \psi}{\partial z} - \frac{1}{\rho} \frac{\partial \rho}{\partial r} \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{\partial \psi}{\partial r} = - \rho r \omega \tag{18}$$

This equation is transformed to ξ, η coordinates as

$$z2\ \frac{\partial^2\psi}{\partial\xi^2} + z3\ \frac{\partial\psi}{\partial\xi} + 2z1\ z4\ \frac{\partial^2\psi}{\partial\xi\partial\eta} + z5\ \frac{\partial^2\psi}{\partial\eta^2} + z6\ \frac{\partial\psi}{\partial\eta} + z8\ \frac{\partial^2\psi}{\partial\xi^2} + z9\ \frac{\partial\psi}{\partial\xi} +$$

$$2z7\ z10\ \frac{\partial^2\psi}{\partial\xi\partial\eta} + z11\ \frac{\partial^2\psi}{\partial\eta^2} + z12\ \frac{\partial\psi}{\partial\eta} - \frac{1}{\rho}\left(z1\frac{\partial\rho}{\partial\xi} + z4\ \frac{\partial\rho}{\partial\eta}\right)\left(z1\frac{\partial\psi}{\partial\xi} + z4\ \frac{\partial\psi}{\partial\eta}\right) -$$

$$\frac{1}{\rho}\left(z7\ \frac{\partial\rho}{\partial\xi} + z10\ \frac{\partial\rho}{\partial\eta}\right)\left(z7\ \frac{\partial\psi}{\partial\xi} + z10\ \frac{\partial\psi}{\partial\eta}\right) - \frac{1}{r}\left(z7\ \frac{\partial\psi}{\partial\xi} + z10\ \frac{\partial\psi}{\partial\eta}\right) + \rho r\omega = 0.$$

$$(z7 + z8)\frac{\partial^2\psi}{\partial\xi^2} + (z5 + z11)\ \frac{\partial^2\psi}{\partial\eta^2} - \Big[- z3 - z9 + \frac{z1}{\rho}\left(z1\frac{\partial\rho}{\partial\xi} + z4\frac{\partial\rho}{\partial\eta}\right) +$$

$$\frac{z7}{\rho}\left(z7\ \frac{\partial\rho}{\partial\xi} + z10\ \frac{\partial\rho}{\partial\eta}\right) + \frac{z7}{r}\Big]\ \frac{\partial\psi}{\partial\xi} - \Big[- z6 - z12 + \frac{z4}{\rho}\left(z1\ \frac{\partial\rho}{\partial\xi} + z4\ \frac{\partial\rho}{\partial\eta}\right) +$$

$$\frac{z10}{\rho}\left(z7\ \frac{\partial\rho}{\partial\xi} + z10\ \frac{\partial\rho}{\partial\eta}\right) + \frac{z10}{r}\Big]\ \frac{\partial\psi}{\partial\eta} + \Big[2(z1\ z4 + z10)\ \frac{\partial^2\psi}{\partial\xi\partial\eta} + \rho r\omega\Big] = 0. \quad (19)$$

The standard coefficients are defined by

$$a_{1s} = z2 + z8$$

$$a_{2s} = z5 + z11$$

$$b_{1s} = \frac{z1}{\rho}\left(z1\ \frac{\partial\rho}{\partial\xi} + z4\ \frac{\partial\rho}{\partial\eta}\right) + \frac{z7}{\rho}\left(z7\ \frac{\partial\rho}{\partial\xi} + z10\ \frac{\partial\rho}{\partial\eta}\right) - \left(z3 + z9 - \frac{z7}{r}\right)$$

$$b_{2s} = \frac{z4}{\rho}\left(z1\ \frac{\partial\rho}{\partial\xi} + z4\ \frac{\partial\rho}{\partial\eta}\right) + \frac{z10}{\rho}\left(z7\ \frac{\partial\rho}{\partial\xi} + z10\ \frac{\partial\rho}{\partial\eta}\right) - \left(z6 + z12 - \frac{z10}{r}\right)$$

$$d_s = 2(z1z4 + z7z10)\ \frac{\partial^2\psi}{\partial\xi\partial\eta} + \rho r\omega$$

