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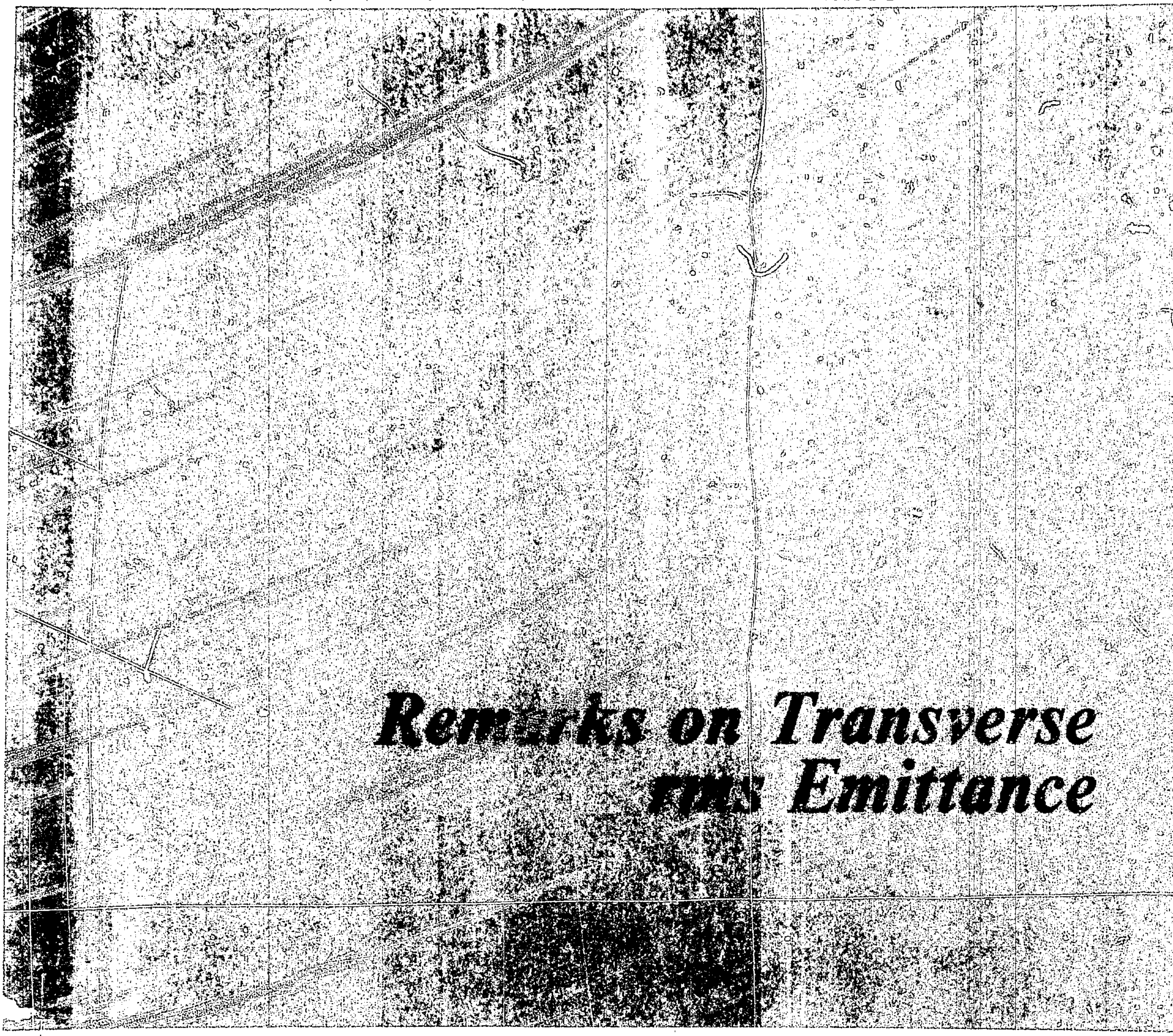

Remarks on Transverse rms Emittance

Los Alamos
Los Alamos National Laboratory
Los Alamos, New Mexico 87545

REMARKS ON TRANSVERSE RMS EMITTANCE

by

Michael Pabst

ABSTRACT

This report presents some considerations on the radial rms emittance of linear accelerator beams. In the first part we show that, in case of acceleration, a constant normalized rms emittance results if we describe each beam particle by the same linear differential equation. The second part shows how the rms emittance varies if the beam particles have a distribution in betatron frequency of approximately the nominal value.

