GLR-178

A THIN ROD HEAT FLUX TRANSDUCER POSITIONED IN THE EARTH HAVING A UNIFORM TEMPERATURE GRADIENT: A CLOSED FORM SOLUTION

For

ERDA

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CONTENTS

INTRODUCTION

One of the geothermal sensors that is being developed by Geoscience for ERDA in connection with the measurement of the geothermal heat flux is a thin rod heat flux transducer. It is to be lowered in a drillhole where signals from this sensor will be measured and translated into a geothermal heat flux. A mathematical temperature solution has been developed for an idealized thin rod positioned in an infinite solid with a uniform temperature gradient. This solution approximates the actual temperature to be encountered in the transducer when located in a drill hole.

The solution is a closed form type for a thin rod wherein radial temperature differences are small compared to axial ones; the system can be classified as a fin with a variable environmental temperature gradient. The resulting temperature field is evaluated for a range of system parameters that are of interest in geothermal heat flux measurement.

A description of how the mathematical results obtained will be used to extract the unknown geothermal heat flux and earth thermal conductivity from the experimental temperature information obtained by the heat flux transducer is also presented. In addition, the solution is extended to the case where there is a water or air annulus space between the transducer and the drill-hole.

DERIVATION

The thermal analysis presented below is based on a thin rod, positioned in an infinite solid which is characterized by a uniform temperature gradient (see Figure 1). The problem is to determine the temperature field in the thin rod. A heat balance on a differential element of the thin rod which is transferring heat to or from the surrounding infinite solid through a radial thermal resistance is given by the classical fin equation,

$$\frac{d^2t}{dz^2} = \frac{P}{R_\infty kA}(t - t_\infty) \qquad (1)$$

where

- t, rod temperature (above the rod midpoint temperature datum)
- z, distance along rod
- P, perimeter of the rod
- A, cross sectional area of the rod
- k, thermal conductivity of the rod
- R_∞, equivalent radial thermal resistance of the solid surrounding the rod
- t_∞, the linear lateral temperature variation (above the rod midpoint temperature datum) in the solid at a radial distance sufficiently great so that the presence of the rod does not effect it.

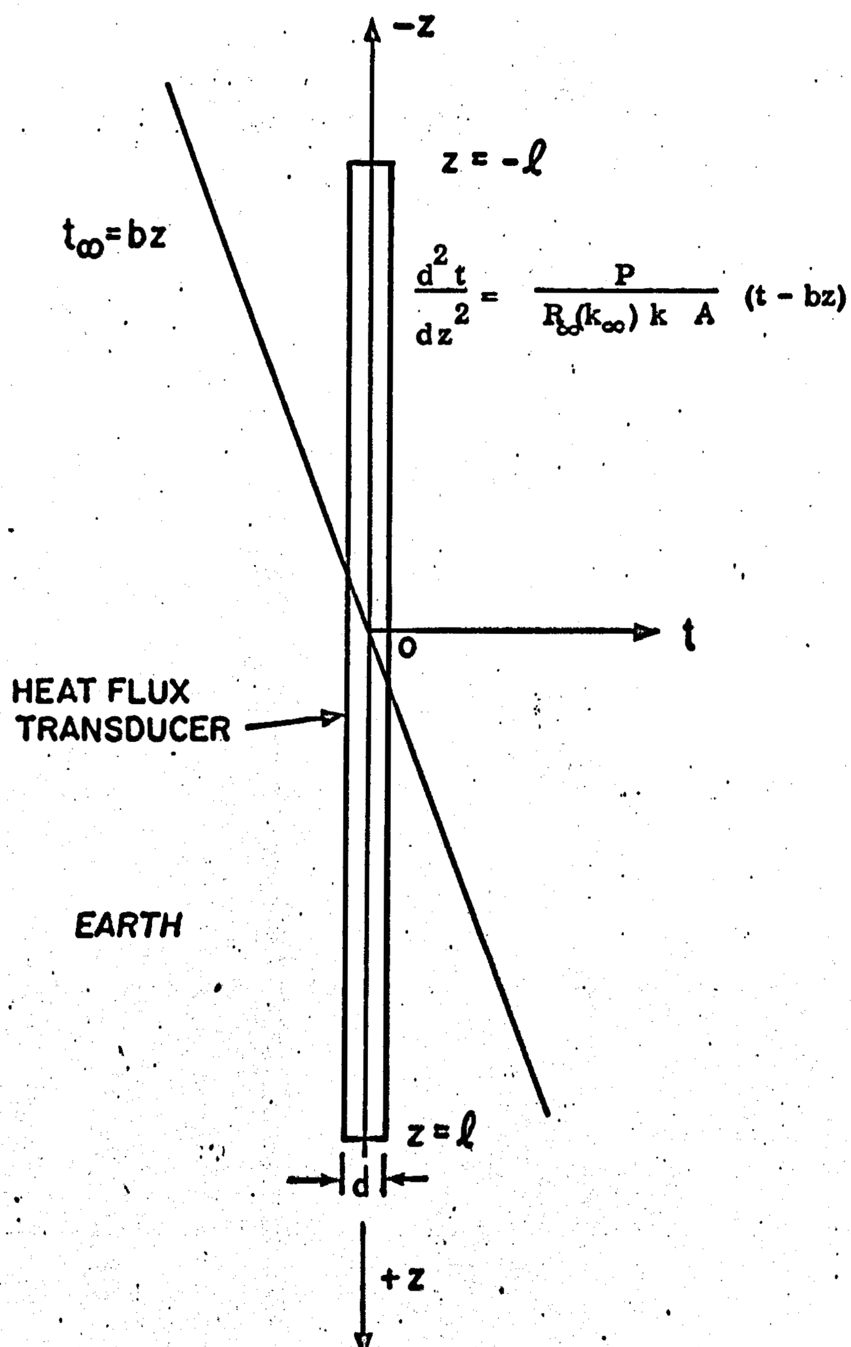

Figure 1. Idealized thin rod transducer system

The temperature variation in the solid is given by,

$$t_\infty = bz \qquad (2)$$

where the parameter, b, is the undisturbed vertical temperature gradient of the earth.

Thus, Equation 1 can be expressed as

$$\left(\frac{d^2}{dz^2} - C\right) t = -Bz \qquad (3)$$

where

$$C = \frac{P}{R_\infty kA}$$

$$B = \frac{Pb}{R_\infty kA}$$

The complimentary solution of Equation (3), t_c, is

$$t_c = c_1 e^{-\sqrt{C}z} + c_2 e^{\sqrt{C}z} \qquad (4)$$

The particular solution of Equation (3), t_p, can be obtained using the method of undetermined coefficients, namely, let

$$t_p = c_3 z^2 + c_4 z + c_5 \qquad (5)$$

Substituting Equation (5) into (3) yields,

$$\left(\frac{d^2}{dz^2} - C\right)\left(c_3 z^2 + c_4 z + c_5\right) = - Bz$$

or

$$2c_3 - Cc_3 z^2 - Cc_4 z - Cc_5 = - Bz \tag{6}$$

Thus, the coefficients in Equation (5) became,

$$\left.\begin{aligned} c_3 &= 0 \\ c_4 &= \frac{B}{C} \\ c_5 &= 0 \end{aligned}\right\} \tag{7}$$

Therefore, the complete solution of Equation (3) is the sum of (4) and (5),

$$t = c_1 e^{-\sqrt{C}z} + c_2 e^{\sqrt{C}z} + \frac{B}{C} z \tag{8}$$

One boundary condition for this problem is $t = 0$ at $z = 0$. Thus,

$$c_2 = -c_1 \tag{9}$$

and $$t = c_1 \left(e^{-\sqrt{C}z} - e^{\sqrt{C}z} \right) + \frac{B}{C} z \tag{10}$$