GOVERNING EQUATION FOR TURBULENT KINETIC ENERGY (k)

The first equation in the so-called "k-ε1" model for turbulence transport is

$$\frac{\partial}{\partial z}(\rho urk) + \frac{\partial}{\partial r}(\rho vrk) - \frac{\partial}{\partial z}\left(\frac{\mu}{\sigma_k} r \frac{\partial k}{\partial z}\right) - \frac{\partial}{\partial r}\left(\frac{\mu}{\sigma_k} r \frac{\partial k}{\partial r}\right) + rS_k = 0 \qquad (20)$$

The standard coefficients are defined by

$$a_{1k} = z2 + z8$$

$$a_{2k} = z5 + z11$$

$$b_{1k} = \frac{1}{\mu}\ z1(\rho u\sigma_k - z1 \frac{\partial\mu}{\partial\xi} - z4 \frac{\partial\mu}{\partial\eta}) +$$

$$z7(\rho v\sigma_k - z7 \frac{\partial\mu}{\partial\xi} - z10 \frac{\partial\mu}{\partial\eta}) - \mu(z3 + z9 + \frac{z7}{r})]$$

$$b_{2k} = \frac{1}{\mu}\left\{ z4(\rho u\sigma_k - z1 \frac{\partial\mu}{\partial\xi} - z4 \frac{\partial\mu}{\partial\eta}) + \right.$$

$$\left. z10(\rho v\sigma_k - z7 \frac{\partial\mu}{\partial\xi} - z10 \frac{\partial\mu}{\partial\eta}) - \mu(z6 + z12 + \frac{z10}{r})\right\}$$

$$d_k = 2(z1z4 + z7z10)\frac{\partial^2 k}{\partial\xi\partial\eta} - \frac{\sigma_k}{\mu} S_k$$

S_k is the physical net source term for k, namely,

$$S_k = \rho\varepsilon - \mu\left\{2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{v}{r}\right)^2\right] + \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right)^2\right\}$$

It transforms to

$$S_k = \rho\varepsilon - \mu\left\{2\left[\left(z1 \frac{\partial u}{\partial\xi} + z4 \frac{\partial u}{\partial\eta}\right)^2 + \left(z7 \frac{\partial v}{\partial\xi} + z10 \frac{\partial v}{\partial\eta}\right)^2 + \left(\frac{v}{r}\right)^2\right] + \right.$$

$$\left.\left(z7 \frac{\partial u}{\partial\xi} + z10 \frac{\partial u}{\partial\eta} + z1 \frac{\partial v}{\partial\xi} + z4 \frac{\partial v}{\partial\eta}\right)^2\right\}$$

GOVERNING EQUATION FOR TURBULENT KINETIC ENERGY DISSIPATION (ε)

The second equation in the so-called "k-ε1" turbulence transport model is

$$\frac{\partial}{\partial z}(\rho u r \varepsilon) + \frac{\partial}{\partial r}(\rho v r \varepsilon) - \frac{\partial}{\partial z}\left(\frac{\mu}{\sigma_\varepsilon} r \frac{\partial \varepsilon}{\partial z}\right) - \frac{\partial}{\partial r}\left(\frac{\mu}{\sigma_\varepsilon} r \frac{\partial \varepsilon}{\partial r}\right) + r S_\varepsilon = 0 \qquad (21)$$