I. CONSERVATION OF RMS EMITTANCE

Let us consider a beam where the particle motion in one direction does not couple to other directions. It then follows from Liouville's theorem that the normalized total emittance for each degree of freedom is a constant of time. In this report we discuss the normalized rms emittance.

For an on-axis beam, let the rms emittance be defined by

$$E_{rms}^2 = \overline{x^2}\;\overline{x'^2} - \overline{xx'}^2 , \tag{1}$$

where the bar denotes the average over the different particles in one bunch.

We assume the same equation of motion for all particles in the beam:

$$x'' + P(s)x' + Q(s)x = 0 . \tag{2}$$

The quantity s is the axial coordinate.

If $P(s) \equiv 0$ we have Hill's equation. In the case of acceleration, we assume $P(s)$ to be given by

$$P(s) = \frac{1}{\beta_s \gamma_s} \frac{d}{ds} (\beta_s \gamma_s) \quad , \tag{3}$$

where β_s and γ_s are the relativistic factors of the synchronous particle.

Every solution of Eq. (2) now can be represented by a linear transformation of the initial values (Appendix A):

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} a(s) & b(s) \\ a'(s) & b'(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \quad . \tag{4}$$

The transformation is the same for all possible solutions. Inserting Eq. (4) into Eq. (1), we get

$$E_{rms}(s) = \Delta(s)\, E_{rms}(0) \quad , \tag{5}$$

where $\Delta(s) = a(s)b'(s) - b(s)a'(s)$ is the determinant of the transformation matrix in Eq. (4). Therefore, the rms emittance change depends only linearly on the determinant $\Delta(s)$.

In case of no acceleration, $P(s) \equiv 0$, the determinant equals 1 for all s (Appendix B). Therefore the rms emittance is constant.

If acceleration is included and $P(s)$ is given by Eq. (3), the determinant is a function of s (Appendix B):

$$\Delta(s) = \frac{\beta_s(0)\gamma_s(0)}{\beta_s(s)\gamma_s(s)} \quad . \tag{6}$$

The rms emittance then is damped by $1/(\beta_s \gamma_s)$. The normalized rms emittance defined by

$$E^n_{rms} = \beta_s \gamma_s\, E_{rms} \tag{7}$$

is therefore constant:

$$E^n_{rms} = \beta_s(0)\gamma_s(0)\ E_{rms}(0) \quad . \tag{8}$$

We should remark that, for having constant normalized rms emittance, a linear equation of motion is needed. Only the total normalized emittance stays constant in nonlinear cases, if the equation of motion could be derived from a Hamiltonian function. Generally, the corresponding normalized rms emittance will not be constant anymore.

Therefore, constant normalized rms emittance is a consequence of a linear differential equation and not of Liouville's theorem. In other words, Liouville's theorem could apply to the normalized rms emittance only in case of linear forces.

II. RMS EMITTANCE GROWTH CAUSED BY PARTICLES WITH DIFFERENT PHASE ADVANCE

A. Introduction

In this section we discuss the influence of particles having different phase advance per cell on the rms emittance. We try to understand the contribution this effect makes on the rms emittance growth seen in multiparticle simulations at the beginning of drift-tube linacs (DTL).*

Our assumptions for the following calculations are

- only one dimension is considered, no coupling to other directions of motion;
- the particle trajectories are described in smooth approximation with linear space charge and constant velocity;
- the averaged phase advance and phase spread do not change; and
- the distribution of particles as a function of the phase advance is uncorrelated to the particles' initial condition.

* R. A. Jameson, Los Alamos National Laboratory AT-Division, furnished this information.

B. Definition

The rms emittance is defined by

$$\varepsilon_{rms}^2 = \overline{x^2} \cdot \overline{x'^2} - \overline{xx'}^2 \quad , \tag{9}$$

where the bar denotes the average over different particles in one bunch. The prime denotes the derivative of x with respect to the cell number. Usually $\overline{x^2}$ is defined by

$$\overline{x^2} = \frac{1}{N} \sum_{i=1}^{N} x_i^2 \quad , \tag{10}$$

where N is the number of particles per bunch. Because we assume now a different phase advance for different particles, we replace Eq. (10) by

$$\widehat{x^2} = \int \overline{x^2(\sigma)} \; f(\sigma) d\sigma \quad , \tag{11}$$

where

$$\overline{x^2(\sigma)} = \frac{1}{n_\sigma} \sum_{i=1}^{n_\sigma} x_i^2(\sigma) \tag{12}$$

is the average over all particles with the same phase advance σ. The quantity n_σ is the number of particles having the phase advance σ, and f(σ) is the particles' distribution as a function of phase advance and f(σ) is normalized to 1.