The second boundary condition for this problem defines the heat loss from the end of the rod (at $z = \ell$), namely,

$$-k\left(\frac{dt}{dz}\right)_{\ell} A = \frac{1}{R_e}\left(t_{\ell} - b\ell\right) A \tag{11}$$

where R_e is the equivalent end thermal resistance of the solid surrounding the rod. Upon substituting Equation (10) into (11), there results

$$-k\left[c_1\left(-\sqrt{C}e^{-\sqrt{C}\ell} - \sqrt{C}e^{\sqrt{C}\ell}\right) + \frac{B}{C}\right] =$$

$$\frac{1}{R_e}\left[c_1\left(e^{-\sqrt{C}\ell} - e^{\sqrt{C}\ell}\right) + \frac{B}{C}\ell - b\ell\right]$$

or

$$c_1 = \frac{-b}{-\sqrt{C}\left(e^{-\sqrt{C}\ell} + e^{\sqrt{C}\ell}\right) + \frac{1}{kR_e}\left(e^{-\sqrt{C}\ell} - e^{\sqrt{C}\ell}\right)} \tag{12}$$

It is also necessary to define the cylindrical and hemispherical thermal resistances of the solid surrounding the transducer, namely, R_∞ and R_e, respectively. The resistance for the annulus surrounding the rod is,

$$R_\infty = \frac{r_i}{K_\infty} \ln \frac{r_o}{r_i} \tag{13}$$

where

r_o, radius at which the undisturbed linear temperature field in the infinite solid exists

r_i, radius of the rod

k_∞, thermal conductivity of the infinite solid

The thermal resistance of the hemispherical shell at the end of the rod is,

$$R_e = \frac{r_i^2 \left(\frac{1}{r_i} - \frac{1}{r_o} \right)}{K_\infty} \tag{14}$$

Thus, the complete temperature solution for the thin rod transducer is given by Equations (10), (12), (13) and (14).

The solution developed above has also been extended to a composite thin rod which consists of a sensor rod surrounded by an annulus of water or air (the space between the sensor and borehole). Again, it is postulated that radial temperature differences in the rod and annulus are small and negligible compared to axial values. From elementary parallel flow conduction concepts, it can be shown that the composite (or equivalent) thermal conductivity of a rod with an annulus, k_c, is

$$k_c = k_r \nu_r + k_a \nu_a \tag{15}$$

where

k_r, rod thermal conductivity

k_a, annulus thermal conductivity

ν_r, rod volume fraction of the composite rod

ν_a, annulus volume fraction of the composite rod

PARAMETRIC RESULTS

To illustrate the variation of the temperature field in a rod heat flux transducer positioned vertically in the earth, Equations (10), (12), (13), and (14) were evaluated for the following range of parameters:

Geothermal heat flux ($k_\infty b$) = 2 x 10^{-6} cal/sec cm^2 = 2 HFU

2ℓ = 6 feet (1.83 meters)

d = 4 inches (0.102 meters)

k range: 0.05 to 100 Btu/hr ft°F (0.0002 to 0.40 cal/sec cm°C)

k_∞ range: 0.1 to 1.0 Btu/hr ft°F (0.0004 to 0.004 cal/sec cm°C)

r_o range: 1 to 9 feet (0.305 to 2.743 meters)

The results of these evaluations can be seen in Figures 2 to 13.

Legend Information For

Figure 2 - 13

Curve	$K_{rod}\left(\frac{Btu}{hr\ ft^2\ °F/ft}\right)$	$K_{rod}\left(\frac{cal}{sec\ cm^2\ °C/cm}\right)$
A	0.05	0.0002
B	0.1	0.0004
C	0.5	0.002
D	1.0	0.004
E	5	0.02
F	10	0.04
G	50	0.2
H	100	0.4

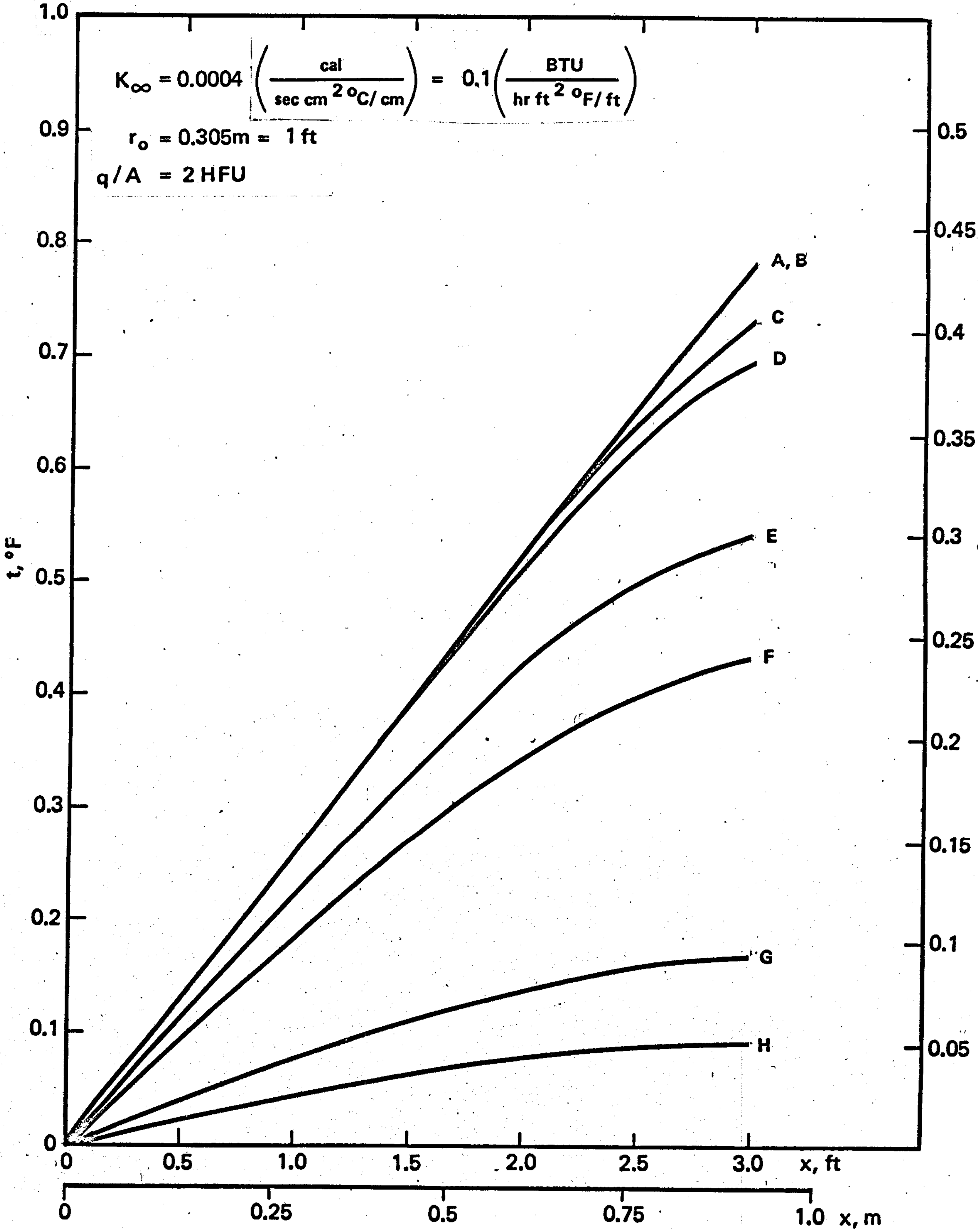

Figure 2. Thin rod temperature profiles for K_∞ = 0.1 Btu/hr ft^2°F/ft and r_0 = 1 ft