The standard coefficients are defined by

$$a_{1\varepsilon} = z2 + z8$$

$$a_{2\varepsilon} = z5 + z11$$

$$b_{1\varepsilon} = \frac{1}{\mu}\Big\{z1(\rho u \sigma_\varepsilon - z1 \frac{\partial \mu}{\partial \xi} - z4 \frac{\partial \mu}{\partial \eta}) +$$

$$z7(\rho v \sigma_\varepsilon - z7 \frac{\partial \mu}{\partial \xi} - z10 \frac{\partial \mu}{\partial \eta}) - \mu\left(z3 + z9 + \frac{z7}{r}\right)\Big\}$$

$$b_{2\varepsilon} = \frac{1}{\mu}\Big\{z4(\rho u \sigma_\varepsilon - z1 \frac{\partial \mu}{\partial \xi} - z4 \frac{\partial \mu}{\partial \eta}) +$$

$$z10(\rho v \sigma_\varepsilon - z7 \frac{\partial \mu}{\partial \xi} - z10 \frac{\partial \mu}{\partial \eta}) - \mu\left(z6 + z12 + \frac{z10}{r}\right)\Big\}$$

$$d_\varepsilon = 2(z1z4 + z7z10) \frac{\partial^2 \varepsilon}{\partial \xi \partial \eta} - \frac{\sigma_\varepsilon}{\mu} S_\varepsilon$$

S_ε is the physical source term for ε, namely,

$$S_\varepsilon = C_2 \frac{\rho \varepsilon^2}{\kappa} - C_1 \frac{\varepsilon}{\kappa} \mu \left\{2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{v}{r}\right)^2\right] + \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right)^2\right\}$$

It transforms to

$$S_\varepsilon = C_2 \frac{\rho \varepsilon^2}{\kappa} - C_1 \frac{\varepsilon}{\kappa} \mu \Big\{2\Big[\left(z1 \frac{\partial u}{\partial \xi} + z4 \frac{\partial u}{\partial \eta}\right)^2 + \left(z7 \frac{\partial v}{\partial \xi} + z10 \frac{\partial v}{\partial \eta}\right)^2 + \left(\frac{v}{r}\right)^2\Big] +$$

$$\left(z7 \frac{\partial u}{\partial \xi} + z10 \frac{\partial u}{\partial \eta} + z1 \frac{\partial v}{\partial \xi} + z4 \frac{\partial v}{\partial \eta}\right)^2\Big\}$$

GOVERNING EQUATION FOR PRESSURE IN AN INCOMPRESSIBLE FLOW (P)

$$\left(\frac{\partial^2 P}{\partial z^2} + \frac{\partial^2 P}{\partial r^2}\right) + \frac{1}{r}\left(\frac{\partial P}{\partial r}\right) = S_P \tag{22}$$

where

$$S_P = \omega^2 - u\left(\frac{\partial \omega}{\partial r} + \frac{\omega}{r}\right) + v\left(\frac{\partial \omega}{\partial z}\right) - \left[\frac{\partial^2}{\partial z^2}\left(\frac{u2+v2}{2}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{u2+v2}{2}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{u2+v2}{2}\right)\right]$$

The equation taken from Ref. 3 is transformed to stretched coordinates (ξ, η) as follows.