$$\int f(\sigma) d\sigma = 1 \quad . \tag{13}$$

The integrals' integration range in Eqs. (11) and (13) depends on the properties of f(σ). Because we have replaced Eq. (10) by Eq. (11), we also have to redefine the rms emittance by

$$\varepsilon_{rms}^2 = \widehat{x^2} \cdot \widehat{x'^2} - (\widehat{xx'})^2 \quad . \tag{14}$$

C. Emittance

The single-particle motion in smooth approximation with linear space charge is described by the equation of a simple harmonic oscillator:

$$\frac{d^2x_i}{d\eta^2} + \sigma^2 x_i = 0 \quad . \tag{15}$$

Here $\eta = s/L$, where L is the cell length. For an Alvarez (FODO), we have $L = 2\beta\lambda$ with the wavelength λ and the axial particle velocity β.

We express the solution of Eq. (15) by a linear transformation of the initial conditions x_{i0} and x_{i0}':

$$\begin{pmatrix} x_i \\ x_i' \end{pmatrix} = \begin{pmatrix} \cos\sigma\eta & \frac{\sin\sigma\eta}{\sigma} \\ -\sigma\sin\sigma\eta & \cos\sigma\eta \end{pmatrix} \begin{pmatrix} x_{i0} \\ x_{i0}' \end{pmatrix} \quad . \tag{16}$$

Inserting Eq. (16) into Eq. (14) we could express the rms emittance as a function of the initial beam parameters. Therefore we calculate $\widehat{x^2}$, $\widehat{x'^2}$, and $\widehat{(xx')^2}$ by using Eqs. (11), (12), and (16). For example, we have for $\widehat{x^2}$:

$$\begin{aligned} \widehat{x^2} = & \int \frac{1}{n_\sigma} \sum_{i=1}^{n_\sigma} x_{i0}^2(\sigma) \cos^2\sigma\eta \; f(\sigma)\, d\sigma \\ & + 2\int \frac{1}{n_\sigma} \sum_{i=1}^{n_\sigma} x_{i0}(\sigma) \; x_{i0}'(\sigma) \frac{1}{\sigma} \sin\sigma\eta \cos\sigma\eta \; f(\sigma) d\sigma \\ & + \int \frac{1}{n_\sigma} \sum_{i=1}^{n_\sigma} \left[x_{i0}'(\sigma)\right]^2 \frac{1}{\sigma^2} \sin^2\sigma\eta \; f(\sigma) d\sigma \quad . \end{aligned} \tag{17}$$

Because we assumed no correlation between the initial conditions and $f(\sigma)$, we have for each σ:

$$\frac{1}{n_\sigma}\sum_{i=1}^{n_\sigma} x_{i0}^2(\sigma) = \frac{1}{N}\sum_{i=1}^{N} x_{i0}^2 = \overline{x_0^2} \ ,$$

$$\frac{1}{n_\sigma}\sum_{i=1}^{n_\sigma} x_{i0}(\sigma)x'_{i0}(\sigma) = \frac{1}{N}\sum_{i=1}^{N} x_{i0}x'_{i0} = \overline{x_0x'_0} \ , \qquad (18)$$

$$\frac{1}{n_\sigma}\sum_{i=1}^{n_\sigma} \left[x'_{i0}(\sigma)\right]^2 = \frac{1}{N}\sum_{i=1}^{N} x'^2_{i0} = \overline{x'^2_0} \ .$$

Therefore, we get for $\widehat{x^2}$:

$$\widehat{x^2} = \overline{x_0^2}\, I_1 + 2\,\overline{x_0x'_0}\, I_2 + \overline{x'^2_0}\, I_3 \ . \qquad (19)$$

Similarly, we calculate for $\widehat{x'^2}$ and $\widehat{xx'}$:

$$\widehat{x'^2} = \overline{x_0^2}\, I_4 - 2\,\overline{x_0x'_0}\, I_5 + \overline{x'^2_0}\, I_1 \ , \qquad (20)$$

$$\widehat{xx'} = -\overline{x_0^2}\, I_5 + \overline{x_0x'_0}\,(I_1 - I_6) + \overline{x'^2_0}\, I_2 \ . \qquad (21)$$