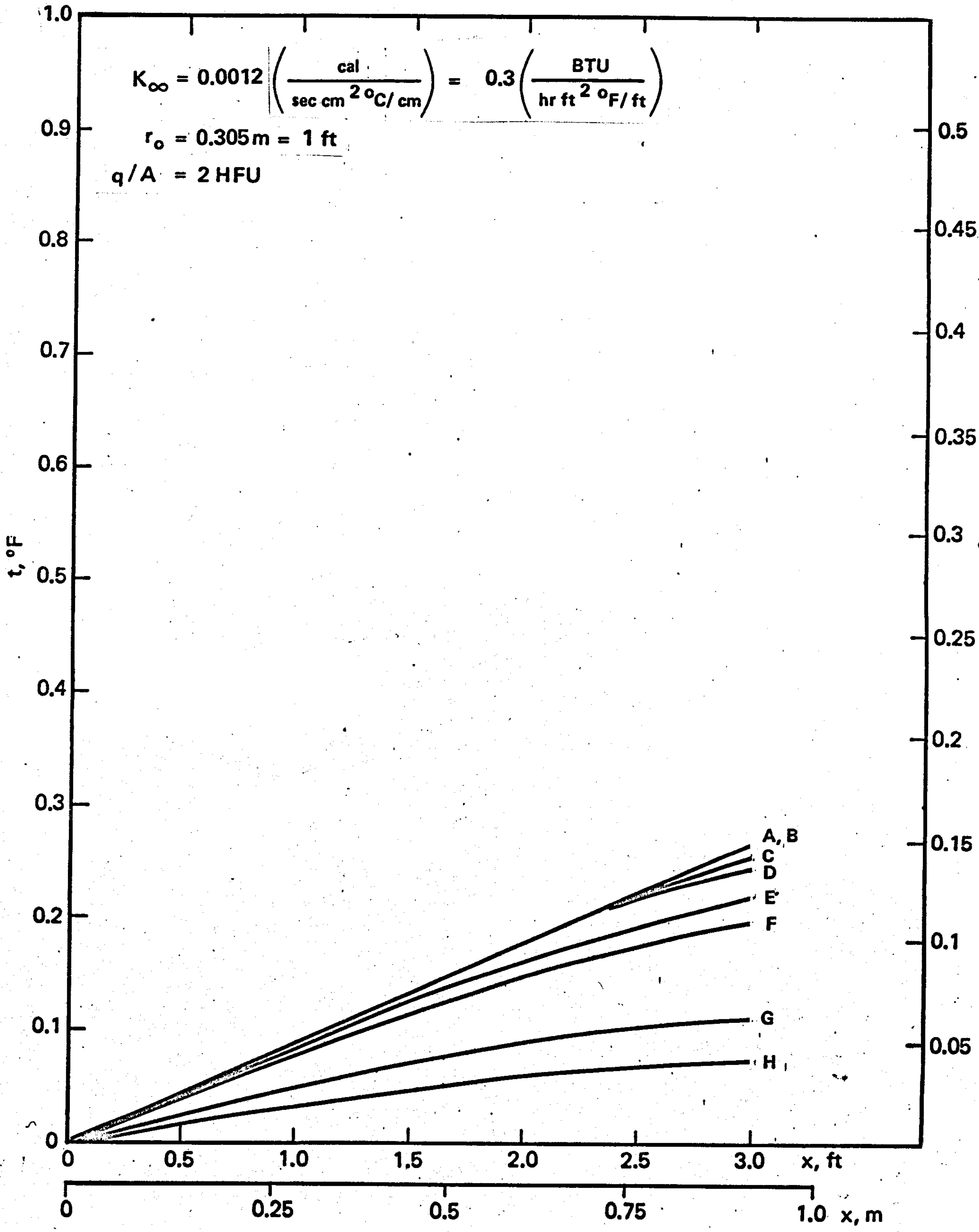

Figure 3. Thin rod temperature profiles for K_∞ = 0.3 Btu/hr ft^2°F/ft and r_o = 1 ft

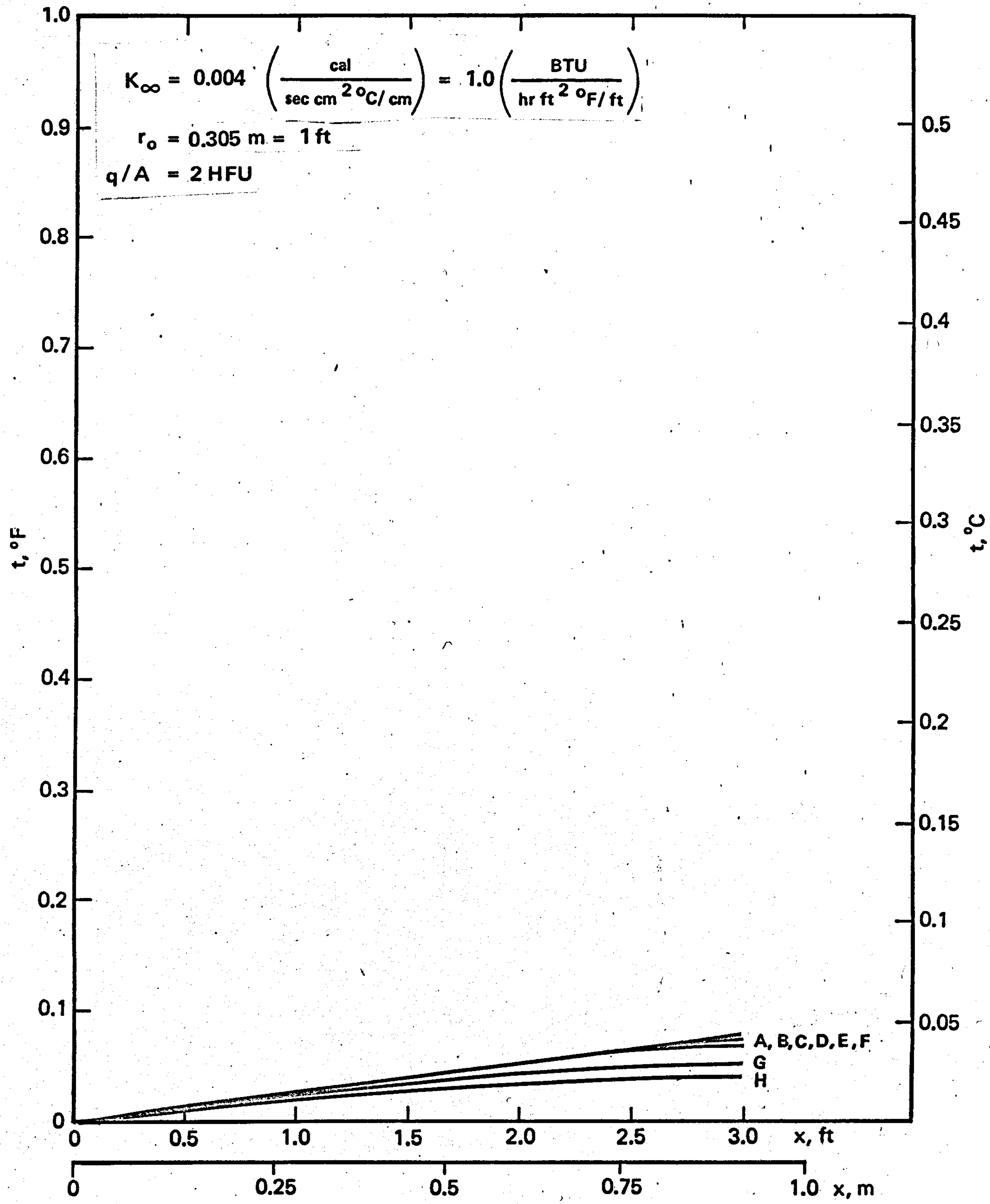

Figure 4. Thin rod temperature profiles for K_∞ = 1.0 Btu/hr ft^2°F/ft and r_0 = 1 ft

