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MINIMAL RELATIVISTIC THREE-PARTICLE EQUATIONS

James Lindesay

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James Lindesay

Stanford Linear Accelerator Center
Stanford University
Stanford, California 94305

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ABSTRACT

A minimal self-consistent set of covariant and unitary three-
particle equations is presented. Numerical results are obtained for
three-particle bound states, elastic scattering and rearrangement of
bound pairs with a third particle, and amplitudes for breakup into
states of three free particles.

The mathematical form of the three-particle bound state equations
is explored; constraints are set upon the range of eigenvalues and
number of eigenstates of these one parameter equations. The behavior
of the number of eigenstates as the two-body binding energy decreases
to zero in a covariant context generalizes results previously obtained
non-relativistically by V. Efimov.

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CHAPTER I

INTRODUCTION

In the study of three-particle scattering theory, many subtleties of quantum mechanics require careful attention for calculations. Through the understanding of the three-particle scattering problem, one can gain insight into the foundations of quantum mechanics as a predictive science. The three-particle problem probes the properties of two-particle systems, as well as serves as a guide to the understanding of N-particle problems. A description of certain physical systems often described as three-particle bound states (for instance barvons, the triton, etc.) could extend theoretical insight into some of the fundamental symmetries of nature.

The formulation of a consistent three-body quantum mechanics was achieved in a non-relativistic context by Faddeev[1] with considerable mathematical rigor. Faddeev defined a specific channel decomposition of the system through which physical observables could be extracted, within a Hamiltonian scattering theory. Thus, given the Faddeev equations, with specific two-body scattering input, one in principle obtains self consistent three-body scattering amplitudes.

As numerical and analytic techniques developed to study the three-particle problem, the differences between two and three particle dynamics became more apparent. Through an analysis of the configuration space equations for the wave function, H. P. Noyes[2] noted that long range effects occur in the three-body system even if all pairwise interactions are short range. This effect, called by Noyes "the eternal

triangle," changes the interaction between a given pair, if a third (interacting) particle is brought into the system anywhere, regardless of the range of the forces involved. By examining the scaling behavior of the Hamiltonian eigenstates in the case of resonantly interacting particles, Efimov[3,4] determined that the actual number N of three body bound state solutions can become large as the magnitude of the scattering length $|a|$ for a pair becomes large relative to the scale of forces r_0:

$$N \approx \frac{1}{\pi} \log \left(|a|/r_0\right) \qquad (1.1)$$

for $E_N \ll 1/r_0^2$, low three-body binding energy. This effect would result in a logarithmic growth in the number of three-body bound states as the two-body pairwise binding energy decreases to zero. Efimov[5] subsequently demonstrated that effectively there is a long range $(1/R^2)$ potential which is responsible for the effect. The range of the bound states is large compared to r_0. These results are consistent with the eternal triangle effect discussed by Noyes, within the context of the kinematics. These effects appear in the low energy limits, and thus should be consistently reproduced in a covariant formalism. A rigorous treatment of this effect will be presented within the model to be examined in Section III-A.

The development of a relativistically covariant formalism requires that additional sets of constraints be satisfied, as well as introducing the complicated analytic structure of relativistic kinematics into the theory. An important question is the choice of the covariant propagator which reduces correctly to the non-relativistic

situation. Often calculations have been done using the Blankenbecler-Sugar prescription.[6] Brayshaw[7] used a propagator which involved a linear difference in the four-momentum variables, such that the form of the equations satisfy certain clustering properties. A propagator similar to that of Brayshaw is used in the development of Section II-A.

What will be presented is a general formulation of a consistent relativistic quantum mechanics, along with an investigation into the properties of a particular model involving separable, zero-range two-body interactions. The formulation of the relativistic problem in terms similar to those presented by Faddeev will be the topic of Sections II-A and II-B. Section II-C will present the particular development of the model being explored. The numerical and analytical results of the model will be examined in Section III.

CHAPTER II

FORMULATION OF BASIC EQUATIONS

A. General Development of Covariant Three-Particle Equations

The system to be considered will consist of three distinguishable particles specified by momentum variables and a mass shell condition. The particles will be labeled by Latin indices a,b,c,..., which will take on values from 1 to 3. Four-vectors will be denoted with arrows and three-vectors will be underscripted

$$\vec{k}_a = (\epsilon_a, \underline{k}_a)$$

$$\vec{k}_a \cdot \vec{k}_a = m_a^2 \ , \quad \text{where} \quad \vec{A} \cdot \vec{B} = A^0 B^0 - \underline{A} \cdot \underline{B} \tag{2.1}$$

1. Covariant States

The non-interacting eigenstates, which can be represented as a direct product of single particle states, will be denoted $k_1\epsilon_1; k_2\epsilon_2; k_3\epsilon_3\rangle$ and will satisfy the following conditions:

$$\vec{P}_{NI} | k_1\epsilon_1; k_2\epsilon_2; k_3\epsilon_3 \rangle = \left(\sum_{a=1}^{3} \vec{k}_a\right) | k_1\epsilon_1; k_2\epsilon_2; k_3\epsilon_3 \rangle$$

$$\langle k_1\epsilon_1; k_2\epsilon_2; k_3\epsilon_3 | k_{10}\epsilon_{10}; k_{20}\epsilon_{20}; k_{30}\epsilon_{30} \rangle = \prod_{a=1}^{3} \epsilon_a (\underline{k}_a, m_a) \delta^3(\underline{k}_a - \underline{k}_{a0}) \tag{2.2}$$

$$1 = \int \frac{d^3k_1 d^3k_2 d^3k_3}{\epsilon_1 \epsilon_2 \epsilon_3} | k_1\epsilon_1; k_2\epsilon_2; k_3\epsilon_3 \rangle \langle k_1\epsilon_1; k_2\epsilon_2; k_3\epsilon_3 |$$

where $\epsilon_a(\underline{k}_a, m_a) \equiv \sqrt{\underline{k}_a^2 + m_a^2} = \epsilon_a$ is the mass shell condition.

The physical problem will be examined in terms of boundary states which satisfy the asymptotic conditions. Without loss of generality in the formalism, it will be assumed that only one bound state can exist for each of the pairs. The results will be easily generalizable to include the entire discrete finite spectrum of each of the subsystems in the case this condition does not hold. The possible asymptotic situations consist of bound pairs with a third non-interacting particle, or three non-interacting particles. These boundary states will be represented as follows:

$$| \Phi_a: k_a\epsilon_a; \psi_a(\underline{k}_a^P, \epsilon_{\nu_a}) \rangle$$

$$| \Phi_0: \langle k_1\epsilon_1; k_2\epsilon_2; k_3\epsilon_3 \rangle \vec{P}_{(0)} \rangle$$
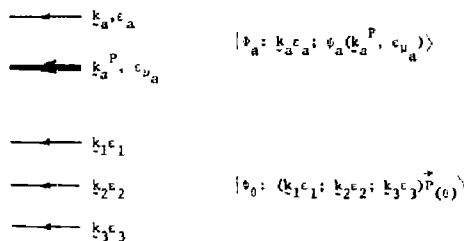
where ψ_a represents the bound state of the two particles other than a; these particles will be labeled a- and a+. The energy of the bound state is $\epsilon_a = \sqrt{\underline{k}_a^{P\,2} + \underline{\epsilon}_a^2}$, where $\underline{k}_a^P$ is the three-momentum of the pair state. The total four-momentum in the state $| \Phi_0 \rangle$ is represented by $\vec{P}_{(0)}$. The boundary states satisfy the following conditions:

$$(\underline{k}_1 \epsilon_1; \underline{k}_2 \epsilon_2; \underline{k}_3 \epsilon_3 | \Phi_0 ; (\underline{k}_{10} \epsilon_{10}; \underline{k}_{20} \epsilon_{20}; \underline{k}_{30} \epsilon_{30}), \vec{P}_{(0)}) = \prod_{a=1}^{3} \epsilon_a \delta^3 (\underline{k}_a - \underline{k}_{a0})$$

$$(\underline{k}_1 \epsilon_1; \underline{k}_2 \epsilon_2; \underline{k}_3 \epsilon_3 | \Phi_a ; \underline{k}_{a0} \epsilon_{a0}; \psi_a (\underline{k}_{a0}^P, \epsilon_{\mu_a0})) = \tag{2.3}$$

$$\epsilon_{a0} \delta^3 (\underline{k}_a - \underline{k}_{a0}) \epsilon_{\mu_a0} \delta^3 (\underline{k}_{a+} + \underline{k}_{a-} - \underline{k}_{a0}^P) \psi_a (\underline{k}_{a+}, \underline{k}_{a-}; \nu_a)$$

where $\psi_a (\underline{k}_{a+}, \underline{k}_{a-}; \nu_a)$ is the pair bound-state covariant wave function.

The fully interacting scattering states will be representable in terms of these boundary states. The fully interacting states will be eigenstates of the total four-momentum operator:

$$\vec{P}_{FI} | \psi_0^{(\pm)}; (\underline{k}_{10} \epsilon_{10}; \underline{k}_{20} \epsilon_{20}; \underline{k}_{30} \epsilon_{30}), \vec{P}_{(0)}$$
$$= \vec{P}_{(0)} | \psi_0^{(\pm)}; (\underline{k}_{10} \epsilon_{10}; \underline{k}_{20} \epsilon_{20}; \underline{k}_{30} \epsilon_{30}), \vec{P}_{(0)})$$

$$\vec{P}_{FI} | \psi_a^{(\pm)}; \underline{k}_{a0} \epsilon_{a0}; \psi_a (\underline{k}_{a0}^P \epsilon_{\nu_a0}) = \vec{P}_{a0} | \psi_a^{(\pm)}; \underline{k}_{a0} \epsilon_{a0}; \psi_a (\underline{k}_{a0}^P, \epsilon_{\nu_a0}) \tag{2.4}$$

where $\vec{P}_{(0)} = \sum_c \vec{k}_{a0}$ and $\vec{P}_{a0} = (\epsilon_{ac} + \epsilon_{\nu_a0}, \underline{k}_{a0} + \underline{k}_{a0}^P)$.