The integrals $I_1,\ldots,I_6$ are listed in Table I. For the rms emittance we get

$$E^2_{rms}(n) = A\left(\overline{x_0^2}\right)^2 + B\left(\overline{x'^2_0}\right)^2 + C\,\overline{x_0^2}\cdot\overline{x'^2_0}$$
$$- D\left(\overline{x_0x'_0}\right)^2 + 2E\,\overline{x_0^2}\cdot\overline{x_0x'_0} + 2F\,\overline{x'^2_0}\cdot\overline{x_0x'_0} \ . \qquad (22)$$

TABLE I

LIST OF INTEGRALS

	Values at $\eta = 0$	Values at $\eta = \infty$
$I_1(\eta) = \int \cos^2\sigma\eta\, f(\sigma)d\sigma$	1	$\frac{1}{2}$
$I_2(\eta) = \int \frac{1}{\sigma} \cos\sigma\eta\, \sin\sigma\eta\, f(\sigma)d\sigma$	0	0
$I_3(\eta) = \int \frac{1}{\sigma^2} \sin^2\sigma\eta\, f(\sigma)d\sigma$	0	$\frac{1}{2}\alpha_{-2}$
$I_4(\eta) = \int \sigma^2 \sin^2\sigma\eta\, f(\sigma)d\sigma$	0	$\frac{1}{2}\alpha_2$
$I_5(\eta) = \int \sigma \sin\sigma\eta\, \cos\sigma\eta\, f(\sigma)d\sigma$	0	0
$I_6(\eta) = \int \sin^2\sigma\eta\, f(\sigma)d\sigma$	0	$\frac{1}{2}$
$A = I_1 I_4 - I_5{}^2$	0	$\frac{1}{4}\alpha_2$
$B = I_1 I_3 - I_2{}^2$	0	$\frac{1}{4}\alpha_{-2}$
$C = I_1{}^2 + I_3 I_4 + 2I_2 I_5$	1	$\frac{1}{4}(1+\alpha_2\alpha_{-2})$
$D = 4I_2 I_5 + I_1{}^2 + I_6{}^2 - 2I_1 I_6$	1	0
$E = I_2 I_4 - I_5 I_6$	0	0
$F = I_2 I_6 - I_3 I_5$	0	0

α_n is defined: $\alpha_n = \int \sigma^n f(\sigma)d\sigma$

The factors A,...,F are combinations of the integrals $I_1,\ldots,I_6$. They are listed in Table I. Properties of the integrals are discussed in Appendix C. Before we go into a detailed study of Eq. (22), we show some simple properties.

(a) If $f(\sigma)$ is a δ-function, $f(\sigma) = \delta(\sigma - \sigma_0)$.

In that case, we get

$$E_{rms}(\eta) = E_{rms}(0) \quad . \tag{23}$$

If the particles all have the same tune σ_0 and their motion is described by Eq. (15), the rms emittance is constant. This result agrees with the results discussed in Sec. I.

(b) The behavior of E_{rms} at η equals infinity.

If we assume $f(\sigma)$ to be nonzero and integrable on the interval $a \leq \sigma \leq b$ with $a,b > 0$, we could apply the theorem of Riemann-Lebesgue to the integrals $I_1,\ldots,I_6$ (see also Appendix C).

We then get for the emittance at infinity,

$$E_{rms}(\infty) = \frac{1}{4}\ \alpha_2\left(\overline{x_0^2}\right)^2 + \frac{1}{4}\ \alpha_{-2}\left(\overline{x_0'^2}\right)^2 + \frac{1}{4}\left(1 + \alpha_2 \cdot \alpha_{-2}\right)\ \overline{x_0^2}\ \ \overline{x_0'^2} \quad , \tag{24}$$

where α_n are moments defined by

$$\alpha_n = \int_a^b \sigma^n\ f(\sigma)d\sigma \quad . \tag{25}$$

The result does not depend on terms including x_0x_0'. Therefore $E_{rms}(\infty)$ does not depend on the initial correlation of x_0 and x_0'.