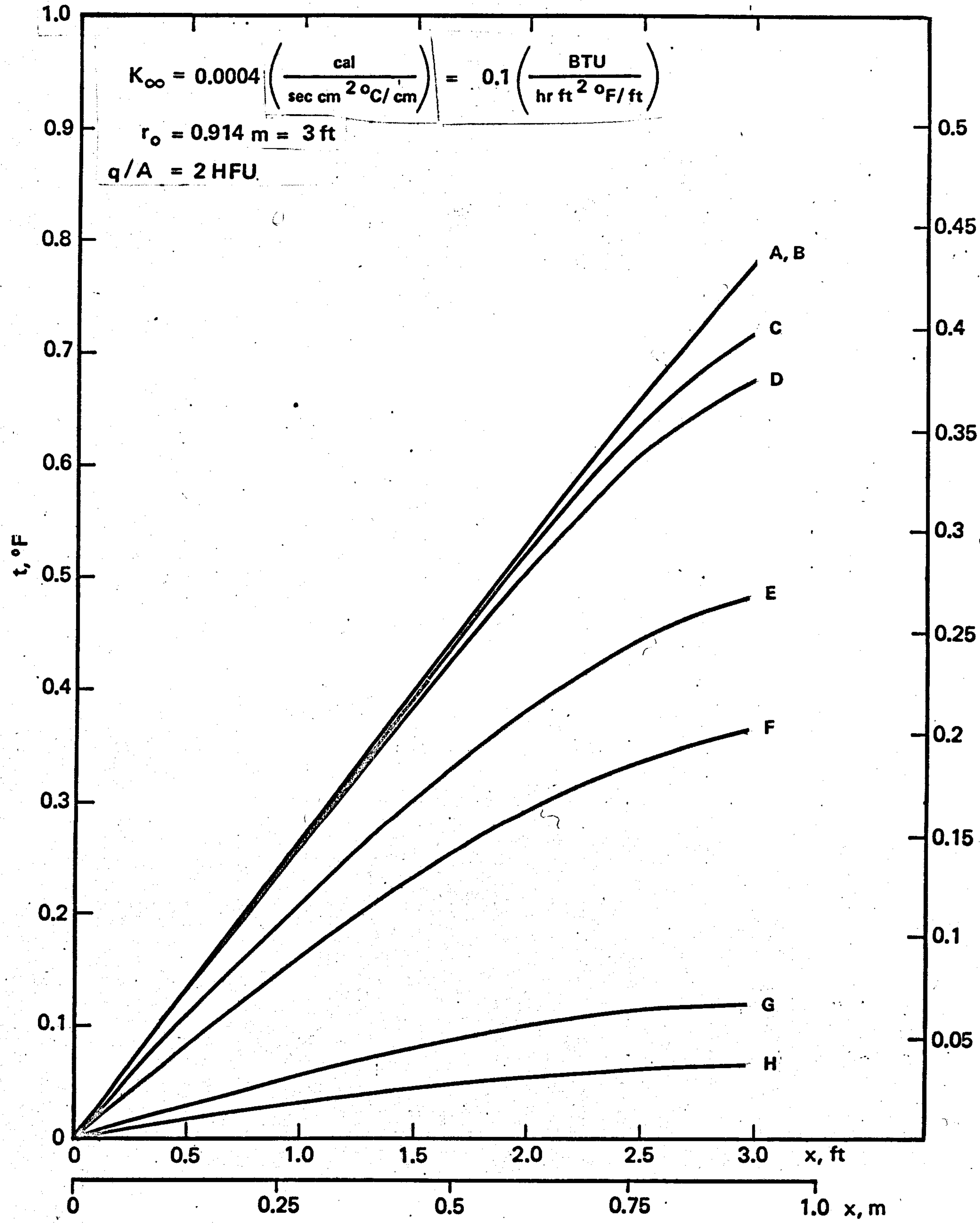

Figure 5. Thin rod temperature profiles for K_∞ = 0.1 Btu/hr ft^2°F/ft and r_o = 3 ft

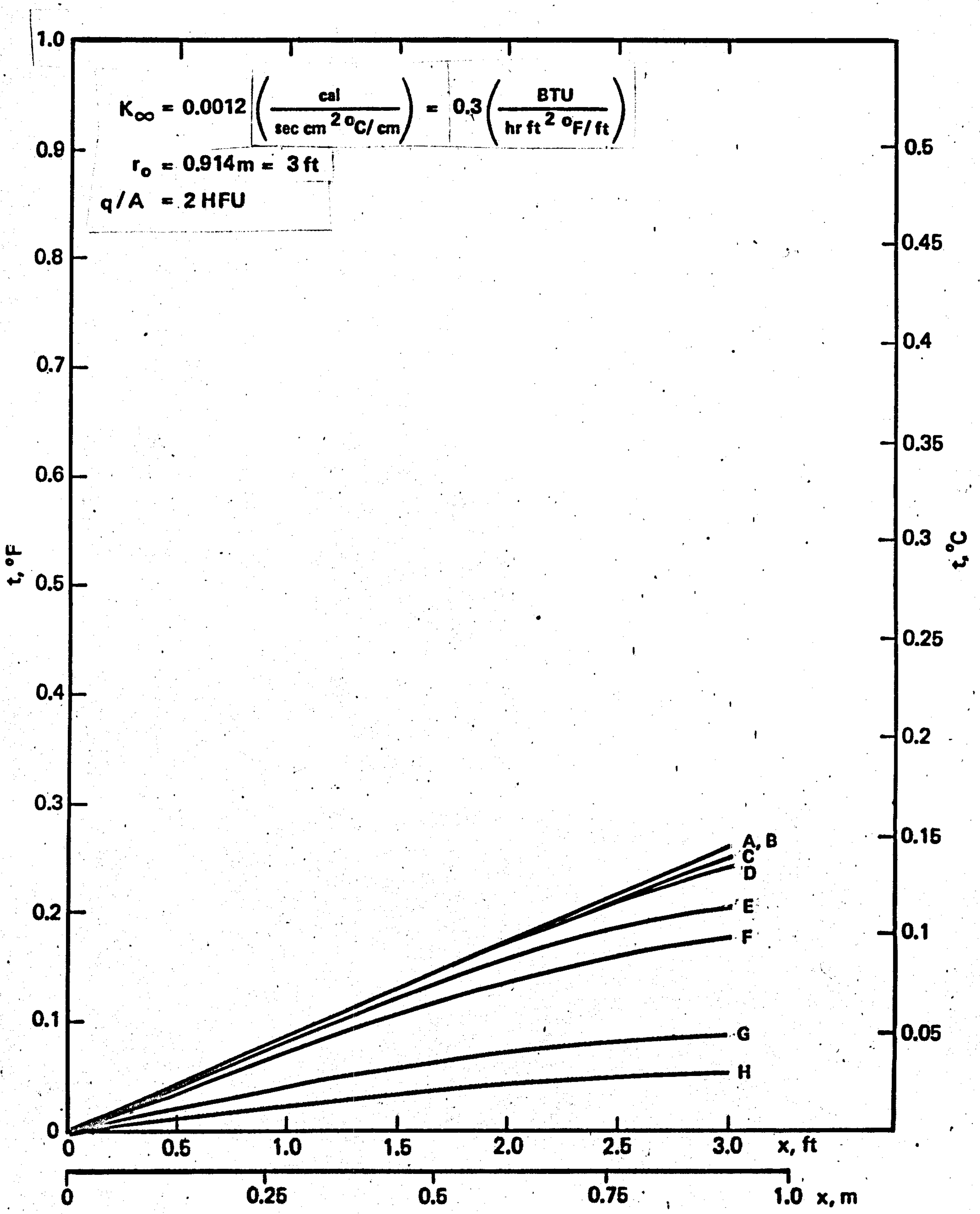

Figure 6. Thin rod temperature profiles for K_{∞} = 0.3 Btu/hr ft^2°F/ft and r_0 = 3 ft