The difference between the interacting four-momentum and the non-interacting four-momentum will define a quantity which determines the nature of the interactions

$$\vec{P}' = \vec{P}_{FI} - \vec{P}_{NI} = (H', \underline{P}') + (H', 0)$$

for eigenstates of three momentum. In the three-particle center of momentum system (3-CMS), the states take on the following form.

$$P_{FI}^0 | \psi_0^{(\pm)}; (\underline{k}_1 \epsilon_1; \underline{k}_2 \epsilon_2; \underline{k}_3 \epsilon_3), (M, \underline{0}) = H | \psi_0^{(\pm)}; (\underline{k}_1 \epsilon_1; \underline{k}_2 \epsilon_2; \underline{k}_3 \epsilon_3), (M, 0))$$
$$\tag{2.5}$$

$$P_{FI}^0 | \psi_a^{(\pm)}; \underline{k}_a \epsilon_a; \psi_a (\underline{k}_a^P \epsilon_{\nu_a}) = (c_a + c_{\nu_a}) | \psi_a^{(\pm)}; \underline{k}_a \epsilon_a; \psi_a (\underline{k}_a^P, \epsilon_{\nu_a}))$$

In the case of only pairwise interactions the term H' can be decomposed into pairwise functions as follows:

$$H' = \sum_{a=1}^{3} H_a' = P_{FI}^0 - P_{NI}^0$$

where

$$(P_{NI}^0 + H_a') | \Phi_a; \underline{k}_a \epsilon_a; \psi_a (-\underline{k}_a \epsilon_{\nu_a})) = (\epsilon_a + \epsilon_a) | \Phi_a; \underline{k}_a \epsilon_a; \psi_a (-\underline{k}_a, \epsilon_{\nu_a}))$$
$$\tag{2.6}$$

2. Properties of Resolvants

To study the relationships between the eigenstates, use will be made of the resolvants of the four-momentum operators defined as follows:

$$G_{NI}(\hat{Q}) = [\hat{Q} \cdot (\vec{P}_{NI} - \hat{Q})]^{-1} Q^{0.3} (\underline{P}_{NI} - \underline{Q})$$

$$G_a(\hat{Q}) = [\hat{Q} \cdot (\vec{P}_a - \hat{Q})]^{-1} Q^{0.3} (\underline{P}_a - \underline{Q}) \tag{2.7}$$

$$G_{FI}(\hat{Q}) = [\hat{Q} \cdot (\vec{P}_{FI} - \hat{Q})] Q^{0.3} (\underline{P}_{FI} - \underline{Q})$$

where $\vec{P}_a = \vec{P}_{NI} + \vec{P}_a'$

which will be well defined equations for $\text{Im } Q^0 \neq 0$. In the three-particle CMS, these can be expressed:

$$G_{NI}(Z, 0) = [P_{NI}^0 - Z]^{-1,3} (\underline{P}_{NI}) = R_?(Z)^{-3} (\underline{P}_{NI})$$

$$G_a(Z, 0) = [P_{NI}^0 + H_a' - Z]^{-1,3} (\underline{P}_A) = R_A(Z)^{-3} (\underline{P}_A) \tag{2.8}$$

$$G_{FI}(Z, 0) = [P_{NI}^0 - Z]^{-1,3} (\underline{P}_{FI}) = R_F(Z)^{-3} (\underline{P}_{FI})$$

As previously mentioned, since all systems of covariant states are three-momentum eigenstates, the following holds

$$P_{\chi'} = P_a = P_{FI} \qquad P$$

Using these relations, the fully interacting eigenstates which satisfy the asymptotic boundary conditions can be expressed formally as follows:

$$\psi^{(+)}(M,0) = \lim \psi : M \cdot i \cdot)$$

(2.9a)

$$\psi_a^{(+)}(\epsilon_a + \epsilon_{\nu a},0) = \lim \psi_a : \epsilon_a + \epsilon_{\nu a} \cdot i \cdot$$

where

$$\psi_i : Z_i = |\phi_i : M - R_i(Z)H'_i \psi_i : Z$$

$$\psi_a : Z = |\phi_a : \epsilon_a + \epsilon_{\nu a} - R_a(Z)\left[\sum_b \bar\tau_{ab} H'_b\right] \psi_a : Z \cdot$$

(2.9b)

$$\bar\tau_{ab} \quad 1 - \delta_{ab} \quad , \qquad \phi_a : M \qquad \psi(M,0)/$$

and momentum arguments in the states have been suppressed. These equations can be reexpressed in the form

$$|\psi_0^{(+)} : (k_1 c_1; k_2 c_2; k_3 c_3),(M,0)$$

$$= \lim_{\eta \to 0}(\cdot i\eta)R_F(M \cdot i\eta)_i i_0 : (k_1 c_1; k_2 c_2; k_3 c_3),(M,0)$$

$$: U_0^{(+)}(M,0)|\phi_0 : (k_1 c_1; k_2 c_2; k_3 c_3),(M,0)$$

$$|\psi_a^{(+)} : (k_a; a; \nu_a(-k_a \nu_a)) = \lim_{\eta \to 0}(\cdot i\eta)R_F(\epsilon_a + \epsilon_{\nu a} \cdot i\eta) \tau_a : k_a; a; \nu_a(-k_a \nu_a)/$$

$$: U_a^{(+)}(\epsilon_a + \epsilon_{\nu a},0)|\phi_a : k_a; a; \nu_a(-k_a \nu_a)$$

(2.10)

It is advantageous to examine some properties of the resolvants, since these will have direct relevance to the properties of the system. Suppose that only ϵ pair interacts, and the third particle a_0 acts only as a "spectator." One's physical intuition would suspect that this particular three-body system should behave just as though the spectator were not present, and that in the mathematical expressions describing the process the parameters involving the spectator should not alter the two-particle observables. The behavior of any of the resolvants is

$$G(\vec{Q}) = |\dot{Q} \cdot (\vec{P}_{a_2} + \vec{k}_{a_0} - \dot{Q})|^{-1} Q^0 \delta^3(P_{a_0} + k_{a_0} - Q)$$

(2.11)

where $\dot{P}_a$ is the four-momentum operator of the (a +, a_0-) pair subsystem. In the case being described (which will be called "clustering") the momentum k_B will remain unchanged. The center of momentum system for the pair (Z_a -CMS) will be described by the four-vector $\dot{Q}_{2a}$-CMS with components

$$\dot{Q}_{2-CMS} = (Z_a + \epsilon_a , k_{a_0})$$

(2.12)

and thus the resolvants can be denoted

$$G(\vec{Q}_{2-CMS}) : G^{(2_a)}_{a_0}(Z_a) = (P^0_{a_0} - Z_{a_0})^{-1} \delta^3(P_{a})$$

(2.13)

where $\dot{P}_a$ operates only in the two-particle space. Examining this expression it appears that the form of this resolvant is identical to what would appear in a purely two-particle space. Since the scattering

eigenstates of the full system depend through the resolvant upon the boundary states, then if the boundary states satisfy these cluster conditions so will the full scattering states.

The operators $\vec{P}_{F1}$ and $\vec{P}_{N1}$ are self adjoint if they are to be physical observables. This property reflects itself in the following properties of the resolvants (Hilbert's identity):

$$R(Z_1) - R(Z_2) = (Z_1 - Z_2) R(Z_1) R(Z_2)$$

$$G(Z_1, \underline{P}_{(0)}) - G(Z_2, \underline{P}_{(0)}) = (Z_1 - Z_2) R(Z_1) R(Z_2) \delta^3(\underline{P} - \underline{P}_{(0)})$$

(2.14)

Other properties also follow directly from the definition:

$$R^+(Z) = R(Z^*) \quad \text{star denotes complex conjugate}$$

$$R_F(Z) = R_0(Z) - R_0(Z) H' R_F(Z) = R_0(Z) - R_F(Z) H' R_0(Z)$$

$$R_F(Z) = R_a(Z) - R(Z)\left[\sum_b \vec{\delta}_{ab} H'_b\right] R_a(Z)$$

$$R_a(Z) = R_0(Z) - R_0(Z) H'_a R_a(Z)$$

(2.15)

These relations will be useful in determining amplitudes for the various physical processes.

3. Amplitudes and Channel Decomposition

The covariant probability amplitude for scattering from asymptotic initial to final states is given by

$$\langle \psi_\alpha^{(+)}(\vec{P}_{sf}) | \psi_\beta^{(-)}(\vec{P}_{si}) \rangle =$$

$$\langle \phi_\alpha(\vec{P}_{sf}) | U_\alpha^{(+)\dagger}(\vec{P}_{sf}) U_\beta^{(-)}(\vec{P}_{si}) | \phi_\beta(\vec{P}_{si}) \rangle$$

(2.16)

where Greek indices α, β, ... will take on values from , and momentum arguments of the states have been suppressed. Here, $\vec{P}_\alpha$ refers to the four-momentum of the particular system examined. The scattering operator will be related to the transition operator in the following way

$$S(\vec{P}_{(0)}) = 1 + 2\pi i \xi^4 (\vec{P}_s - \vec{P}_{(0)}) \Lambda (\vec{P}_{(0)})$$

(2.17)

The components of the scattering operator between boundary states can be obtained from Eq. (2.16) as

$$S_{\alpha\beta}(\vec{Q}) = U_\alpha^{(+)\dagger}(\vec{Q}) U_\beta^{(-)}(\vec{Q})$$

(2.18)

From Eq. (2.15), the following identities can be shown

$$\left[1 - R_F(Z)\sum_b \vec{\delta}_{ab} H'_b\right]\left[1 + R_a(Z)\sum_b \vec{\delta}_{ab} H'_b\right] = 1$$

$$\left[1 - R_F(Z) H'\right]\left[1 + R_0(Z) H'\right] = 1$$

(2.19)

Using these identities with Eqs. (2.9b) it follows that

$$|\psi_a : Z_1\rangle - |\psi_a : Z_2\rangle = -[R_F(Z_1) - R_F(Z_2)]\sum_b \vec{\delta}_{ab} H'_b|\phi_a : \epsilon_a + \epsilon_{b_a}\rangle$$

$$|\psi : Z_1\rangle - |\psi : Z_2\rangle = -[R_F(Z_1) - R_F(Z_2)] H'|\phi : H\rangle$$