We now study the rms emittance change for $0 \leq \eta < \infty$. The initial particle distribution in x, x' phase space is assumed to be elliptic and symmetric in x and x'. The initial emittance should be described by an upright ellipse ($\overline{x_0 x_0'} = 0$) and should be matched to the averaged value $\bar{\sigma}$ of the phase advance. We then have

$$\overline{x_0^2} = \frac{1}{\bar{\sigma}} E_{rms}(0) \tag{26}$$

and

$$\overline{x_0'^2} = \bar{\sigma}\, E_{rms}(0) \quad . \tag{27}$$

For the distribution $f(\sigma)$ we assume a step function:

$$f(\sigma) = \begin{cases} \frac{1}{b-a} \; , & 0 < a \leqslant \sigma \leqslant b \\ 0 & \text{elsewhere} \end{cases} \tag{28}$$

Mean value σ and variance s^2 then are given by

$$\bar{\sigma} = \int \sigma \; f(\sigma) \; d\sigma = \frac{a+b}{2} \quad , \tag{29a}$$

$$s^2 = \int (\sigma - \bar{\sigma})^2 f(\sigma) \; d\sigma = \frac{(b-a)^2}{12} \quad . \tag{29b}$$

With these assumptions, it is possible to calculate the integrals analytically. The integrals are listed in Appendix C.

If η tends to infinity, we get

$$\frac{E^2_{rms}(\infty)}{E^2_{rms}(0)} = 1 + \frac{1}{4} \left(\frac{s^2}{\bar{\sigma}^2} + 7 \frac{s^2}{\bar{\sigma}^2 - 3s^2} \right) \; , \tag{30}$$

or if $3s^2 \ll \bar{\sigma}^2$,

$$\frac{E_{rms}(\infty)}{E_{rms}(0)} \simeq 1 + \left(\frac{s}{\bar{\sigma}}\right)^2 \quad . \tag{31}$$

For several cases we calculated the rms-emittance growth, using f as given by Eq. (28).

Figure 1 shows the results compared to multiparticle simulations (see footnote, page 3). In Fig. 1b and 1c, comparison is possible only up to cell 10 because the average phase advance $\bar{\sigma}$ strongly changes after cell 10. The oscillatory behavior is reproduced by the analytic calculations. A different distribution $f(\sigma)$ might change only slightly the analytic values. The strong increase of the multiparticle simulations' normalized rms emittance has its source in the equipartitioning. The initial longitudinal rms emittance was chosen a factor 5 larger than the initial transverse rms emittance. The equipartitioning could not be described within our one-dimensional description. Therefore the analytic calculated rms emittance does not increase so strongly.

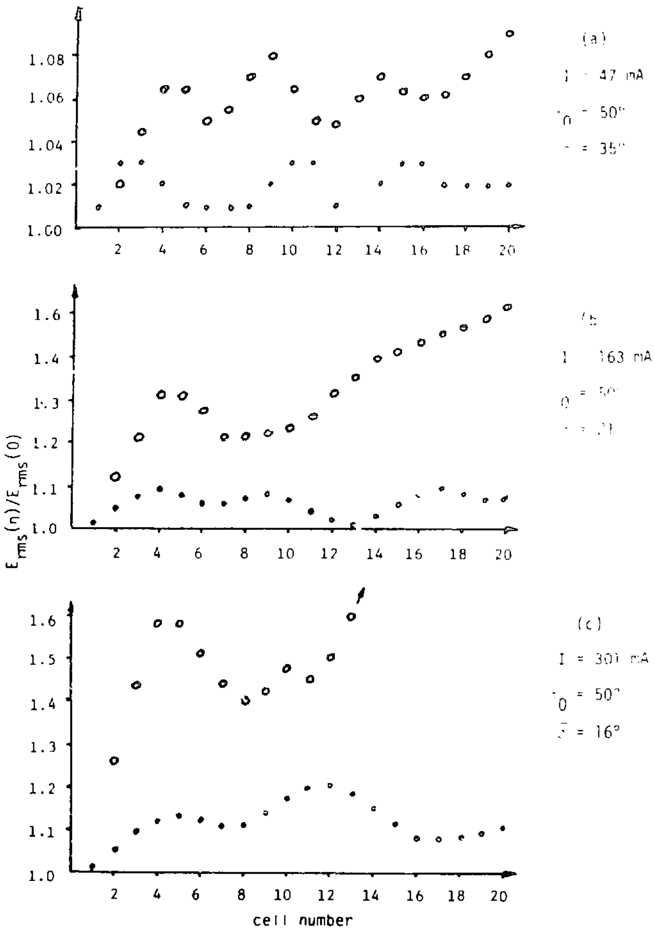

Fig. 1. rms-emittance growth
o:= multiparticle simulations
•:= analytic calculations with $f(\sigma)$ given by Eq. (28).
S = 5° in all cases