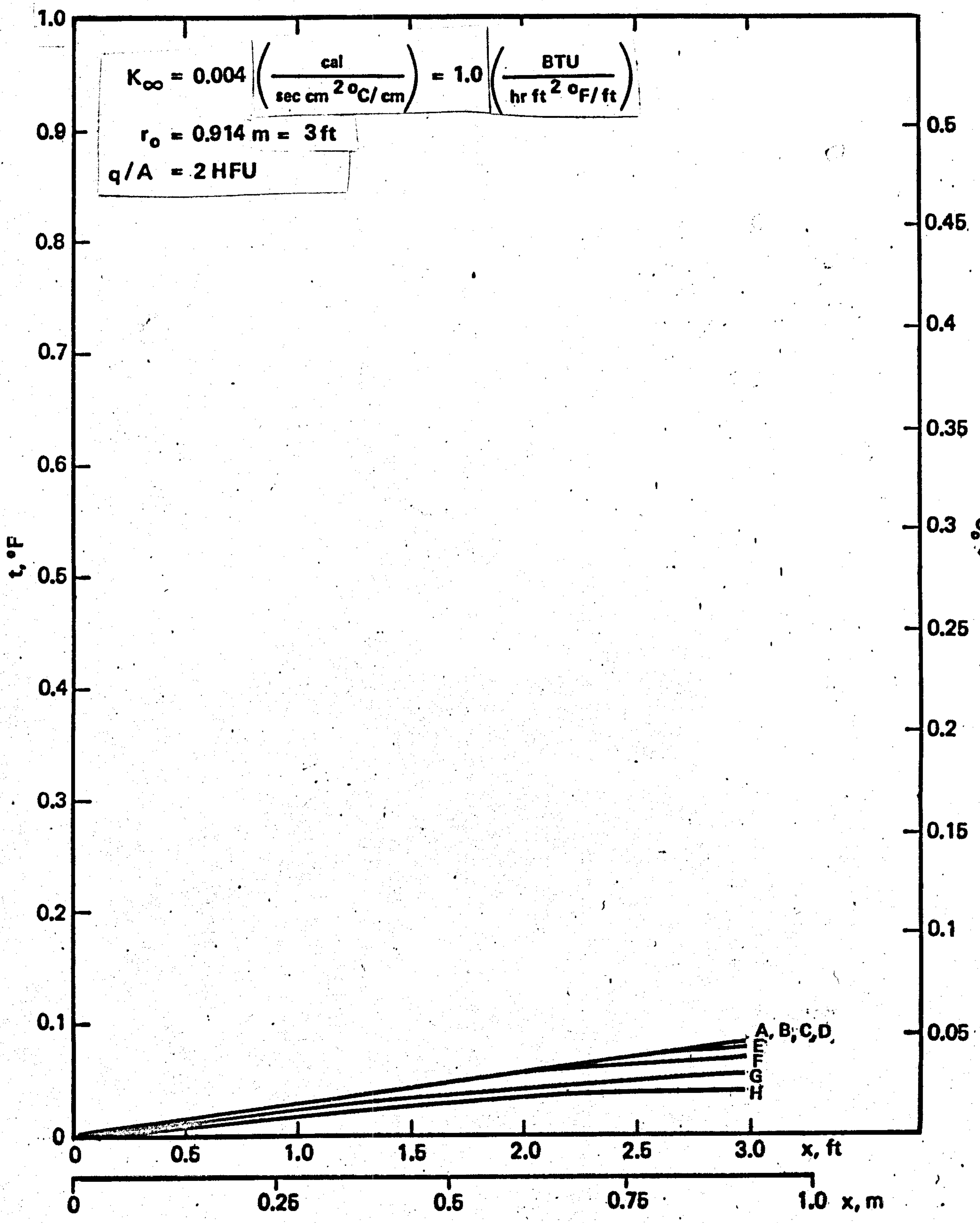

Figure 7. Thin rod temperature profiles for K_∞ = 1.0 Btu/hr ft^2°F/ft and r_o = 3 ft

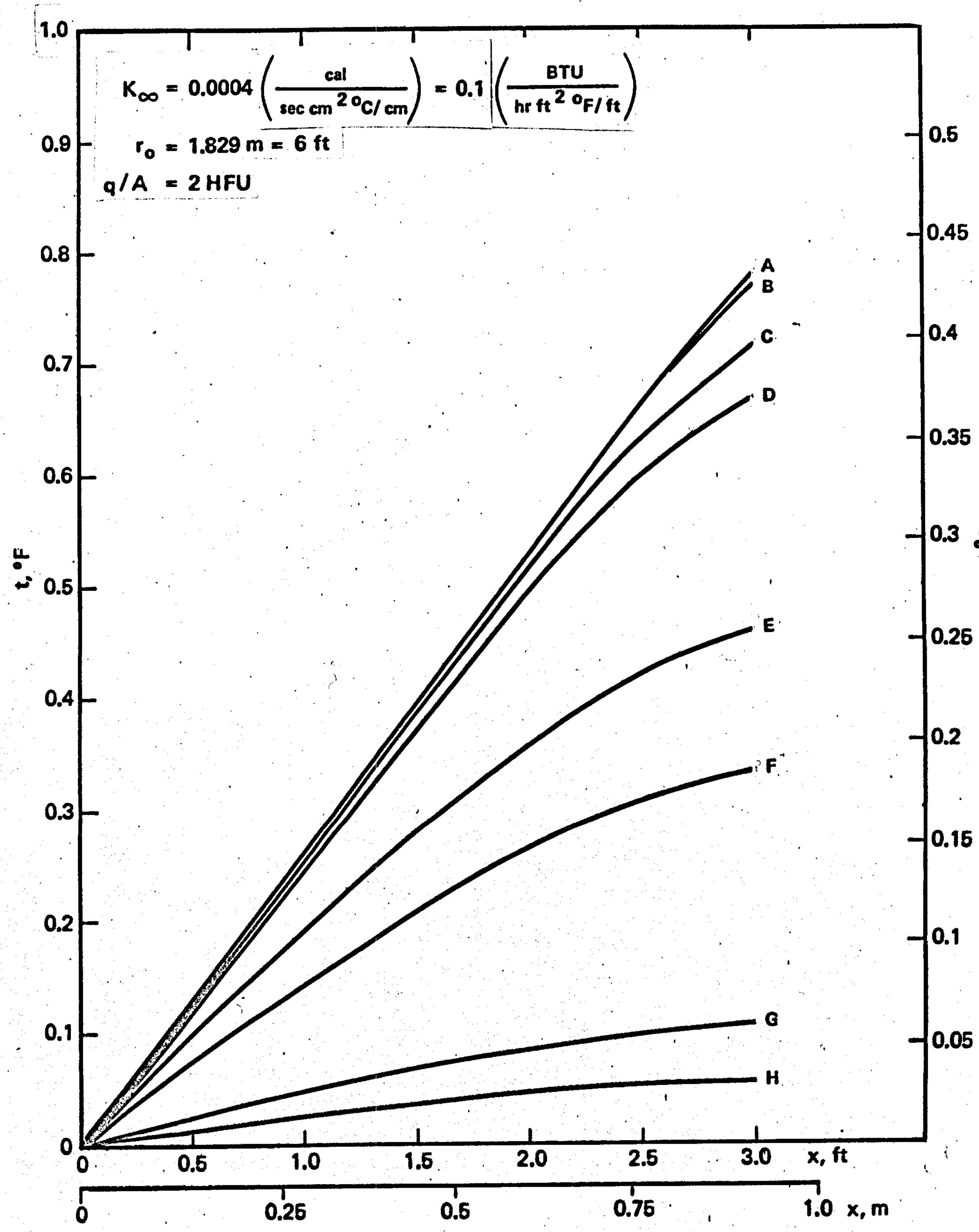

Figure 8. Thin rod temperature profiles for K_∞ = 0.1 Btu/hr ft^2°F/ft and r_o = 6 ft