(2.20)

The overlap amplitude can now be expressed:

$$\psi_a : Z_1|\psi_\beta : Z_2\rangle = \langle \psi_a : Z_1|\psi_\beta : Z_1\rangle +$$

$$\psi_c : Z_1|[R_F(Z_1) - R_F(Z_2)][\vec{P}_{F1}^0 - \vec{P}']|\psi_\beta : \vec{P}_{(0)}\rangle$$

(2.21)

where the operator $\vec{P}_0 = \vec{P}_{N1}$. The physical overlap amplitudes are:

$$\langle \Psi_\alpha^{(+)} : P_s^0 ; \Psi_\beta^{(-)} : P_{(0)}^0 = \langle \phi_1^{(+)} : P_s^0 | \Psi_\beta^{(+)} : P_{(0)}^0 +$$

$$2\pi i \delta(P_s^0 - P_{(0)}^0)\langle \Psi_\alpha^{(+)} : P_s^0 | [\, \mathbb{P}_{FI}^0 - \mathbb{P}_\beta^0]|\phi_\beta : P_{(0)}^0 \tag{2.22}$$

Thus, the covariant probability amplitude into boundary states can be written as follows:

$$\langle \Psi_\alpha^{(+)}(\vec{P}_s)|\Psi_\beta^{(-)}(\vec{P}_{(0)})\rangle = \langle \Psi_\alpha^{(+)}(\vec{P}_s)|\Psi_\beta^{(+)}(\vec{P}_{(0)})\rangle +$$

$$2\pi i \delta^4(\vec{P}_s - \vec{P}_{(0)}) \langle \phi_\alpha(\vec{P}_s)|A_{\alpha\beta}^{(+)}(\vec{P}_s)|\phi_\beta(\vec{P}_{(0)})\rangle \tag{2.23}$$

where the amplitude $\langle \phi_\alpha|A_{\alpha\beta}|\phi_\beta\rangle$ represents the transition amplitude, and is expressed:

$$\delta^3(P_s)\langle \phi_\alpha(\vec{P}_s)|A_{\alpha\beta}^{(+)}(\vec{P}_s)| : \Psi_\beta : P_{(0)}^0\rangle = \langle \Psi_\alpha^{(+)}(\vec{P}_s)|[\,\mathbb{P}_{FI}^0 - \mathbb{P}_\beta^0]|\phi_\beta : P_{(0)}^0$$

or

$$P_{(0)}^0 \delta^3(P_s - P_{(0)}) \langle \phi_\alpha(\vec{P}_s)|A_{\alpha\beta}^{(+)}(\vec{P}_s)|\phi_\beta(\vec{P}_{(0)})\rangle = \tag{2.24}$$

$$\langle \Psi_\alpha^{(+)}(\vec{P}_s)|\vec{P}_{(0)} \cdot \, [\,\vec{P}_{FI} - \vec{P}_\beta]|\phi_\beta(\vec{P}_{(0)})\rangle$$

The states $|\Psi_\alpha^{(+)}(\vec{P}_s)\rangle$ will shortly be shown to be covariantly ortho-normal, and thus the amplitudes involving $A_{\alpha\beta}$ represent the physical transition amplitudes.

To develop equations for the amplitudes, consider the Lippman-Schwinger equation for the operator T in the 3-CMS:

$$\delta^3(P - Q)T(\vec{Q}) = H' - H'R_0(Q^0)\delta^3(P - Q)T(\vec{Q}) \tag{2.25}$$

$$Q = Q$$

The channel decomposition proposed by Faddeev[1] for a system with only pairwise interactions,

$$H' = \sum_{a=1}^3 H_a' \tag{2.26}$$

involves the definition of T as:

$$T(Q^0, 0) = \sum_{a,b=1}^3 T_{ab}(Q^0, 0) \tag{2.27}$$

These components satisfy the sets of equations

$$\delta^3(P - Q) T_{ab}(\vec{Q}) = \delta_{ab} H_a' - H_a' R_F(Q^0) H_b' \tag{2.28}$$

which can be reexpressed using Eq. (2.15) as:

$$\delta^3(P - Q) T_{ab}(\vec{Q}) = \delta_{ab} H_a' - H_a' R_0(Q^0) \sum_c \delta^3(P - Q) T_{cb}(\vec{Q}) \tag{2.29}$$

Operators t_a can be defined for the subsystems which satisfy

$$\delta^3(P - Q) t_a(\vec{Q}) = H_a' - H_a' R_a(Q^0) H_a'$$

$$= H_a' - H_a' R_0(Q^0)\delta^3(P - Q) t_a(\vec{Q}) \tag{2.30}$$

These equations can be rewritten

$$[1 + H_a' R_0(Q^0)]\delta^3(P - Q) T_{ab}(\vec{Q}) = H_a'\Big[\delta_{ab} - R_0(Q^0) \sum_c \bar{\delta}_{ac}\delta^3(P - Q) T_{cb}(\vec{Q})\Big]$$

$$[1 + H_a' R_0(Q^0)]\delta^3(P - Q) t_a(\vec{Q}) = H_a' \tag{2.31}$$

and the equations for T_{ab} can be cast into a form dependent only on the quantity t_a

$$T_{ab}(\vec{Q}) = \delta_{ab} t_a(\vec{Q}) - t_a(\vec{Q}) G_{NI}(\vec{Q}) \sum_c \bar{\delta}_{ac} T_{cb}(\vec{Q})$$

$$T(\vec{Q}) = \sum_{a,b} T_{ab}(\vec{Q}) \tag{2.32}$$

Using Eqs. (2.25) and (2.15), the fully interacting resolvent can be expressed in terms of the operator T

$$R_F(Q^o) = R_0(Q^o) - R_0(Q^o) \delta^3(\vec{P} - \vec{Q}) T(\vec{Q}) R_0(Q^o)$$

or $\qquad G_{FI}(\vec{Q}) = G_{NI}(\vec{Q}) - G_{NI}(\vec{Q}) T(\vec{Q}) G_{NI}(\vec{Q})$ $\qquad$ (2.33a)

Similarly, the "channel" resolvent can be expressed in terms of t_a

$$R_a(Q^o) = R_0(Q^o) - R_0(Q^o) \delta^3(\vec{P} - \vec{Q}) t_a(\vec{Q}) R_0(Q^o)$$

or $\qquad G_a(\vec{Q}) = G_{NI}(\vec{Q}) - G_{NI}(\vec{Q}) t_a(\vec{Q}) {}_{NI}(\vec{Q})$ $\qquad$ (2.33b)

The relations (2.14) imply conditions on the operators T_{ab} through Eq. (2.29). Using the easily verifiable relations (for $Q = Q$)

$$R_0(Q^o) \sum_c \delta^3(\vec{P} - \vec{Q}) T_{cb}(\vec{Q}) = R_F(Q^o) H_b'$$

$$\sum_c T_{ac}(\vec{Q}) \delta^3(\vec{P} - \vec{Q}) R_0(Q^o) = H_a' R_F(Q^o) \tag{2.34}$$

the form of the Hilbert identity in relation to the operators T_{ab} becomes

$$\delta^3(\vec{P}) T_{ab}(Z,Q) - \delta^3(\vec{P}) T_{ab}(Z_2,Q) \tag{2.35}$$

$$= (Z_2 - Z_1) \sum_c T_{ac}(Z_1,Q) \delta^3(\vec{P}) R_0(Z_1) R(Z_2) \delta^3(\vec{P}) \sum_c T_{cb}(Z_2,Q)$$

Likewise

$$\delta^3(\vec{P}) t_a(Z_1,Q) - \delta^3(\vec{P}) t_a(Z_2,Q) \tag{2.36}$$

$$= (Z_2 - Z_1) t_a(Z_1,Q) \delta^3(\vec{P}) R_0(Z_1) R_0(Z_2) \delta^3(\vec{P}) t_a(Z_2,Q)$$

B. Physical Observables

1. Interacting Eigenstates

Once the singularity structure of the operators is determined, the relationship of the operators with the physical observables of the system can be extracted. Examining Eq. (2.32) the expressions for T_{ab} can be diagrammatically represented:

$$T_{ab} = \delta_{ab} t_a - \sum_c t_a \bar{\delta}_{ac} G_{NI} T_{cb}$$
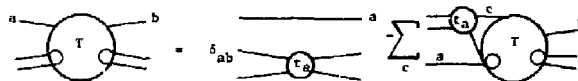
The structure of the fully (three-particle) connected piece will be examined.

$$W_{ab}(\vec{Q}) = T_{ab}(\vec{Q}) - \delta_{ab} t_a(\vec{Q}) \tag{2.37}$$

A formal relationship between operators and observables can be established using Eq. (2.10) expressed in the form

$$|\psi_0^{(\pm)}: M) = \lim_{\eta \to 0} R_F(M \pm i\eta) \left[\rho_{NI}^o - M \mp i\eta \right] |\phi_0: M) \tag{2.38}$$

$$|\psi^{(\pm)}\rangle: \epsilon_c + \epsilon_{\mu c} = \lim_{\eta \to 0} R_F(\epsilon_c + \epsilon_{\mu c} \pm i\eta) \left[P_c^0 - \epsilon_c - \epsilon_{\mu c} \pm i\eta \right] |\phi_c\rangle: \epsilon_c + \epsilon_{\mu c}\rangle \tag{2.39}$$

The resolvant can be expressed, using Eq. (2.33a):

$$R_F(Z) = R_c(Z) - R_c(Z) \sum_{ab} \delta^3(P) T_{ab}(Z,Q) R_c(Z) \tag{2.40}$$

Using these expressions, the fully interacting eigenstate which asymptotically represents three non-interacting particles can directly be represented

$$|\psi_0^{(\pm)}(k_1\epsilon_1; k_2\epsilon_2; k_3\epsilon_3),(M,Q)\rangle = \left[1 - \lim_{\eta \to 0} R_0(M \pm i\eta) \right. \tag{2.41}$$
$$\left. \cdot \sum_{ab} \delta^3(P) T_{ab}(M \pm i\eta,Q) \right] |\phi_v: (k_1\epsilon_1; k_2\epsilon_2; k_3\epsilon_3),(M,Q)\rangle$$