$$\begin{aligned}
&\left(z2\frac{\partial^2 P}{\partial \xi^2} + z5\frac{\partial^2 P}{\partial \eta^2} + z3\frac{\partial P}{\partial \xi} + z6\frac{\partial P}{\partial \eta} + 2z1\ z4\frac{\partial^2 P}{\partial \xi \partial \eta} + z8\frac{\partial^2 P}{\partial \xi^2} + z11\frac{\partial^2 P}{\partial \eta^2} + z9\frac{\partial P}{\partial \xi} + \right.\\
&\left. z12\frac{\partial P}{\partial \eta} + 2z7\ z10\frac{\partial^2 P}{\partial \xi \partial \eta}\right) + \frac{1}{r}\left(z7\frac{\partial P}{\partial \xi} + z10\frac{\partial P}{\partial \eta}\right) - \omega^2 + u\left(z7\frac{\partial \omega}{\partial \xi} + z10\frac{\partial \omega}{\partial \eta} + \frac{\omega}{r}\right) -\\
&v\left(z1\frac{\partial \omega}{\partial \xi} + z4\frac{\partial \omega}{\partial \eta}\right) + \Big\{u\left(z2\frac{\partial^2 u}{\partial \xi^2} + z5\frac{\partial^2 u}{\partial \eta^2} + z3\frac{\partial u}{\partial \xi} + z6\frac{\partial u}{\partial \eta} + 2z1\ z4\frac{\partial^2 u}{\partial \xi \partial \eta}\right) +\\
&\left(z1\frac{\partial u}{\partial \xi} + z4\frac{\partial u}{\partial \eta}\right)^2 + v\left(z2\frac{\partial^2 v}{\partial \xi^2} + z5\frac{\partial^2 v}{\partial \eta^2} + z3\frac{\partial v}{\partial \xi} + z6\frac{\partial v}{\partial \eta} + 2z1\ z4\frac{\partial^2 v}{\partial \xi \partial \eta}\right) +\\
&\left(z1\frac{\partial v}{\partial \xi} + z4\frac{\partial v}{\partial \eta}\right)^2 + u\left(z8\frac{\partial^2 u}{\partial \xi^2} + z11\frac{\partial^2 u}{\partial \eta^2} + z9\frac{\partial u}{\partial \xi} + z12\frac{\partial u}{\partial \eta} + 2z7\ z10\frac{\partial^2 u}{\partial \xi \partial \eta}\right) +\\
&\left(z7\frac{\partial u}{\partial \xi} + z10\frac{\partial u}{\partial \eta}\right)^2 + v\left(z8\frac{\partial^2 v}{\partial \xi^2} + z11\frac{\partial^2 v}{\partial \eta^2} + z9\frac{\partial v}{\partial \xi} + z12\frac{\partial v}{\partial \eta} + 2z7\ z10\frac{\partial^2 v}{\partial \xi \partial \eta}\right) +\\
&\left(z7\frac{\partial v}{\partial \xi} + z10\frac{\partial v}{\partial \eta}\right)^2 + \frac{1}{r}\left[u\left(z7\frac{\partial u}{\partial \xi} + z10\frac{\partial u}{\partial \eta}\right) + v\left(z7\frac{\partial v}{\partial \xi} + z10\frac{\partial v}{\partial \eta}\right)\right]\Big\} = 0
\end{aligned} \tag{23}$$

or

$$(z2 + z8)\frac{\partial^2 P}{\partial \xi^2} + (z5 + z11)\frac{\partial^2 P}{\partial \eta^2} - \left(-z3 - z9 - \frac{z7}{r}\right)\frac{\partial P}{\partial \xi} - \left(-z6 - z12 - \frac{z10}{r}\right)\frac{\partial P}{\partial \eta} + dp \tag{24}$$

where

$$dp = \Big\{\Big\{\ 2(z1z4 + z7z10)\frac{\partial^2 P}{\partial \xi \partial \eta} + u[(z2 + z8)\frac{\partial^2 u}{\partial \xi^2} + (z5 + z11)\frac{\partial^2 u}{\partial \eta^2} + (z3 + z9)\frac{\partial u}{\partial \xi} +$$

$$(z6 + z12) \frac{\partial u}{\partial \eta} + 2(z1z4 + z7z10)\frac{\partial^2 u}{\partial \xi \partial \eta}] + v [(z2 + z8) \frac{\partial^2 v}{\partial \xi^2} + (z5 + z11) \frac{\partial^2 u}{\partial \eta^2} +$$

$$(z3 + z9) \frac{\partial v}{\partial \xi} + (z6 + z12) \frac{\partial v}{\partial \eta} + 2(z1z4 + z7z10) \frac{\partial^2 v}{\partial \xi \partial \eta}] + (z1 \frac{\partial u}{\partial \xi} + z4 \frac{\partial u}{\partial \eta})^2 +$$