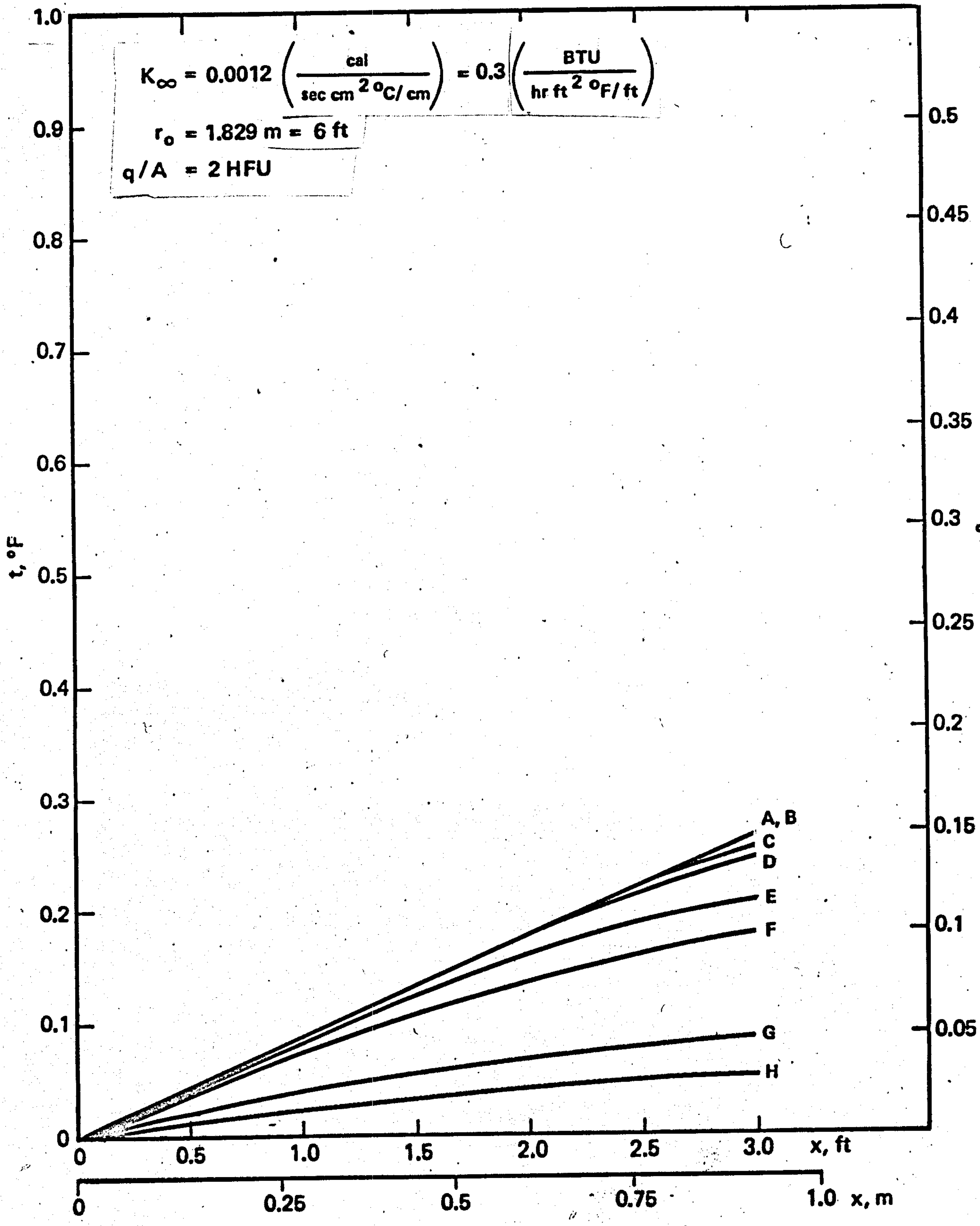

Figure 9. Thin rod temperature profiles for K_∞ = 0.3 Btu/hr ft^2°F/ft and r_0 = 6 ft

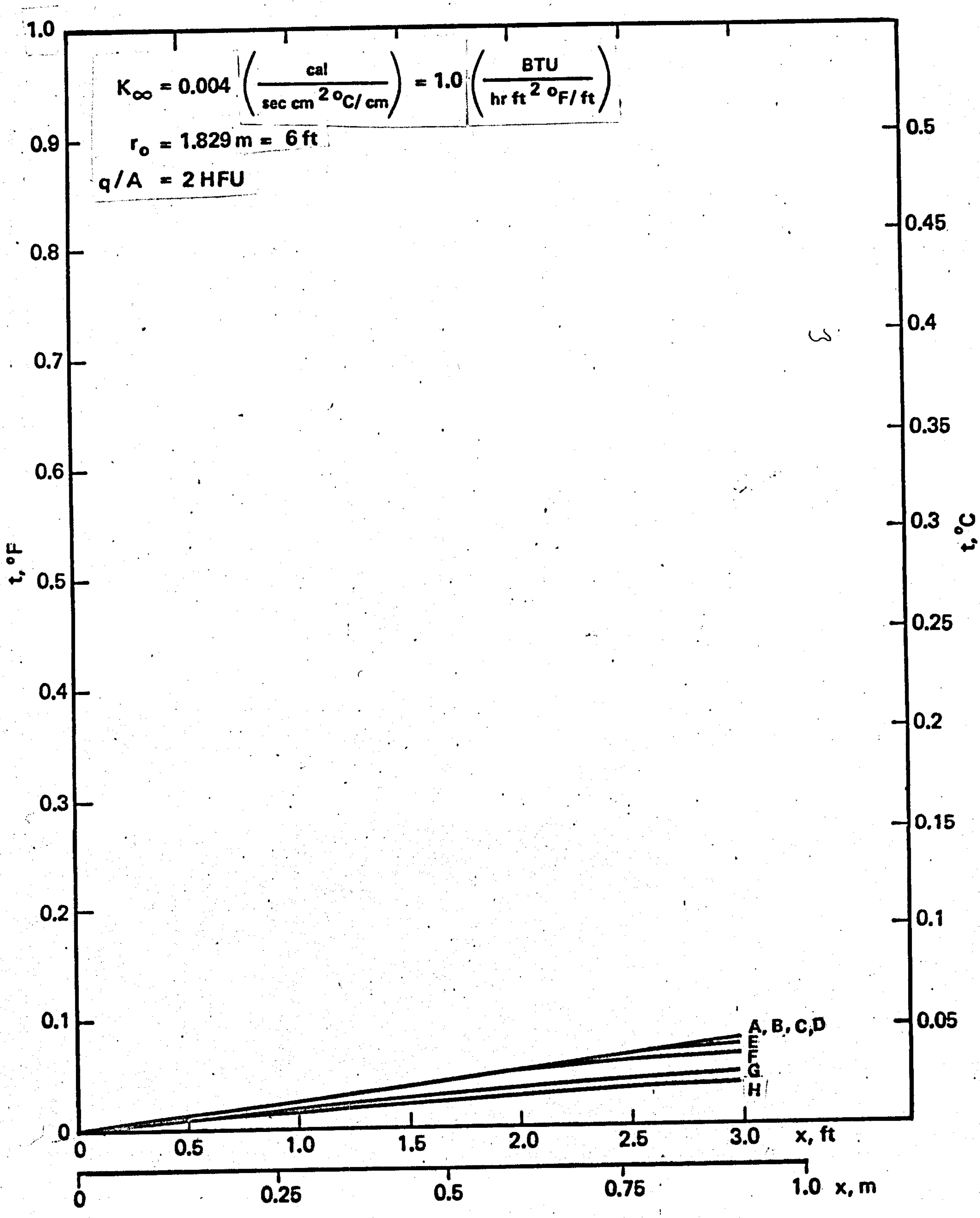

Figure 10. Thin rod temperature profiles for K_∞ = 1.0 Btu/hr ft^2°F/ft and r_0 = 6 ft