To determine formal expressions for fully interacting eigenstates which asymptotically represent a bound pair with a third particle, use will be made of the full resolvant expressed in terms of the "channel" resolvants

$$R_F(Z) = R_0(Z) + \sum_a \left[R_a(Z) - R_0(Z) \right] - R_0(Z) \sum_{ab} \delta^3(P) W_{ab}(Z,Q) R_0(Z) \tag{2.42}$$

The actions of the resolvants R_0 and R_a on the boundary states can be determined

$$\lim_{\eta \to 0} R_0(\epsilon_c + \epsilon_{\mu c} \pm i\eta) \left[P_c^0 - \epsilon_c - \epsilon_{\mu c} \mp i\eta \right] |\phi_c: \epsilon_c + \epsilon_{\mu c}\rangle = 0 \tag{2.43}$$

$$\lim_{\eta \to 0} R_a(\epsilon_c + \epsilon_{\mu c} \pm i\eta) \left[P_c^0 - \epsilon_c - \epsilon_{\mu c} \mp i\eta \right] |\phi_c: \epsilon_c + \epsilon_{\mu c}\rangle \tag{2.44}$$
$$= \delta_{ac} |\phi_c: \epsilon_c + \epsilon_{\mu c}\rangle$$

Use will be made of the operator K_{ab} formally defined by

$$W_{ab}(\hat{Q}) G_{NI}(\hat{Q}) \equiv K_{ab}(\hat{Q}) G_b(\hat{Q}) \tag{2.45}$$

With these relations, the fully interacting eigenstates are

$$|\psi_b^{(\pm)}: \underline{k}_b\epsilon_b; \epsilon_b(-\underline{k}_b\epsilon_{\mu b})\rangle = \left[1 - \lim_{\eta \to 0} R_0(\epsilon_b + \epsilon_{\mu b} \pm i\eta) \right. \tag{2.46}$$
$$\left. \cdot \sum_a \delta^3(P) K_{ab}(\epsilon_b + \epsilon_{\mu b} \pm i\eta,Q) \right] |\phi_b: \underline{k}_b\epsilon_b; \psi_b(-\underline{k}_b\epsilon_{\mu b})\rangle$$

Thus the operators which transform the boundary states into fully interacting eigenstates can be expressed

$$U_0^{(\pm)}(\vec{P}_{(0)}) = 1 - \lim_{\eta \to 0} G_{NI}(P_{(0)}^0 \pm i\eta, P_{(0)}) \sum_{ab} T_{ab}(P_{(0)}^0 \pm i\eta, P_{(0)})$$

$$U_b^{(\pm)}(\epsilon_b + \epsilon_{\mu b}, \underline{k}_b + \underline{k}_b^P) \tag{2.47}$$
$$= 1 - \lim_{\eta \to 0} G_{NI}(\epsilon_b + \epsilon_{\mu b} \pm i\eta, \underline{k}_b + \underline{k}_b^P) \sum_a K_{ab}(\epsilon_b + \epsilon_{\mu b} \pm i\eta, \underline{k}_b + \underline{k}_b^P)$$

Next, formal expressions will be developed for operators which directly yield physical amplitudes. These amplitudes will be extracted from operators $\mathcal{A}_{\alpha\beta}$ defined by

$$G_{FI}(\vec{Q}_1) G_{FI}(\vec{Q}_2) \equiv G_\alpha(\vec{Q}_1) \mathcal{A}_{\alpha\beta}(\vec{Q}_1; \vec{Q}_2) G_\beta(\vec{Q}_2) \tag{2.48}$$

Expectation values of $\mathcal{A}_{\alpha\beta}$ between boundary states contain information on physical observables through Eqs. (2.10) and (2.16). The following formal definitions will be made:

$$G_{NI}(\vec{Q}) W_{ab}(\vec{Q}) G_{NI}(\vec{Q}) \equiv G_{NI}(\vec{Q}) K_{ab}(\vec{Q}) G_b(\vec{Q})$$
$$\equiv G_a(\vec{Q}) \tilde{K}_{ab}(\vec{Q}) G_{NI}(\vec{Q}) \tag{2.49}$$
$$\equiv G_a(\vec{Q}) \mathcal{K}_{ab}(\vec{Q}) G_b(\vec{Q})$$

Using expressions (2.40) and (2.42), the following sets of equations can be shown valid

$$R_F(Z_1)R_F(Z_2) = R_\alpha(Z_1)\mathscr{A}_{\alpha\beta}(Z_1,0; Z_2,0)R_\beta(Z_2)$$

$$= R_0(Z_1)\left\{1 + \left[\frac{1}{Z_1-Z_2} - R_0(Z_2)\right]\delta^3(P)T(Z_2,0) - \delta^3(P)T(Z_1,0)\right.$$

$$\left. \times\left[\frac{1}{Z_1-Z_2} + R_0(Z_1)\right]\right\}R_0(Z_2)$$

$$= R_0(Z_1)R_0(Z_2) + \sum_a\left[R_a(Z_1)R_a(Z_2) - R_0(Z_1)R_0(Z_2)\right] + R_0(Z_1)$$

$$\times \sum_{ab}\left\{\left[\frac{1}{Z_1-Z_2} - R_0(Z_2)\right]\delta^3(P)K_{ab}(Z_2,0) - \delta^3(P)K_{ab}(Z_1,0)\right.$$

$$\left. \times\left[\frac{1}{Z_1-Z_2} + R_0(Z_1)\right]\right\}R_b(Z_2)$$

$$= R_0(Z_1)R_0(Z_2) + \sum_a\left[R_a(Z_1)R_a(Z_2) - R_0(Z_1)R_0(Z_2)\right] + \sum_{ab}R_a(Z_1)$$

$$\times\left\{\left[\frac{1}{Z_1-Z_2} - R_a(Z_2)\right]\delta^3(P)\tilde{K}_{ab}(Z_2,0) - \delta^3(P)\tilde{K}_{ab}(Z_1,0)\right.$$

$$\left. \times\left[\frac{1}{Z_1-Z_2} + R_0(Z_1)\right]\right\}R_0(Z_2)$$

$$= R_0(Z_1)R_0(Z_2) + \sum_a\left[R_a(Z_1)R_a(Z_2) - R_0(Z_1)R_0(Z_2)\right]$$

$$+ \sum_{ab}R_a(Z_1)\left\{\left[\frac{1}{Z_1-Z_2} - R_a(Z_2)\right]\delta^3(P)\mathscr{K}_{ab}(Z_2,0) - \delta^3(P)\mathscr{K}_{ab}(Z_1,0)\right.$$

$$\left. \times\left[\frac{1}{Z_1-Z_2} + R_b(Z_1)\right]\right\}R_b(Z_2) \tag{2.50}$$

With these relations, by cleverly choosing the limits on the parameters Z_1 and Z_2, the amplitudes in Eq. (2.25) can be determined

$$\langle\phi_0; (k_1c_1; k_2c_2; k_3c_3),\vec{P}_{(F)}|A_{00}^{(+)}(\vec{P}_{(F)})|\phi_0;(k_{10}c_{10}; k_{20}c_{20}; k_{30}c_{30}),\vec{P}_{(F)}\rangle$$

$$= -\sum_{ab}\langle k_1c_1; k_2c_2; k_3c_3|T_{ab}^{(+)}(\vec{P}_{(F)})|k_{10}c_{10}; k_{20}c_{20}; k_{30}c_{30}\rangle$$

where
$$\sum_a \vec{k}_a = \vec{P}_{(F)} = \sum_b \vec{k}_{b0} \quad \text{free asymptotic states}$$

$$\langle\phi_0; (k_1c_1; k_2c_2; k_3c_3),\vec{P}_{(B)}|A_{0b}^{(+)}(\vec{P}_{(B)})|\phi_b; k_{b0}c_{b0}; \psi_b(k_{b0}^P c_{\nu b0})\rangle$$

$$= -\sum_a \langle k_1c_1; k_2c_2; k_3c_3|K_{ab}^{(+)}(\vec{P}_{(B)})|\phi_b; k_{b0}c_{b0}; \psi_b(k_{b0}^P c_{\nu b0})\rangle$$

where
$$\sum_a \vec{k}_a = \vec{P}_{(B)} = \vec{k}_{b0} + \vec{k}_{b0}^P \quad \text{breakup}$$

$$\langle\phi_a; k_a c_a; \psi_a(k_a^P c_{\nu a})|A_{a0}^{(+)}(\vec{P}_{(C)})|\phi_0;(k_{10}c_{10}; k_{20}c_{20}; k_{30}c_{30}),\vec{P}_{(C)}\rangle$$

$$= -\sum_b \langle\phi_a; k_a c_a; \psi_a(k_a^P c_{\nu a})|K_{ab}^{(+)}(\vec{P}_{(C)})|k_{10}c_{10}; k_{20}c_{20}; k_{30}c_{30}\rangle$$

where
$$\vec{k}_a + \vec{k}_a^P = \vec{P}_{(C)} = \sum_b \vec{k}_{b0} \quad \text{coalescence}$$

$$\langle\phi_a; k_a c_a; \psi_a(k_a^P c_{\nu a})|A_{ab}^{(+)}(\vec{P}_{(E)})|\phi_b; k_{b0}c_{b0}; \psi_b(k_{b0}^P c_{\nu b0})\rangle$$

$$= -\langle\phi_a; k_a c_a; \psi_a(k_a^P c_{\nu a})|\mathscr{K}_{ab}^{(+)}(\vec{P}_{(E)})|\phi_b; k_{b0}c_{b0}; \psi_b(k_{b0}^P c_{\nu b0})\rangle$$

where
$$\vec{k}_a + \vec{k}_a^P = \vec{P}_{(E)} = \vec{k}_{b0} + \vec{k}_{b0}^P \quad \text{elastic and rearrangement scattering}$$

$$\tag{2.51}$$

2. Primary Singularities of the Amplitudes

The singularity structure of these amplitudes remains to be examined. Consider first the operator t_a. From expression (2.30), the behavior of t_a in the vicinity of a bound pair can be determined