$$z7 \frac{\partial u}{\partial \xi} + z10 \frac{\partial u}{\partial \eta})^2 + (z1 \frac{\partial v}{\partial \xi} + z4 \frac{\partial v}{\partial \eta})^2 + (z7 \frac{\partial v}{\partial \xi} + z10 \frac{\partial v}{\partial \eta})^2 + \frac{1}{r} [u(z7 \frac{\partial u}{\partial \xi} + z10 \frac{\partial u}{\partial \eta}) +$$

$$v (z7 \frac{\partial v}{\partial \xi} + z10 \frac{\partial v}{\partial \eta})] - \omega_2 + u(z7 \frac{\partial \omega}{\partial \xi} + z10 \frac{\partial \omega}{\partial \eta} + \frac{\omega}{r})\}\}$$

The standard coefficients are defined by

$a_{1P} = z2 + z8$

$a_{2P} = z5 + z11$

$$b_{1P} = -(z3 + z9 + \frac{z7}{r})$$

$$b_{2P} = -(z6 + z12 + \frac{z10}{r})$$

$dP = \{\{ \quad \}\}$ from pages A-10 and A-11.

THE VELOCITY RECOVERY EQUATIONS

(i) for u

$$\frac{\partial \psi}{\partial r} = \rho u r \tag{25}$$

or

$$u = \frac{1}{\rho r} \frac{\partial \psi}{\partial r} \tag{26}$$

Therefore, in transformed coordinates

$$u = \frac{1}{\rho r} \left(z7 \frac{\partial \psi}{\partial \xi} + z10 \frac{\partial \psi}{\partial \eta} \right) \tag{27}$$

(ii) for v

$$\frac{\partial \psi}{\partial z} = - \rho v r \tag{28}$$

or

$$v = - \frac{1}{\rho r} \frac{\partial \psi}{\partial z} \tag{29}$$

Therefore, in transformed coordinates,

$$v = - \frac{1}{\rho r} \left(z1 \frac{\partial \psi}{\partial \xi} + z4 \frac{\partial \psi}{\partial \eta} \right) \tag{30}$$

APPENDIX B

Coordinate Transformation Technique

Numerical Solution Procedure

The numerical solution procedure is written in terms of stretched coordinates. The coordinate stretching required in the present study was provided by the transformation functions of the form

$$\xi = g(z,r)$$

$$\eta = f(z,r)$$

For these stretching functions, it turns out that by the chain-rule of differentiation,

1 $$\frac{\partial}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}$$

2 $$\frac{\partial^2}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}\right)$$

$$= \frac{\partial^2 \xi}{\partial z^2}\frac{\partial}{\partial \xi} + \frac{\partial \xi}{\partial z}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \xi}\right) + \frac{\partial^2 \eta}{\partial z^2}\frac{\partial}{\partial \eta} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \eta}\right)$$

$$= \frac{\partial^2 \xi}{\partial z^2}\frac{\partial}{\partial \xi} + \frac{\partial \xi}{\partial z}\frac{\partial \xi}{\partial z}\frac{\partial^2}{\partial \xi^2} + \frac{\partial \xi}{\partial z}\frac{\partial \eta}{\partial z}\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2 \eta}{\partial z^2}\frac{\partial}{\partial \eta}$$

$$+ \frac{\partial \eta}{\partial z}\frac{\partial \xi}{\partial z}\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial \eta}{\partial z}\frac{\partial \eta}{\partial z}\frac{\partial^2}{\partial \eta^2}$$

$$= \left(\frac{\partial \xi}{\partial z}\right)^2\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2 \xi}{\partial z^2}\frac{\partial}{\partial \xi} + 2\frac{\partial \eta}{\partial z}\frac{\partial \xi}{\partial z}\frac{\partial^2}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial z}\right)^2\frac{\partial^2}{\partial \eta^2}$$

$$+ \frac{\partial^2 \eta}{\partial z^2}\frac{\partial}{\partial \eta}$$

3 $$\frac{\partial}{\partial r} = \frac{\partial \xi}{\partial r}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial r}\frac{\partial}{\partial \eta}$$