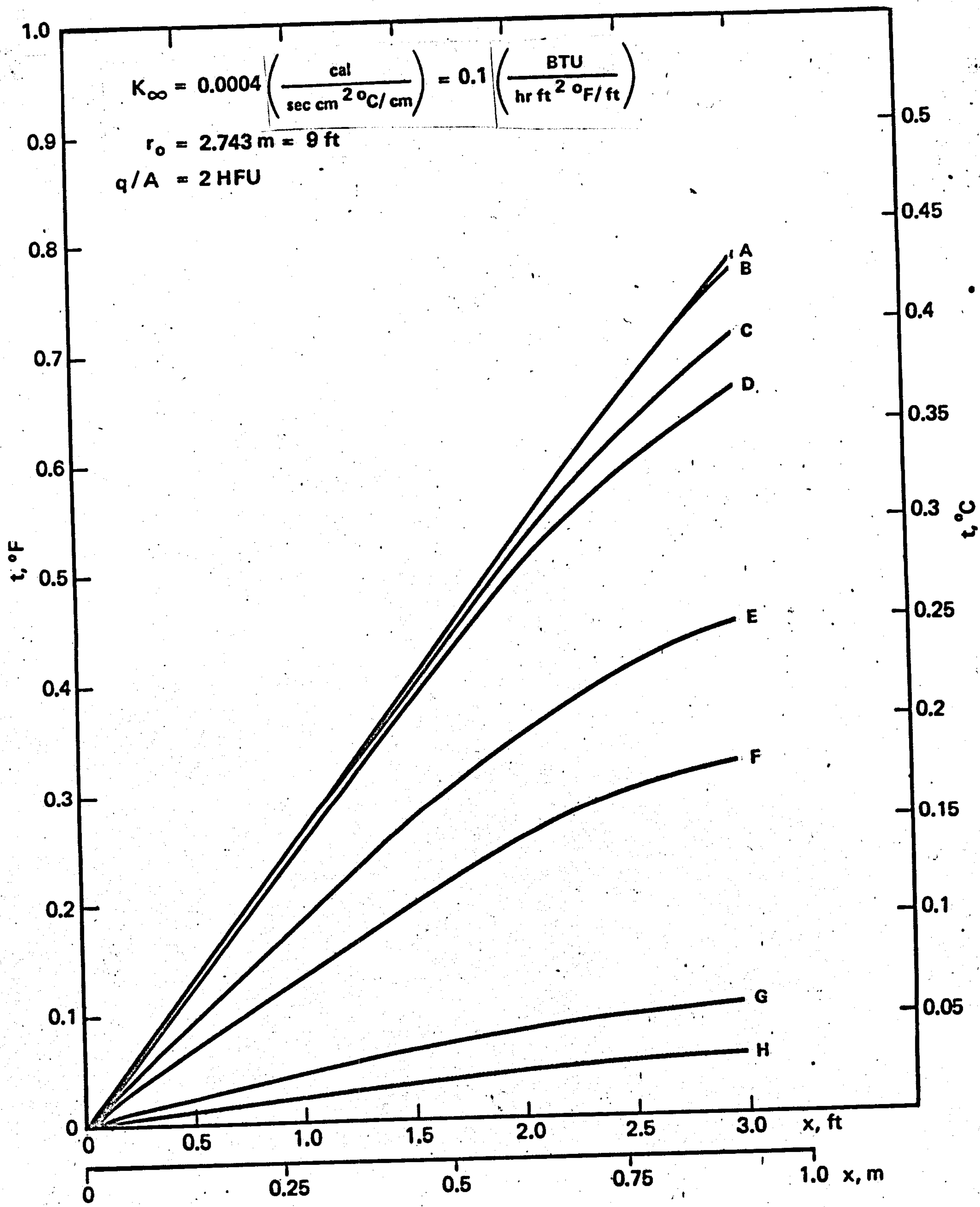

Figure 11. Thin rod temperature profiles for K_∞ = 0.1 Btu/hr ft^2°F/ft and r_0 = 9 ft

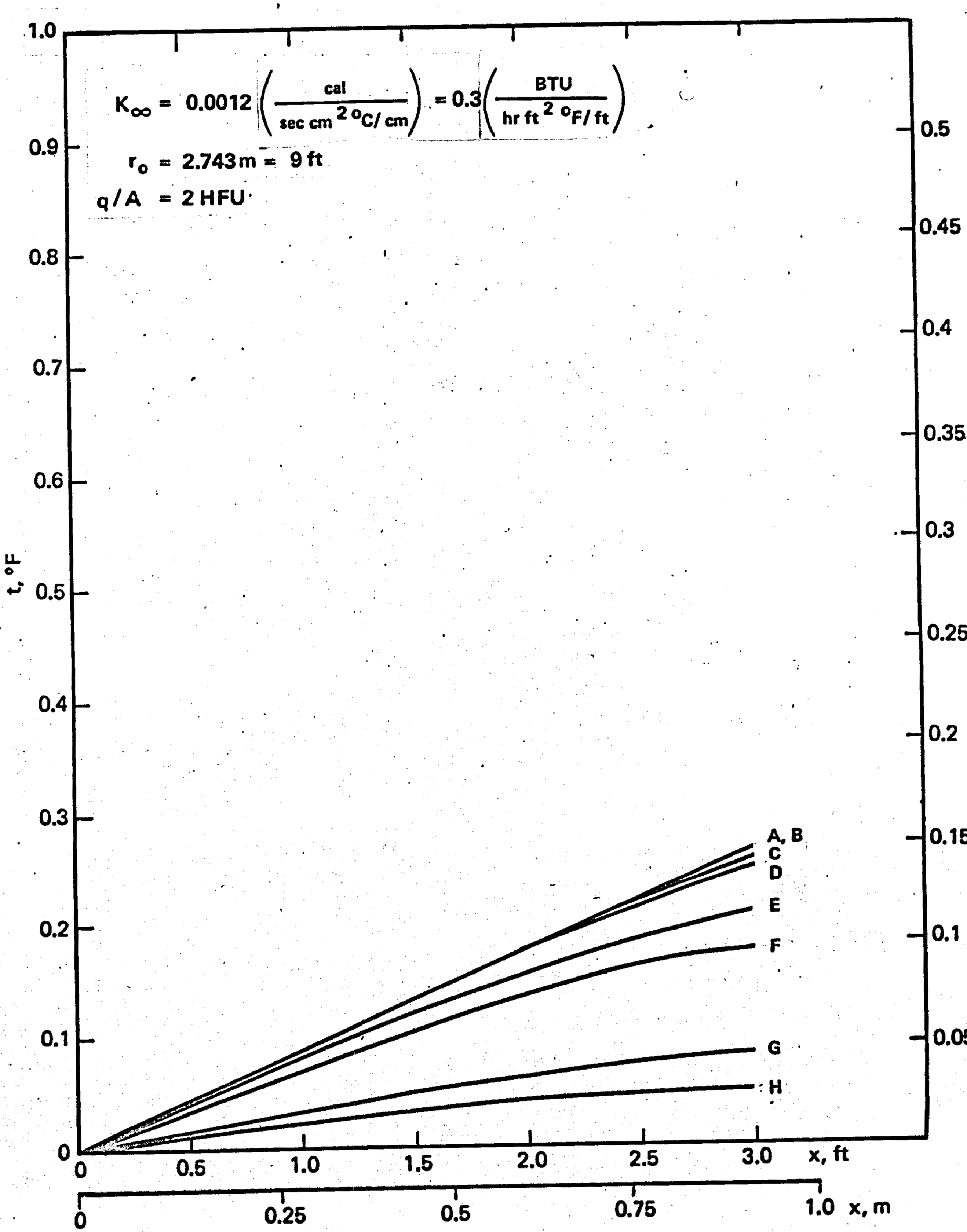

Figure 12. Thin rod temperature profiles for K_∞= 0.3 Btu/hr ft^2°F/ft and r_o = 9 ft