$$\langle k_1 c_1; k_2 c_2; k_3 c_3 | \delta^3(P) t_a(Z,0) | k_{10} c_{10}; k_{20} c_{20}; k_{30} c_{30} \rangle$$

$$= - \int \frac{d^3 k_a' d^3 k_a^{P'}}{c_a' c_{\mu_a}'} \frac{\langle k_1 c_1; k_2 c_2; k_3 c_3 | H_a' | \phi_a; k_a' c_a'; \psi_a(k_a^{P'} c_{\mu_a}') \rangle}{c_a' + c_{\mu_a}' - Z} \tag{2.52a}$$

$$\cdot \langle \phi_a; k_a' c_a'; \psi_a(k_a^{P'} c_{\mu_a}') | H_a' | k_{10} c_{10}; k_{20} c_{20}; k_{30} c_{30} \rangle$$

$$+ \ (\text{term non-singular for } Z + c_a + c_{\mu_a})$$

Using the expressions (2.3), this can be written

$$\lim_{Z \to c_{a0} + c_{\mu_a 0}} (Z - c_{a0} - c_{\mu_a 0})$$

$$\times \langle k_1 c_1; k_2 c_2; k_3 c_3 | \delta^3(P) t_a(Z,0) | k_{10} c_{10}; k_{20} c_{20}; k_{30} c_{30} \rangle \tag{2.52b}$$

$$= c_{a0} \delta^3(k_a - k_{a0}) c_{\mu_a 0} \delta^3(k_{a+} + k_{a-} - k_{a+0} - k_{a-0})$$

$$\times \Big[(c_{a0} + c_{\mu_a 0}) - (c_1 + c_2 + c_3) \Big] \psi_a(k_{a+} k_{a-}; \mu_a) \phi_a^*(k_{a+0} k_{a-0}) \Big[(c_{a0} + c_{\mu_a 0})$$

$$- (c_{10} + c_{20} + c_{30}) \Big]$$

where $\qquad c_{\mu_a 0} = \sqrt{|k_{a+0} + k_{a-0}|^2 + \mu_a^2} - \sqrt{|k_{a0}|^2 + \mu_a^2}$

A normalization condition can be determined for the bound state wave functions. By considering the expression

$$\lim_{Z_1 \to c_a + c_{\mu_a}} \lim_{Z_2 \to c_{a0} + c_{\mu_a 0}} (Z_1 - c_a - c_{\mu_a})(Z_2 - c_{a0} - c_{\mu_a 0})$$

$$\times \langle k_1 c_1; k_2 c_2; k_3 c_3 | R_0(Z_1) \delta^3(P) t_a(Z_1,0) R_0(Z_1)$$

$$\times R_0(Z_2) \delta^3(P) t_a(Z_2,0) R_0(Z_2) | k_{10} c_{10}; k_{20} c_{20}; k_{30} c_{30} \rangle \tag{2.53a}$$

and recalling the relation (2.36), the following normalization condition can be determined

$$\int \frac{d^3 k_{a+} \, d^3 k_{a-}}{c_{a+}' c_{a-}'} c_{\mu_a}' \delta^3(k_{a+}' + k_{a-}' - k_{a+0} - k_{a-0}) \psi_a^*(k_{a+}' k_{a-}'; \mu_a) \psi_a(k_{a+}' k_{a-}'; \mu_a) = 1 \tag{2.53b}$$

where $\qquad c_{\mu_a}' = \sqrt{|k_{a+}' + k_{a-}'|^2 + \mu_a^2}$

To examine the singularity structure of W_{ab}, the equation satisfied by W_{ab} can be determined using Eq. (2.32) to be

$$W_{ab}(\bar{q}) = -\bar{\delta}_{ab} t_a(\bar{q}) G_{NI}(\bar{q}) t_b(\bar{q}) - \sum_c \bar{\delta}_{ac} t_a(\bar{q}) G_{NI}(\bar{q}) W_{cb}(\bar{q}) \tag{2.54}$$

As can be seen by the driving term, the singularities in t_a and t_b will occur to all orders of iteration of Eq. (2.54). These singularities of W_{ab} which appear in all orders of iteration are called "primary singularities." These singularities are due only to the factors from t_a and t_b. Using Eq. (2.52b) with the definitions (2.49), expressions obtained for the operators K_{ab} and H_{ab} will be free of primary singularities

$$\lim_{Z \to c_{b0} + c_{\mu_b 0} \pm i0} (Z - c_{b0} - c_{\mu_b 0})$$

$$\times \langle k_1 c_1; k_2 c_2; k_3 c_3 | W_{ab}(Z,0) | k_{10} c_{10}; k_{20} c_{20}; k_{30} c_{30} \rangle \tag{2.55a}$$

$$= - \langle \phi_0; (k_1 c_1; k_2 c_2; k_3 c_3), (P_{(0)}', 0) | K_{ab}^{(\pm)}(c_{b0} + c_{\mu_b 0}, 0) | \phi_b; k_{b0} c_{b0}; \psi_b(-k_{b0}, c_{\mu_b 0}) \rangle$$

$$\times \psi_b^*(k_{b+0} k_{b-0}; \mu_b) \Big[c_{10} + c_{20} + c_{30} - c_{b0} - c_{\mu_b 0} \Big]$$

where $\qquad c_1 + c_2 + c_3 = P_{(0)}^0 = c_{b0} + c_{\mu_b 0}$

$$\lim_{Z \to \epsilon_a + \epsilon_{\mu_a} \pm i0} (Z - \epsilon_a - \epsilon_{\mu_a})$$

$$\times \langle k_1 c_1; k_2 c_2; k_3 c_3 | W_{ab}(Z,0) | k_{10} c_{10}; k_{20} c_{20}; k_{30} c_{30} \rangle (Z - \epsilon_{b0} - \epsilon_{\mu_b 0})$$

$$= [\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_a - \epsilon_{\mu_a}] \psi_a(k_{a+c-}; \mu_a)$$

$$\times \langle \phi_a; k_a \epsilon_a; \psi_a(-k_a \epsilon_{\mu_a}) | \mathcal{H}_{ab}^{(\pm)}(\epsilon_a + \epsilon_{\mu_a}, 0) | \phi_b; k_{b0} \epsilon_{b0}; \psi_b(-k_{b0} \epsilon_{\mu_b 0}) \rangle$$

$$\times \psi_b^*(k_{b+} k_{b-0}; \mu_b)[\epsilon_{10} + \epsilon_{20} + \epsilon_{30} - \epsilon_{b0} - \epsilon_{\mu_b 0}] \qquad (2.55b)$$

for
$$\epsilon_a + \epsilon_{\mu_a} = \epsilon_b + \epsilon_{\mu_b}$$

The discrete spectrum $\{M_j\}$ of P_{FI}^0 will correspond to poles in the full resolvent $P_T(M_j)$. Thus, the solutions to the homogeneous equation will correspond to the discrete eigenvalues of P_{FI}^0

$$G_{FI}(\hat{Q}) = -G_{NI}(\hat{Q}) T(\hat{Q}) G_{NI}(\hat{Q}) \qquad (2.56)$$

The projector onto the discrete spectrum of $\hat{P}_{FI}$ will be denoted $\mathcal{P}_{discrete}$.

To end the discussion of the fully-interacting system, it should be noted that the relations (2.47) or (2.50) can be used to examine on shell orthogonality and completeness relations. The operator expressions obtained are

$$U_\alpha^{(\pm)\dagger} U_\beta^{(\pm)} = \delta_{\alpha\beta} \mathbb{1}_\alpha \qquad (2.57)$$

where $\mathbb{1}_\alpha$ is the unit operator of boundary state $|\phi_\alpha\rangle$

$$\sum_{\alpha=0}^3 U_\alpha^{(\pm)} U_\alpha^{(\pm)\dagger} = 1 - \mathcal{P}_{discrete} \quad .$$

3. Cross Sections

A final connection will be made with the amplitudes examined and the standard cross section. The probability flux into a subset of final states can be written

$$\Delta(\text{probability flux}) = \mathcal{F} \Delta\sigma \qquad (2.58)$$

where $\mathcal{F}$ is the incoming probability flux per unit area, and $\Delta\sigma$ is the apparent cross sectional area. This expression can be rewritten

$$\Delta\sigma = \sum_{f \in \Delta\Psi_f} \frac{\langle \Psi_i | \Psi_f \rangle \langle \Psi_f | \Psi_i \rangle}{\mathcal{F}\langle \Psi_i | \Psi_i \rangle} \qquad (2.59)$$

With the given state normalization and covariant flux, the standard covariant differential cross section is

$$d\sigma = \prod_{a=1}^N \frac{d^3 k_a}{\epsilon_a} \frac{(2\pi)^4 \delta^4(\vec{P}_{(f)} - \vec{P}_{(1)}) |\Delta_{fi}|^2}{\left[(\vec{k}_{(1)0} \cdot \vec{k}_{(2)0})^2 - (\vec{k}_{(1)0} \cdot \vec{k}_{(1)0})(\vec{k}_{(2)0} \cdot \vec{k}_{(2)0}) \right]^{1/2}} \qquad (2.60)$$

where $k_{(j)0}$ is the four momentum of the incoming subsystem j.