4 $$\frac{\partial^2}{\partial r^2} = \left(\frac{\partial \xi}{\partial r}\right)^2\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2 \xi}{\partial r^2}\frac{\partial}{\partial \xi} + 2\frac{\partial \eta}{\partial r}\frac{\partial \xi}{\partial r}\frac{\partial^2}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial r}\right)^2\frac{\partial^2}{\partial \eta^2} + \frac{\partial^2 \eta}{\partial r^2}\frac{\partial}{\partial \eta}$$

5 $$\frac{\partial^2}{\partial z \partial r} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial r}\right) = \frac{\partial}{\partial z}\left(\frac{\partial \xi}{\partial r}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial r}\frac{\partial}{\partial \eta}\right)$$

$$= \frac{\partial^2 \xi}{\partial z \partial r}\frac{\partial}{\partial \xi} + \frac{\partial \xi}{\partial r}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \xi}\right) + \frac{\partial^2 \eta}{\partial z \partial r}\frac{\partial}{\partial \eta} + \frac{\partial \eta}{\partial r}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \eta}\right)$$

$$= \frac{\partial^2 \xi}{\partial z \partial r}\frac{\partial}{\partial \xi} + \frac{\partial \xi}{\partial r}\frac{\partial \xi}{\partial z}\frac{\partial^2}{\partial \xi^2} + \frac{\partial \xi}{\partial r}\frac{\partial \eta}{\partial z}\frac{\partial^2}{\partial \xi \partial \eta}$$

$$+ \frac{\partial^2 \eta}{\partial z \partial r}\frac{\partial}{\partial \eta} + \frac{\partial \eta}{\partial r}\frac{\partial \xi}{\partial z}\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial \eta}{\partial r}\frac{\partial \eta}{\partial z}\frac{\partial^2}{\partial \eta^2}$$

The Fortran encoded functions which represent the partial derivatives used in coordinate transformations are listed below:

1 z1 = $\frac{\partial \xi}{\partial z}$

2 z2 = $(\frac{\partial \xi}{\partial z})^2$ = z1 x z1

3 z3 = $\frac{\partial^2 \xi}{\partial z^2}$

4 z4 = $\frac{\partial \eta}{\partial z}$

5 z5 = $(\frac{\partial \eta}{\partial z})^2$ = z4 x z4

6 z6 = $\frac{\partial^2 \eta}{\partial z^2}$

7 z7 = $\frac{\partial \xi}{\partial r}$

8 z8 = $(\frac{\partial \xi}{\partial r})^2$ = z7 x z7

9 z9 = $\frac{\partial^2 \xi}{\partial r^2}$

10 z10 = $\frac{\partial \eta}{\partial r}$

11 z11 = $(\frac{\partial \eta}{\partial r})^2$ = z10 x z10

12 z12 = $\frac{\partial^2 \eta}{\partial r^2}$

13 z13 = $\frac{\partial^2 \xi}{\partial z \partial r}$

14 z14 = $\frac{\partial^2 \eta}{\partial z \partial r}$

These coefficients can be arranged in a convenient tabular form that shows the relationship between derivatives in both coordinate systems. This is provided in the following table, Table II.

TABLE II. Coefficients Relating Derivatives of Variables in Both Coordinate Systems.

	$\frac{\partial^2}{\partial\xi^2}$	$\frac{\partial^2}{\partial\eta^2}$	$\frac{\partial}{\partial\xi}$	$\frac{\partial}{\partial\eta}$	$\frac{\partial^2}{\partial\xi\partial\eta}$
$\frac{\partial}{\partial z}$ =	0	0	z1	z4	0
$\frac{\partial}{\partial r}$ =	0	0	z7	z10	0
$\frac{\partial^2}{\partial z^2}$ =	z2	z5	z3	z6	2z1z4
$\frac{\partial^2}{\partial r^2}$ =	z8	z11	z9	z12	2z7z10
$\frac{\partial^2}{\partial z\partial r}$ =	z1z7	z4z10	z13	z14	(z1z10 + z4z7)