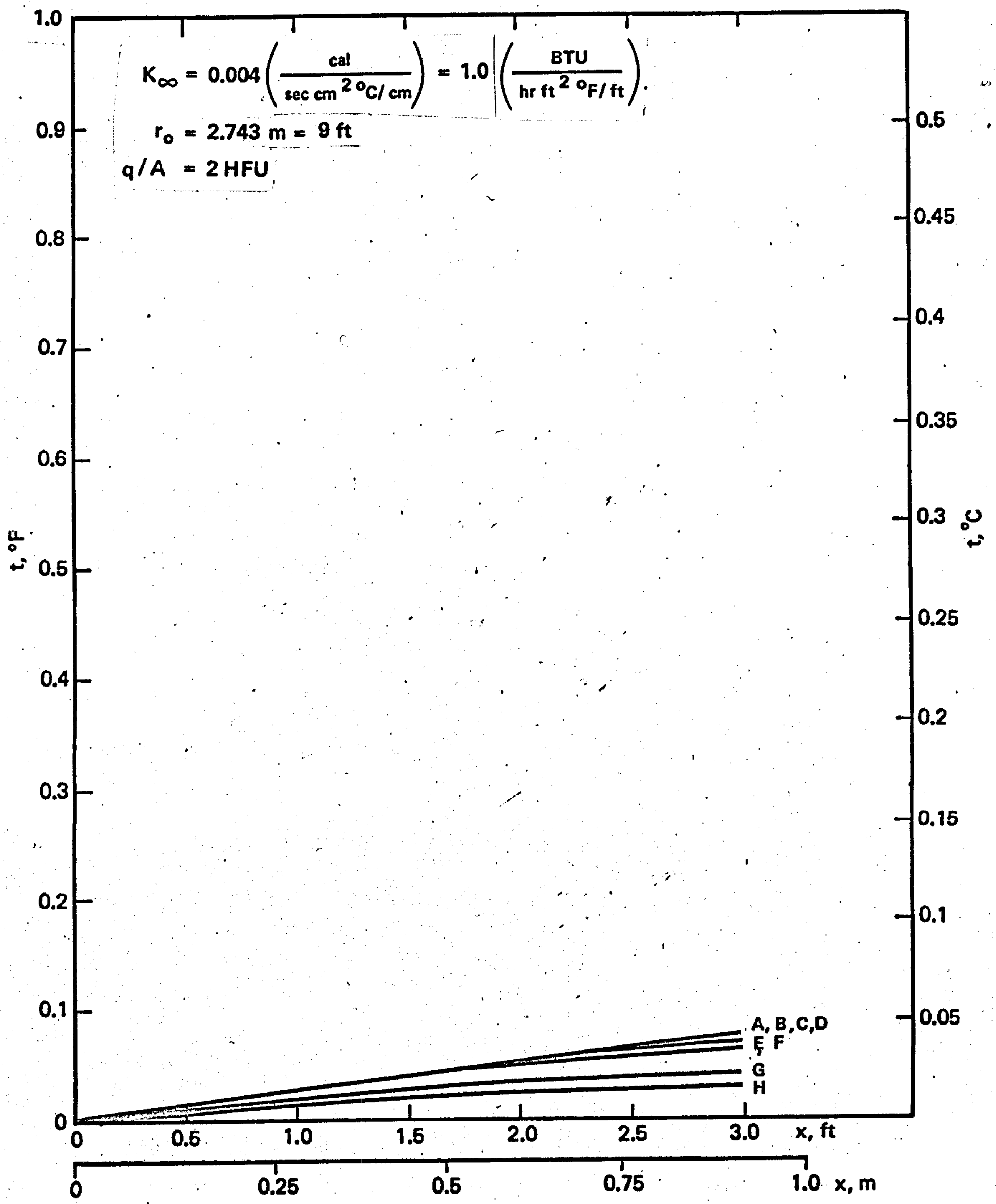

Figure 13. Thin rod temperature profiles for K_{∞} = 1.0 Btu/hr ft^2°F/ft and r_0 = 9 ft

DISCUSSION

A. Application of Solution to Geothermal Heat Flux Measurement

The thin rod solution derived above can be used to relate rod transducer temperature measurements and the sought for geothermal heat flux in the earth where the transducer has been positioned. Specifically, the boundary value problem described above contains two unknowns, namely, the thermal conductivity of the earth, k_∞, and the vertical earth temperature gradient, b (the product of these two quantities being the geothermal heat flux). There are two ways in which the two unknowns can be evaluated from the prediction equations. Either two temperature points along a single transducer of known conductivity are used in the evaluation, or one temperature point each for two different transducers of known thermal conductivity are required.

The results shown in Figures 2 to 13 have been translated into thermopile voltage outputs for k_∞ = 0.3 Btu/hr ft°F (0.0012 cal/sec cm^2°C/cm) and 1.0 Btu/hr ft°F (0.004 cal/sec cm^2°C/cm) for a range of practical junction sets, rod thermal conductivities and typical geothermal heat flux values. The resulting voltages fall in the 0.5 to 1.0 millivolt range and are large enough to be measured by potentiometers without amplification. Details of the thermopile design will appear in a subsequent report.

B. Comments on Model Accuracy

The thermal resistance R_∞ is an important parameter in the rod solution. The variation of this quantity with r_o for a given k_∞ and r_i is shown in Figure 14, together with the corresponding R_e variation. Note that although R_e quickly becomes invarient with r_o, the resistance R_∞ continues to increase with r_o but very slowly at larger r_o values. A calculation shows, for example, that for k = 10 Btu/hr ft°F, the temperature at the end of the rod changes from 0.320°F for r_o = 9 ft. to 0.297°F for r_o = 20 ft. (a seven percent difference).

The limitations of this model will be further discussed in a forthcoming report where the complete heat transfer system is represented by a two-region Laplace equation set which is solved numerically.

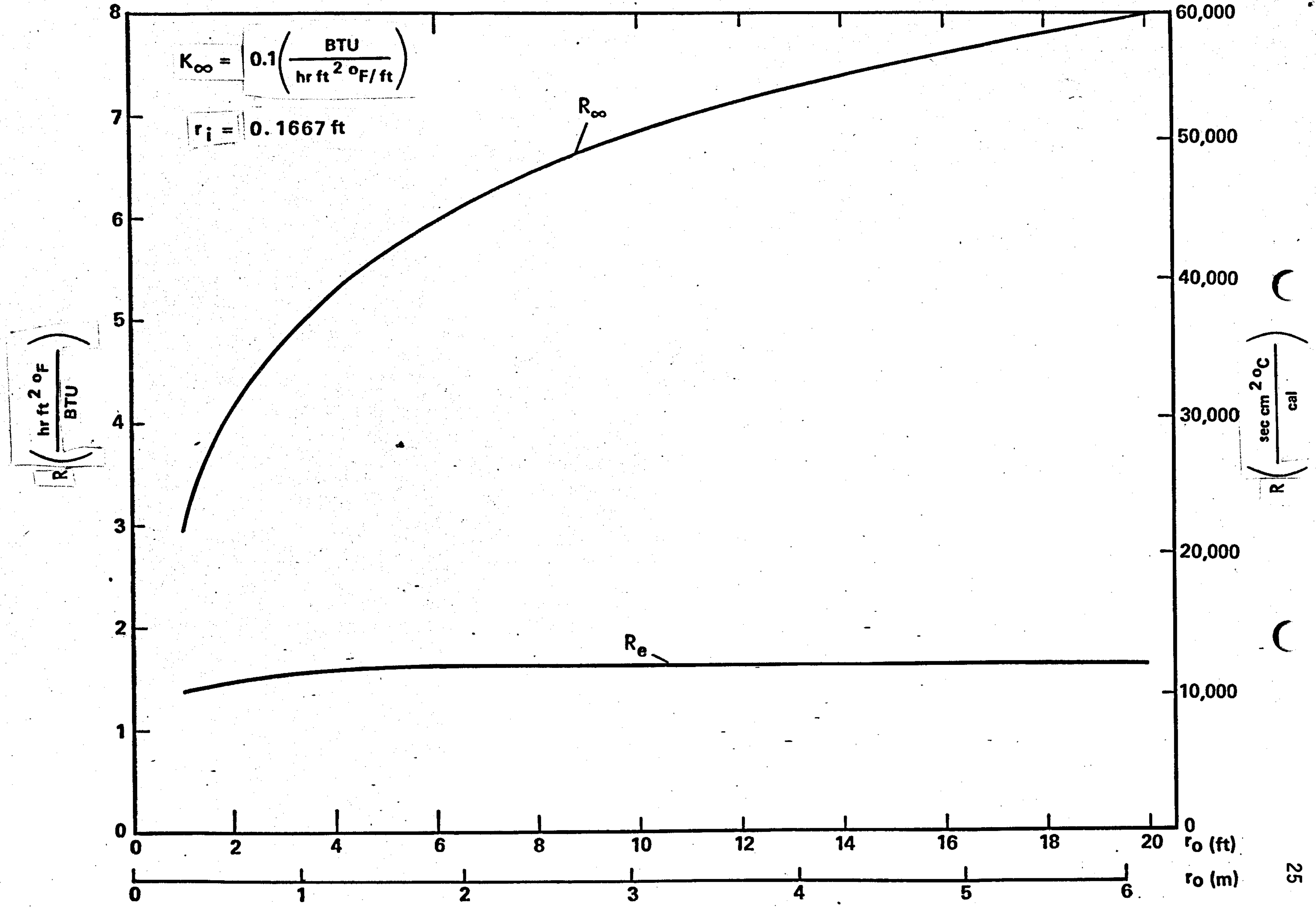

Figure 14. R_∞ and R_e versus r_o