C. Minimal Three-Particle Model

Given the analytic structure of the matrix elements of the operator t_a, one can in principal use Eq. (2.54) to determine a set of equations for the matrix elements of the operators W_{ab}, from which physical observables can be extracted. The equations for T_{ab} and W_{ab} will be recalled for the present discussion

$$T_{ab}(\hat{Q}) = \bar{\delta}_{ab} t_a(\hat{Q}) - \sum_c \bar{\delta}_{ac} t_a(\hat{Q}) G_{NI}(\hat{Q}) T_{cb}(\hat{Q})$$

$$\qquad (2.61)$$

$$W_{ab}(\hat{Q}) = -\bar{\delta}_{ab} t_a(\hat{Q}) G_{NI}(\hat{Q}) t_b(\hat{Q}) - \sum_c \bar{\delta}_{ac} t_a(\hat{Q}) G_{NI}(\hat{Q}) W_{cb}(\hat{Q})$$

The input from the two-particle subsystems is through the scattering operators τ_a which occur in the two-particle space, and from which the operators t_a are obtained as follows

$$\langle \underline{k}_1 \epsilon_1 ; \ \underline{k}_2 \epsilon_2 ; \ \underline{k}_3 \epsilon_3 | \tau_a(\tilde{Q}) | \underline{k}_{10} \epsilon_{10} ; \ \underline{k}_{20} \epsilon_{20} ; \ \underline{k}_{30} \epsilon_{30} \rangle$$

$$= \epsilon_{a0} \delta^3(\underline{k}_a - \underline{k}_{a0}) \langle \underline{k}_{a+} \epsilon_{a+} ; \ \underline{k}_{a-} \epsilon_{a-} | \tau_a(\tilde{Q} - \tilde{k}_{a0}) | \underline{k}_{a+0} \epsilon_{a+0} ; \ \underline{k}_{a-0} \epsilon_{a-0} \rangle$$

$$\equiv \epsilon_{a0} \delta^3(\underline{k}_a - \underline{k}_{a0}) \tau_a(\underline{k}_{a+} \underline{k}_{a-} | \underline{k}_{a+0} \underline{k}_{a-0}; \ \tilde{Q} - \tilde{k}_{a0}) \qquad (2.62)$$

The operators τ_a are the scattering operators which would occur in a purely two-particle theory. The resolvant will have matrix elements

$$\langle \underline{k}_1 \epsilon_1 ; \ \underline{k}_2 \epsilon_2 ; \ \underline{k}_3 \epsilon_3 | G_{MI}(\tilde{Q}) | \underline{k}_{10} \epsilon_{10} ; \ \underline{k}_{20} \epsilon_{20} ; \ \underline{k}_{30} \epsilon_{30} \rangle \qquad (2.63)$$

$$= \frac{\delta^3(\underline{k}_{10} + \underline{k}_{20} + \underline{k}_{30} - Q)}{\epsilon_1 + \epsilon_2 + \epsilon_3 - Q^0} \prod_{a=1}^{3} \epsilon_{a0} \delta^3(\underline{k}_a - \underline{k}_{a0})$$

These equations will be examined in the three-particle center of momentum system. In the 3-CMS the parameter $\tilde{Q}$ will have the on-shell behavior indicated

$$\tilde{Q}_{3-CMS} = (M + 10,0) = \tilde{P}_{(0)}$$

The behavior of the matrix elements of the operator $W_{ab}^{(+)}$ in the free (non-interacting) particle basis with this parametrization is as follows:

$$W_{ab}^{(+)}(\underline{k}_{a+} \underline{k}_{a-} | \underline{k}_{b0} \underline{k}_{b+0} \underline{k}_{b-0}; \ (M,0)) = -\bar{\delta}_{ab}$$

$$\times \frac{\tau_a^{(+)}(\underline{k}_{a+} \underline{k}_{a-} | \underline{k}_{a+0} \underline{k}_{a-0}; \ (M - \epsilon_{a0}, -\underline{k}_{a0})) \tau_b^{(+)}(\underline{k}_{b+} \underline{k}_{b-} | \underline{k}_{b+0} \underline{k}_{b-0}; \ (M - \epsilon_{b0}, \underline{k}_{b0}))}{\epsilon_{ab}(\epsilon_a + \epsilon_b + \epsilon_{ab} - M - i0)}$$

$$- \iint \frac{d^3 k'_{a+} d^3 k'_{a-}}{\epsilon_{a+}' \epsilon_{a-}'} \sum_c \bar{\delta}_{ac} \tau_a^{(+)}(\underline{k}_{a+} \underline{k}_{a-} | \underline{k}'_{a+} \underline{k}'_{a-}; \ (M - \epsilon_a, -\underline{k}_a)) \qquad (2.64)$$

$$\cdot \frac{\delta^3(\underline{k}'_{a+} + \underline{k}'_{a-} + \underline{k}_a)}{\epsilon_a + \epsilon_{a+}' + \epsilon_{a-}' - M - i0} W_{cb}^{(+)}(\underline{k}'_c \underline{k}_{c-} \underline{k}_{ac} | \underline{k}_{b0} \underline{k}_{b+0} \underline{k}_{b-0}; \ (M,0))$$

where $\quad \underline{k}_{12} \equiv \underline{k}_3$

$$\epsilon_{12}^2 \equiv |\underline{k}_1 + \underline{k}_2|^2 + m_3^2 \qquad \text{etc., for 1, 2, 3 cyclic.}$$

1. Equations for Separable Two-Body Input

Equation (2.64) represents a general integral relationship of the elements of W_{ab} to the elements of τ_a. The equation involves an integration over three components of momentum, which could produce numerical complications. To simplify the numerical problem, it will be assumed that the two-body input can be written in a separable form:

$$\tau_a^{(+)}(\underline{k}_{a+} \underline{k}_{a-} | \underline{k}'_{a+} \underline{k}'_{a-}; \ \tilde{Q}_a) = \frac{g_a(\underline{k}_{a+} \underline{k}_{a-}) g_a^*(\underline{k}'_{a+} \underline{k}'_{a-})}{D_a^{(+)}(\tilde{Q}_a \cdot \tilde{Q}_a)} \qquad (2.65)$$

The parameter $\tilde{Q}_a$ will be written in terms of the two-particle on-shell invariant rest energy

$$\tilde{Q}_a \cdot \tilde{Q}_a = (\tilde{P}_{(0)} - \tilde{k}_a) \cdot (\tilde{P}_{(0)} - \tilde{k}_a) \equiv s_a \qquad (2.66)$$

From Eq. (2.64) it directly follows that the elements of W_{ab} can be written in terms of a reduced amplitude $\hat{W}_{ab}$

$$W_{ab}^{(+)}(\underline{k}_{a+} \underline{k}_{a-} | \underline{k}_{b0} \underline{k}_{b+0} \underline{k}_{b-0}; \ (M,\cdot)) \qquad (2.67)$$

$$\equiv \frac{g_a(\underline{k}_{a+} \underline{k}_{a-})}{D_a(s_a)} \hat{W}_{ab}(\underline{k}_a | \underline{k}_{b0}; \ M) \frac{g_b(\underline{k}_{b+0} \underline{k}_{b-0})}{D_b(s_b)}$$

The equation satisfied by the reduced amplitudes is

$$\hat{W}_{ab}(\underline{k}_a | \underline{k}_{b0}; \ M) = -\bar{\delta}_{ab} \frac{g_a^*(\underline{k}_{a+0} \underline{k}_{a-0}) g_b(\underline{k}_{b+} \underline{k}_{b-})}{\epsilon_{ab}(\epsilon_a + \epsilon_b + \epsilon_{ab} - M - i0)} \qquad (2.68)$$

$$- \sum_c \bar{\delta}_{ac} \int \frac{d^3 k'_c}{\epsilon_a' \epsilon_{ac}'} \frac{g_a^*(\underline{k}'_c, -\underline{k}_a - \underline{k}'_c) g_c(\underline{k}_a, -\underline{k}'_c - \underline{k}_a)}{(\epsilon_a + \epsilon_{a0}' + \epsilon_c' - M - i0) D_c(s_c')} \hat{W}_{cb}(\underline{k}'_c | \underline{k}_{b0}; \ M)$$

where

$$\varepsilon_{ac}'^{\,2} = |k_a + k_c'|^2 + m_{ac}^2$$

It is now advantageous to perform an angular momentum decomposition to reduce the number of integration variables. The reduced amplitudes can be written in terms of angular momentum components

$$\hat{W}_{ab}(k_a|k_{b0}; M) = \sum_J \frac{2J+1}{4\pi} P_J(\hat{k}_a \cdot \hat{k}_{b0}) \hat{W}_{ab}^J(k_a|k_{b0}; M) \qquad (2.69)$$

Using some of the properties of the functions $P_J(\xi)$

$$\int_{-1}^{1} P_{J'}(\xi) P_J(\xi) d\xi = \frac{2\delta_{J'J}}{2J+1}$$

$$P_J(\hat{k}_a \cdot \hat{k}_{b0}) \sim \frac{4\pi}{2J+1} \sum_M Y_{JM}(\hat{k}_a) Y_{JM}^*(\hat{k}_{b0}) \qquad (2.70)$$

The equation for the angular momentum components of the reduced amplitudes becomes:

$$\hat{W}_{ab}^J(k_a|k_{b0}; M) = -2\pi\bar\delta_{ab} \int_{-1}^{1} d\xi_{ab} \frac{g_a^*(k_a,k_{b0}; \xi_{ab}) g_b(k_a,k_{b0}; \xi_{ab}) P_J(\xi_{ab})}{\varepsilon_{ab}'(\xi_{ab})[\varepsilon_a + \varepsilon_b + \varepsilon_{ab}'(\xi_{ab}) - M - i0]} +$$

$$-2\pi \sum_c \bar\delta_{ac} \int \frac{k_c'^2 dk_c'}{\varepsilon_c'} \frac{\hat{W}_{cb}^J(k_c'|k_{b0}; M)}{D_c(s_c')} \qquad (2.71)$$

$$\left[\int_{-1}^{1} d\xi_{ac} \frac{g_a^*(k_a,k_c'; \xi_{ac}) R_c(k_a,k_c'; \xi_{ac}) P_J(\xi_{ac})}{\varepsilon_{ac}'(\xi_{ac})[\varepsilon_a + \varepsilon_c' + \varepsilon_{ac}'(\xi_{ac}) - M - i0]} \right]$$

If the kernels R_{ab}^J are defined

$$R_{ab}^J(k_a,k_b; M) = \int_{-1}^{1} d\xi_{ab} \frac{g_a^*(k_a,k_b; \xi_{ab}) g_b(k_a,k_b; \xi_{ab}) P_J(\xi_{ab})}{\varepsilon_{ab}'(\xi_{ab})[\varepsilon_a + \varepsilon_b + \varepsilon_{ab}'(\xi_{ab}) - M - i0]} \qquad (2.72a)$$

then Eq. (2.71) can be expressed

$$\hat{W}_{ab}^J(k_a|k_{b0}; M) = -2\pi\bar\delta_{ab} R_{ab}^J(k_a,k_b; M) + \qquad (2.72b)$$

$$-2\pi \sum_c \bar\delta_{ac} \int dk_c' \frac{k_c'^2 R_{ac}^J(k_a,k_c'; M)}{\varepsilon_c' D_c^{(+)}(s_c')} \hat{W}_{cb}^J(k_c'|k_{b0}; M)$$