Example: $\frac{\partial}{\partial z} = 0\frac{\partial^2}{\partial\xi^2} + 0\frac{\partial^2}{\partial\eta^2} + z1\frac{\partial}{\partial\xi} + z4\frac{\partial}{\partial\eta} + 0\frac{\partial^2}{\partial\xi\partial\eta}$

COORDINATE TRANSFORMATION STRETCHING FUNCTIONS

The physical combustor geometry is transformed to a uniform rectangle in a computational plane (ξ,η) by the following functions.

$$\xi = \frac{z}{zmax} = \frac{z}{L} = X1$$

where L is the combustor length, and

$$\eta = \frac{r - R1(z)}{R2(z)-R1(z)} = X2$$

where R1 (z) and R2 (z) are the inner and outer radii of the combustor walls.

Hence

$$z = L\xi = X1T \quad ; \quad r = R1+\eta(R2-R1) = X2T$$

so

1 $z1 = \frac{\partial \xi}{\partial z} = \frac{1}{L}$

2 $z2 = (\frac{\partial \xi}{\partial z})^2 = z1 \times z1$

3 $z3 = \frac{\partial^2 \xi}{\partial z^2} = 0$

4 $z4 = \frac{\partial \eta}{\partial z} = \frac{(R2-R1)(-R1')-(r-R1)(R2'-R1')}{(R2-R1)^2} = \frac{-R1'}{R2-R1} - \frac{(r-R1)(R2'-R1')}{(R2-R1)^2}$

5 $z5 = (\frac{\partial \eta}{\partial z})^2 = z4 \times z4$

6 $z6 = \frac{\partial^2 \eta}{\partial z^2} = \frac{(R2-R1)(-R1'')+R1'(R2'-R1')}{(R2-R1)^2} - \frac{(R2-R1)2[-R1'(R2'-R1')+(r-R1)(R2''-R1'')]-(r-R1)(R2'-R1')2(R2-R1)(R2'-R1')}{(R2-R1)^4}$

$$= \frac{-R1''}{R2-R1} + 2\,\frac{R1'(R2'-R1')}{(R2-R1)^2} - \frac{(r-R1)(R2''-R1'')}{(R2-R1)^2} + \frac{2(r-R1)(R1'-R1')^2}{(R2-R1)^3}$$

7 z7 $= \frac{\partial \xi}{\partial r} = 0$

8 z8 $= (\frac{\partial \xi}{\partial r})^2 = 0$

9 z9 $= \frac{\partial^2 \xi}{\partial r^2} = 0$

10 z10 $= \frac{\partial \eta}{\partial r} = \frac{1}{R2-R1}$

11 z11 $= (\frac{\partial \eta}{\partial r})^2 =$ z10 x z10

12 z12 $= \frac{\partial^2 \eta}{\partial r^2} = 0$

13 z13 $= \frac{\partial^2 \xi}{\partial z \partial r} = 0$

14 z14 $= \frac{\partial^2 \eta}{\partial z \partial r} = \frac{\partial}{\partial z} (\frac{1}{R2-R1}) = \frac{-(R2'-R1')}{(R2-R1)^2}$

In the previous equations R1´ and R2´ refer to $\frac{dR1}{dz}$ and $\frac{dR2}{dz}$, respectively,

and R1" and R2" refer to $\frac{d^2R1}{dz^2}$ and $\frac{d^2R2}{dz^2}$ respectively.