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APPENDIX A

SOLUTIONS OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS IN MATRIX FORMULATION

We prove that every solution of a second-order linear differential equation could be represented by

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} a(s) & b(s) \\ a'(s) & b'(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} , \tag{A-1}$$

where x_0 and x_0' are the initial values.

Let us assume that we have two linear independent solutions y_1 and y_2. The initial values are given by y_{10}, y_{10}', y_{20}, y_{20}'.

Define two new solutions a(s), b(s) by

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\Lambda}{\Delta} \begin{pmatrix} y_{20}' & -y_{10}' \\ -y_{20} & y_{10} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} , \tag{A-2}$$

where Δ is given by

$$\Delta = y_{10}y_{20}' - y_{20}y_{10}' \quad . \tag{A-3}$$

The initial values of a and b are then

$$\begin{aligned} a(0) &= 1 \ , \ a'(0) = 0 \ , \\ b(0) &= 0 \ , \ b'(0) = 1 \ . \end{aligned} \tag{A-4}$$

Therefore we can express every solution x by

$$x = x_0 a(s) + x_0' b(s) \quad . \tag{A-5}$$

Eq. (A-1) then follows.

The uniqueness of the transfer matrix, Eq. (A-1), follows for all solutions because the initial conditions, Eq. (A-4), determine unique solutions a(s) and b(s).

APPENDIX B

PROPERTIES OF THE TRANSFER MATRIX

We now discuss the s dependence of the transformation matrix's determinant in Eq. (A-1).

$$\Delta(s) = a(s)b'(s) - b(s)a'(s) \quad .$$

The quantities a,b are solutions of the corresponding second-order differential equation:

$$x'' + P(s)x' + Q(s)x = 0 \quad . \tag{B-2}$$

It therefore follows for the derivative $d\Delta(s)/ds$,

$$\Delta'(s) = -P(s)\Delta(s) \quad . \tag{B-3}$$

Integration gives

$$\Delta(s) = C \cdot e^{-\int P(s)ds} \quad , \tag{B-4}$$

with the initial value following from Eq. (B-1),

$$\Delta(0) = 1 \ . \tag{B-5}$$

If $P(s) \equiv 0$, then $\Delta(s)$ does not depend on s and we have $\Delta(s) \equiv 1$.

For P(s) given by Eq. (3), it follows

$$\Delta(s) = \frac{\beta_s(0)\gamma_s(0)}{\beta_s(s)\gamma_s(s)} \quad . \tag{B-6}$$

APPENDIX C

PROPERTIES AND EVALUATION OF THE INTEGRALS $I_1,\ldots,I_6$

In this appendix we discuss some properties of the integrals and related expressions given in Table I.

Applying addition theorems, one gets

$$I_1 = \frac{1}{2} + \int \cos(2\sigma\eta) f(\sigma) d\sigma \quad ,$$

$$I_2 = \frac{1}{2} \int \frac{1}{\sigma} \sin(2\sigma\eta) f(\sigma) d\sigma \quad ,$$

$$I_3 = \frac{1}{2} \alpha_{-2} - \frac{1}{2} \int \frac{1}{\sigma^2} \cos(2\sigma\eta) f(\sigma) d\sigma \quad ,$$

$$I_4 = \frac{1}{2} \alpha_2 - \frac{1}{2} \int \sigma^2 \cos(2\sigma\eta) f(\sigma) d\sigma \quad ,$$

$$I_5 = \frac{1}{2} \int \sigma \sin(2\sigma\eta) f(\sigma) d\sigma \quad ,$$

$$I_6 = \frac{1}{2} - \frac{1}{2} \int \cos(2\sigma\eta) f(\sigma) d\sigma \quad ,$$

where α_2, α_{-2} are explained in Eq. (25).

Some relations between the integrals:

$$I_6 = 1 - I_1 \quad ,$$

$$I_5 = -\frac{1}{2} \frac{d}{d\eta} I_1 \quad ,$$

$$I_4 = \frac{1}{2} \alpha_2 + \frac{1}{4} \frac{d^2}{d\eta^2} I_1$$

Values at $\eta = 0$ and $\eta = \infty$:

For $\eta = 0$, results are shown in Table I.