2. Model Assumptions

The specific model for the two-body input to be explored reduces in the non-relativistic limit to the case in which the pair can interact at zero range only through zero relative angular momentum (s-waves). The covariant model can be represented by amplitudes τ_a given by

$$\tau_a^{(+)}\left(k_a+k_a'|k_a'+k_a'; (\vec{P}_{(0)} - \vec{k}_a)\right) = \frac{g_a \sqrt{s_a}\, \theta(s_a) s_a^2}{\alpha_a(\nu_a) - \sqrt{-|q_a(s_a + i0|^2}} \qquad (2.73)$$

The variable $|q_a|$ represents the pair center-of-mass momentum and the parameter α_a reduces to the inverse scattering length:

$$|q_a(s_a)|^2 = \frac{\left[s_a - (m_{a+} + m_{a-})^2\right]\left[s_a - (m_{a+} - m_{a-})^2\right]}{4 s_a}$$

$$\alpha_a(\nu_a) = \text{sign}\,(m_{a+} + m_{a-} - \nu_a)\left|\frac{\left[\nu_a^2 - (m_{c+} + m_{a-})^2\right]\left[\nu_a^2 - (m_{a+} - m_{a-})^2\right]}{4\nu_a^2}\right|^{1/2}$$

$$\longrightarrow -1/a_a \text{ scattering length.}$$

$$\nu_a \geq |m_{a+} - m_{a-}| \quad , \quad \theta(s_a) = \begin{cases} 0 & s_a < 0 \\ \\ 1 & s_a > 0 \end{cases} \qquad (2.74)$$

The model has two-body bound state poles when the invariant on-shell pair rest energy s_a is equal to the rest mass of the bound state ν_a. The coupling constants g_a are not arbitrary, since restrictions are placed upon them by Eq. (2.62) through Eq. (2.36). This restriction can be expressed

$$\tau_a^{(+)}\left(\underline{k}_{a+}\underline{k}_{a-}|\underline{k}_{a+0}\underline{k}_{a-0}; \vec{P}_{(0)} - \vec{k}_{a0}\right) - \left[\tau_a^{(+)}\left(\underline{k}_{a+}\underline{k}_{a-}|\underline{k}_{a+0}\underline{k}_{a-0}; \vec{P}_{(0)} - \vec{k}_{a0}\right)\right]^*$$

$$= -2\pi i \int \frac{d^3k_{a+}' d^3k_{a-}'}{i_{a+}'i_{a-}'} \tau_a\left(\underline{k}_{a+}\underline{k}_{a-}\ \underline{k}_{a+}'\underline{k}_{a-}'; \vec{P}_{(0)} - \vec{k}_{a0}\right)\lambda^4\left(\vec{k}_{a+}' + \vec{k}_{a-}' - \vec{k}_{a+0} - \vec{k}_{a-0}\right)$$

$$\tau_a^*\left(\underline{k}_{a+}'\underline{k}_{a-}'|\underline{k}_{a+0}\underline{k}_{a-0}; \vec{P}_{(0)} - \vec{k}_{a0}\right) \tag{2.75}$$

For this model, the only integration involved is the two-particle phase space

$$\rho_a(s_a) = 4\pi \frac{|q_a(s_a)|}{\sqrt{s_a}} \tag{2.76}$$

From this, it is easily determined that the functions required for Eq. (2.71) have the form

$$|g_a|^2 = (1/2\pi)^2$$

$$\frac{1}{D_a^{(+)}(s_a)} = \frac{\sqrt{s_a}\,\theta(s_a)}{\alpha_a(\nu_a) - \sqrt{-|q_a(s_a + i0)|^2}} \tag{2.77}$$

Equations (2.72b, vii) be explored in this model for the case $J = 0$ in the next chapter.

CHAPTER III

RESULTS

To begin the investigation of this model, the kinematics of the equations will be examined. Without sacrificing the formal generality, the case of three kinematically and dynamically identical, but distinguishable, particles will be considered in what follows. With the identification $m_a = m$ and $\nu_a = \mu$ for all a, the structure of equations (2.74) simplifies:

$$|q_a(s_a)|^2 = \frac{s_a}{4} - m^2$$

$$\alpha_a(\mu) = \alpha(\mu) = \text{sign}(2m - \mu)|\frac{\mu^2}{4} - m^2|^{\frac{1}{2}} \tag{3.1}$$

For this system, the dynamic parameters of the model are μ and M, which represent the bound/virtual state mass of the two-particle subsystems and the three-particle center of mass energy, respectively. The dynamical regions are schematically represented in Fig. III-1. The line $M = 3m$ indicates the minimal kinematic situation with which to have three asymptotically non-interacting particles. The vertical line $\mu = 2m$ indicates the kinematic threshold below which asymptotic two-particle bound states may exist. The regions will be explored for which the particle states m represent the lowest energy states, and for which pair creation will not occur.

A. Three-Particle Bound States

The first situation to be investigated will involve the three particles fully bound in a $J = 0$ state. The kinematic region for bound
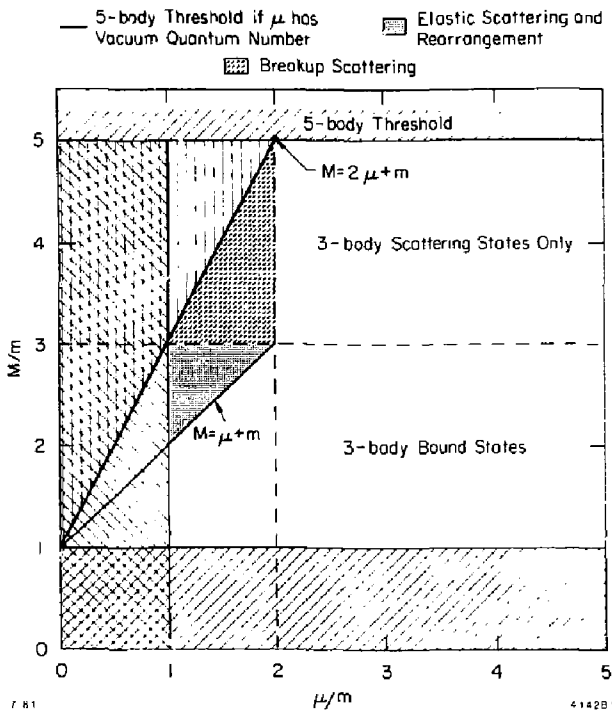
5-body Threshold if μ has Vacuum Quantum Number

Elastic Scattering and Rearrangement

Breakup Scattering

5-body Threshold

$M = 2\mu + m$

3-body Scattering States Only

$M = \mu + m$

3-body Bound States

M/m

μ/m

7 81

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FIG. III-1

states is below three-body scattering threshold ($M \leq 3m$) and bound pair + spectator threshold ($M \leq \mu + m$).

1. Form of Bound State Equations

The discrete spectrum of the fully interacting system will correspond to eigenstates which satisfy the homogeneous scattering equations. The bound state equations for the amplitude $W_{ab}^{J=0}$ for three equal masses will reduce to three identical equations. The kernel $R_{ab}^{J=0}$ can be calculated from Eq. (2.72a), and after a simple integration becomes

$$R_{ab}^{J=0}(k_a, k_b; M) \equiv R(k_a, k_b; M) = R(k_b, k_a; M)$$

$$= \frac{|g|^2}{k_a k_b} \log\left(\frac{\sqrt{m^2 + (k_a + k_b)^2} + \epsilon_a + \epsilon_b - M}{\sqrt{m^2 + (k_a - k_b)^2} + \epsilon_a + \epsilon_b - M}\right) \quad (3.2)$$

Thus, the relation for the reduced amplitudes (2.72b) can be expressed as a single variable homogeneous integral equation

$$\hat{W}_{ab}^{J=0}(k_a | k_{bo}; M) \equiv W_B(k_a; M, \mu)$$

$$W_B(k; M, \mu) = -4\pi \int_0^{\frac{M^2 - m^2}{2M}} dk' \frac{k'^2 \sqrt{m'} \, R(k, k'; M)}{\epsilon' \left[\pm \sqrt{\left|m^2 - \frac{\mu^2}{4}\right|} - \sqrt{m^2 - \frac{s'}{4}}\right]} W_B(k'; M, \mu) \quad (3.3)$$

$$\epsilon' = \sqrt{k'^2 + m^2} \quad , \quad s' = M^2 + m^2 - 2M\epsilon'$$

where the $\pm$ represents the sign of $2m - \mu$. The analytic form of this equation will be examined in the next subsection, and numerical solutions will be presented in the subsection following.