APPENDIX C

Boundary Conditions for Dependent Variables in Transformed Coordinate System

(A) for velocities u and v

(i) At the inlet boundary, the velocities are specified:

$u =$ given or defined as $= v_0 f(r)$

$v = 0$

(ii) At the solid wall, a "no-slip" boundary condition is used

$u = 0$

$v = 0$

(iii) At the exit, we assumed the flow is fully developed, then

$$\frac{\partial u}{\partial z} = 0$$

$$\frac{\partial v}{\partial z} = 0$$

(iv) At the axis, for symmetry,

$$\frac{\partial u}{\partial r} = 0$$

(B) for stream function ψ

(i) At the inlet boundary

$$\psi = \int_o^r \rho u r dr$$

(ii) At the wall

$$\psi = \int_o^{R2} \rho u r dr$$

(iii) At the exit

$$\frac{\partial \psi}{\partial z} = 0$$

(iv) At the axis

$$\psi = 0$$

(C) for vorticity $\omega = \frac{\partial v}{\partial z} - \frac{\partial u}{\partial r}$

which transforms to

$$\omega = z1 \frac{\partial v}{\partial \xi} + z4 \frac{\partial v}{\partial \eta} - z7 \frac{\partial u}{\partial \xi} - z10 \frac{\partial u}{\partial \eta}$$

The value of ω on the boundary points was assigned using this equation and the following finite difference formula for the deviations.

(i) At the inlet boundary (except at the corner)

A forward-difference expression with error of order $(\Delta\xi)^2$ was used to evaluate $\frac{\partial}{\partial \xi}$ in the expression for ω; for example,

$$\phi i' = \frac{-\phi i{+}2 + 4\phi i{+}1 - 3\phi i}{2\Delta\xi}$$

and a central-difference expression with error of order $(\Delta\eta)^2$ was used for $\frac{\partial}{\partial \eta}$

$$\phi j' = \frac{\phi j{+}1 - \phi j{-}1}{2\Delta\eta}$$

(ii) Similarly, at the lower corner, for example

$$\phi i' = \frac{-\phi i{+}2 + 4\phi i{+}1 - 3\phi i}{2\Delta\xi} \quad \text{for } \frac{\partial}{\partial \xi}$$

$$\phi j' = \frac{-\phi j{+}2 + 4\phi j{+}1 - 3\phi j}{2\Delta\eta} \quad \text{for } \frac{\partial}{\partial \eta}$$

(iii) At the upper corner for $\frac{\partial}{\partial \xi}$

$$\phi i' = \frac{-\phi i{+}2 + 4\phi i{+}1 - 3\phi i}{2\Delta\xi}$$

a backward-difference expression with error of order $(\Delta\eta)^2$ was used for $\frac{\partial}{\partial\eta}$

$$\phi_j' = \frac{3\phi_j - 4\phi_{j-1} + \phi_{j-2}}{2\Delta\eta}$$

(iv) At the outer wall, the expression for ω reduces to

$$\omega = z4 \frac{\partial v}{\partial\eta} - z10 \frac{\partial u}{\partial\eta}$$

where the expression used to evaluate the derivatives was

$$\phi_j' = \frac{3\phi_j - 4\phi_{j-1} + \phi_{j-2}}{2\Delta\eta}$$

(v) At the exit.

$$\frac{\partial\omega}{\partial z} = 0$$

or

$$\frac{\partial w}{\partial z} = z1$$

$$z1 \frac{\partial\omega}{\partial\xi} + z4 \frac{\partial\omega}{\partial\eta} = 0$$

A backward-difference expression with error of order $(\Delta\xi)^2$ was used for $\frac{\partial}{\partial\xi}$ and a central-difference expression with error of order $(\Delta\eta)^2$ was used for $\frac{\partial}{\partial\eta}$, it then turns out that

$$\omega_{i,j} = \frac{1}{3}\left[4\,\omega_{i-1,j} - \omega_{i-2,j} - \frac{z4}{z1}\,\frac{\Delta\xi}{\Delta\eta}\left(\omega_{i,j+1} - \omega_{i,j-1}\right)\right]$$

(vi) At the axis

$$\omega = 0$$

In all of the above formulations, ϕ' is the derivative of either u or v with respect to ξ or η (i or j, respectively).

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