For $\eta = \infty$, we apply the theorem of Riemann-Lebesgue.[1]

If $f(\sigma)$ is integrable in the finite interval $[a,b]$, $a,b > 0$, then

$$\lim_{\alpha\to\infty} \int_a^b f(t) \sin \alpha t dt = 0 \quad .$$

Applying this theorem, we get

$$I_1(\infty) = I_2(\infty) = 1/2 \quad ,$$

$$I_2(\infty) = I_5(\infty) = 0 \quad ,$$

$$I_3(\infty) = \alpha_{-2} \quad ,$$

$$I_4(\infty) = \alpha_2 \quad .$$

The resulting expressions for A,...,F are shown in Table I.

Let $f(\sigma)$, $\bar{\sigma}$ and σ^2 be defined by Eqs. (28) and (29a,b). We then calculate for the integrals:

$$I_1 = \frac{1}{2}\left(1 + \frac{\sin(\sqrt{12}\, s\eta)}{\sqrt{12}\, s\, \eta} \cos(2\bar{\sigma}\eta)\right) \quad ,$$

$$I_2 = \frac{1}{2\sqrt{12}\cdot s} \left(\mathrm{Si}(2b\eta) - \mathrm{Si}(2a\eta)\right) \quad ,$$

where the lower and upper bound a,b are given by

$$a = \bar{\sigma} - \frac{\sqrt{12}}{2} s \quad ,$$

$$b = \bar{\sigma} + \frac{\sqrt{12}}{2} s \quad .$$

The quantity Si(x) is the sine-integral:

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt \quad .$$

$$I_3 = \frac{1}{2}\alpha_{-2}$$

$$- \frac{1}{2\sqrt{12}\cdot s}\left(\frac{1}{a}\cos(2a\eta) - \frac{1}{b}\cos(2b\eta)\right)$$

$$+ \frac{\eta}{\sqrt{12}\cdot s}\Big(Si(2b\eta) - Si(2a\eta)\Big) \quad ,$$

where α_{-2} is given by

$$\alpha_{-2} = \frac{1}{\bar{\sigma}^2 - 3s^2} \quad .$$

$$I_4 = \frac{1}{2}\alpha_2 + \frac{1}{4\eta^3}\cdot\frac{\sin(\sqrt{12}\, s\eta)}{\sqrt{12}\cdot s}\cos(2\bar{\sigma}\eta)$$

$$+ \frac{1}{4\eta^2}\left(-\cos(\sqrt{12}\, s\eta)\cos(2\bar{\sigma}\eta) + \frac{2\bar{\sigma}}{\sqrt{12}\, s}\sin(\sqrt{12}\, s\eta)\cdot\sin(2\bar{\sigma}\eta)\right)$$

$$- \frac{1}{4\eta}\left\{\frac{1}{2}\left(\sqrt{12}\, s + \frac{4\bar{\sigma}^2}{\sqrt{12}\, s}\right)\sin(\sqrt{12}\, s\eta)\cos(2\bar{\sigma}\eta)\right.$$

$$\left. + 2\bar{\sigma}\cos(\sqrt{12}\, s\eta)\sin(2\bar{\sigma}\eta)\right\} \quad ,$$

where α_2 is given by:

$$\alpha_2 = \bar{\sigma}^2 + s^2$$

$$I_5 = \frac{1}{4\eta^2} \cdot \frac{\sin(\sqrt{12}\, s\eta)}{\sqrt{12}\, s} \cos(2\bar{\sigma}\eta)$$

$$+ \frac{1}{4\eta} \left\{ - \cos(\sqrt{12}\, s\eta) \cos(2\bar{\sigma}\eta) + \frac{2\bar{\sigma}}{\sqrt{12}\, s} \sin(\sqrt{12}\, s\eta) \sin(2\bar{\sigma}\eta) \right\}$$

$$I_6 = \frac{1}{2} \left\{ 1 - \frac{\sin(\sqrt{12}\, s\eta)}{\sqrt{12}\, s\eta} \cos(2\bar{\sigma}\eta) \right\} .$$

REFERENCE

1. K. S. Miller, Engineering Mathematics (Dover Publications, New York, 1963.)