2. Mathematical Constraints and Predictions

The solutions of Eq. (3.3) will consist of a discrete set of non-degenerate values $\{M_j\}$ for a given two-body bound state mass μ. Alternatively, one may obtain a discrete set of two-body bound state masses $\{\mu_r\}$ which produce a three-body bound state of mass M. There will be a maximum value for the two-body bound state mass μ_{max} above which there will be no three-body bound state solutions (this value will be shown to be finite). As has been mentioned, the minimum value for the mass μ_{min} is determined by the threshold for elastic scattering $\mu + m \rightarrow \mu + m$, which occurs at $M = \mu_{min} + m$.

$$\alpha(\mu_{min}(M)) = \alpha(M - m) = \frac{1}{2}\sqrt{(3m - M)(m + M)}$$

$$= \sqrt{m^2 - \frac{s(k=0)}{4}} \qquad (3.4)$$

The form of the Eqs. (3.3) allow all parameters to be scaled relative to the finite mass m. In studying the analytic form of the equations, it is convenient to use this scale freedom, along with the symmetry of the equation, to define the following:

$$z \equiv \frac{k}{m} \quad , \quad \tilde{M} \equiv \frac{M}{m} \quad , \quad \kappa(\tilde{M}) \equiv \frac{\tilde{M}^2 - 1}{2\tilde{M}}$$

$$\lambda_r(\tilde{M}) \equiv \frac{\alpha(\mu_r(M)) - \alpha(\mu_{min}(M))}{\alpha(\mu_{max}(M)) - \alpha(\mu_{min}(M))} \quad , \quad 0 \leq \lambda_r \leq 1$$

$$\frac{k}{\alpha(\mu_r) - \sqrt{m^2 - \frac{s}{4}}} \sqrt{\frac{\sqrt{s}}{c}} \; W_B(k; M, \mu_r) \equiv V(z; \tilde{M}, \lambda_r)$$

$$h(z; \tilde{M}) = -\left[\frac{\sqrt{m^2 - \frac{s}{4}} - \frac{1}{2}\sqrt{(3m - M)(m + M)}}{\alpha(\mu_{max}(M)) - \alpha(\mu_{min}(M))} \right] \geq 0$$

$$U(z, z'; \tilde{M}) = U(z', z; \tilde{M}) = -\frac{4\pi |g|^2 m}{\alpha(\mu_{max}(M)) - \alpha(\mu_{min}(M))}$$

$$\times \sqrt{\frac{\sqrt{ss'}}{cc'}} \; \ln \cdot \frac{\sqrt{1 + (z + z')^2} + \sqrt{1 + z^2} + \sqrt{1 + z'^2} - \tilde{M}}{\sqrt{1 + (z - z')^2} + \sqrt{1 + z^2} + \sqrt{1 + z'^2} - \tilde{M}} \qquad (3.5)$$

finite for $\tilde{M} \leq 3$

With these definitions, Eq. (3.3) can be expressed

$$\lambda_r \, V(z; \tilde{M}, \lambda_r) = \int_0^{\kappa(\tilde{M})} dz' \; U(z, z'; \tilde{M})V(z'; \tilde{M}, \lambda_r) - h(z; \tilde{M})V(z; \tilde{M}, \lambda_r) \qquad (3.6)$$

The form (3.6) is particularly useful, since the following relation is seen to be valid

$$(\lambda_r - \lambda_s) \int_0^{\kappa(\tilde{M})} dz \; V(z; \tilde{M}, \lambda_r)V(z; \tilde{M}, \lambda_s) = 0 \qquad (3.7)$$

This condition amounts (for non-degenerate eigenvalues) to an orthogonality condition for the functions $V(z; \tilde{M}, \lambda_r)$. These functions can be normalized to satisfy the condition

$$\int_0^{\kappa(\tilde{M})} dz \; V(z; \tilde{M}, \lambda_r)V(z; \tilde{M}, \lambda_s) = \delta_{rs} \qquad r, s = 1, \ldots, N(\tilde{M}) \qquad (3.8)$$

where $N(M)$ is the number of three-body bound states of energy M.

Since the system generates a denumerable set of orthogonal functions, the parameters of the equations can be explored more readily than might otherwise have been the case. To obtain relationships between the parameters, it is advantageous to define functions which sum over the dynamical parameter λ_r

$$\Lambda(z,z';\, \tilde{M}) \equiv \sum_{r=1}^{N(\tilde{M})} V(z;\, \tilde{M},\lambda_r)V(z';\, \tilde{M},\lambda_r)$$

$$\Delta(z,z';\, \tilde{M}) \equiv \sum_{r=1}^{N(\tilde{M})} \lambda_r(\tilde{M})V(z;\, \tilde{M},\lambda_r)V(z';\, \tilde{M},\lambda_r)$$

$$\Gamma(z,z';\, \tilde{M}) \equiv \sum_{r=1}^{N(\tilde{M})} \frac{V(z;\, \tilde{M},\lambda_r)V(z';\, \tilde{M},\lambda_r)}{\lambda_r} \tag{3.9}$$

$$\mathrm{Tr}\ U^2(\tilde{M}) \equiv \int_0^{\kappa(\tilde{M})} dz \int_0^{\kappa(\tilde{M})} dz'\ U^2(z,z';\, \tilde{M})$$

These functions are easily related through the integral equation (3.6). Using Eq. (3.8) and simple algebra, the following conditionals are obtained.

$$N(\tilde{M}) \leq \frac{1}{2}\left[\mathrm{Tr}\ U^2(\tilde{M}) + \sum_{r=1}^{N(\tilde{M})} \frac{1}{\lambda_r^2(\tilde{M})}\right]$$

$$\sum_{r=1}^{N(\tilde{M})} \lambda_r(\tilde{M}) \leq \frac{1}{2}[\mathrm{Tr}\ U^2(\tilde{M}) + 1] \tag{3.10}$$

$$\sum_{r=1}^{N(\tilde{M})} \lambda_r^2(\tilde{M}) \leq \mathrm{Tr}\ U^2(\tilde{M})$$

Since $\mathrm{Tr}\ U^2(\tilde{M})$ is always finite within the kinematic region being studied, these equations set finite bounds on the parameters, except for $N(\tilde{M})$. In addition, for the specific problem at hand, the following is true.

$$\frac{d\lambda_r(\tilde{M})}{d\tilde{M}} \geq 0 \tag{3.11}$$

This implies that the bound state trajectories $\lambda_r(\tilde{M})$ are monotonically increasing, and have one end point along the line $\lambda = 0$, and the other along the three-body continuum threshold $\tilde{M} = 3$, which are the boundaries of the kinematic region. To obtain an estimate for the number of bound states, the equation for $N(\tilde{M})$ will be examined. Since $\Lambda(z,z;\, \tilde{M})$ is a positive semi-definite quantity, the following inequality holds:

$$N(\tilde{M}) = \int_0^{\kappa(M)} dz\ \Lambda(z,z;\, \tilde{M}) \geq \int_0^{\zeta} dz\ \Lambda(z,z;\, \tilde{M}) \tag{3.12}$$

$$\text{for any } \zeta < \kappa(\tilde{M})$$

One of the forms for the expression $\Lambda(z,z;\, \tilde{M})$ can be obtained directly from Eq. (3.6)

$$\Lambda(z,z';\, \tilde{M}) = \frac{1}{h(z;\, \tilde{M})}\left[\int_0^{\kappa(\tilde{M})} dz''\ U(z,z'';\, \tilde{M})\Lambda(z',z'';\, \tilde{M}) - \Delta(z,z';\, \tilde{M})\right]$$

$$\Lambda(z,z;\, \tilde{M}) \equiv \frac{I(z;\, \tilde{M})}{h(z;\, \tilde{M})} \xrightarrow{z\to 0} 0 \tag{3.13}$$

The behavior of this expression is particularly interesting near the rest energy of the three particles. A binding parameter $\hat{e}$ will be defined to examine this case:

$$e = 3m - M \quad , \quad \tilde{e} = 3 - \tilde{M} \qquad (3.14)$$

The behavior of $\zeta(z,z; M)$ for small z and e relative to unity (but otherwise arbitrary) is dominated by the factor $n(z; \tilde{M})$

$$\zeta(z; \tilde{M}) \xrightarrow[\substack{\text{small } z \\ \text{small } \tilde{e}}]{} \left| \frac{m}{\alpha(\mu_{max}(3m))} \right| \left[\sqrt{\frac{3}{4} z^2 + \tilde{e}} - \sqrt{\tilde{e}} \right] \qquad (3.15)$$

For small $\tilde{e}$, the factor $\kappa(\tilde{M}) \to 4/3$. If one sets the parameter ξ to be small compared to unity, but arbitrary compared to $\tilde{e}$, the expressions (3.12) and (3.13) indicate a scaling behavior of the number of bound states with the parameter $\tilde{e}$:

$$x \equiv \frac{z}{\sqrt{\tilde{e}}}$$

$$N \geq \int_0^{\xi/\sqrt{\tilde{e}}} \sqrt{\tilde{e}} \, dx \, \frac{I(\sqrt{\tilde{e}} \, x; \tilde{M})}{h(\sqrt{\tilde{e}} \, x; \tilde{M})} \xrightarrow[\tilde{e} \to 0]{} \sqrt{\frac{4}{3}} \left| \frac{\alpha(\mu_{max})}{c} \right| I(\xi; \tilde{M}) \, \log \frac{\xi}{\sqrt{\tilde{e}}} \qquad (3.16)$$

If bound state solutions exist, then the function $I(\xi; \tilde{M})$ does not identically vanish as a function of ξ or $\tilde{M}$. Therefore, these equations have at least a logarithmic growth in the number of solutions as the three-body continuum threshold is approached

$$N(e) \geq (\text{slowly varying non-zero function}) \times \log\left(\frac{m}{e}\right)$$

$$\text{as } \frac{e}{m} \to 0 \qquad (3.17)$$

This result is determined by the non-relativistic kinematics, and is consistent with the results obtained by Efimov[3] in Eq. (1.1), if one relates the scattering length to the two-body binding using Eq. (2.74) and associates e as the three-particle binding energy. The actual

numerical solutions exhibit the behavior discussed, and will be displayed in the next section.

3. Numerical Bound State Solutions

The bound state trajectories have been calculated, and are consistent with the conditions (3.10), (3.11) and (3.12). The integral equation (3.3) was reduced to a discrete matrix equation using Gaussian quadratures,[8] with Jacobi polynomials as weight functions. Stable solutions for the lowest lying states were obtained using relatively low matrix order (about 8×8). The lowest lying states are exhibited in Figs. III-2a and b. Figure III-2b is an enlargement of the non-relativistic region of Fig. III-2a.

The binding energy of all trajectories remains finite in this model, due to the finite kinematics. The kinematics of all states is non-relativistic, except for the lowest lying state. Most of the lowest lying trajectory λ_1 is within the relativistic domain of the region, although it lies very close to the threshold for pair-particle scattering ($\mu + m \to \mu + m$). The finite binding of all trajectories differs from various non-relativistic models (cf. Dodd[9]), for which the lowest lying states may become bound indefinitely. This behavior is exhibited in Fig. III-2b by the trajectory λ_1 before the relativistic kinematics become manifest. There is an accumulation of essentially non-relativistic states in the region $3m - M \to 0$, $|2m - \mu| \to 0$, consistent with the condition (3.17) and with non-relativistic models.

B. Elastic and Rearrangement Scattering

The region of Fig. III-1 below three-body breakup threshold ($M < 3m$) for which bound pairs scatter with the third particle, will
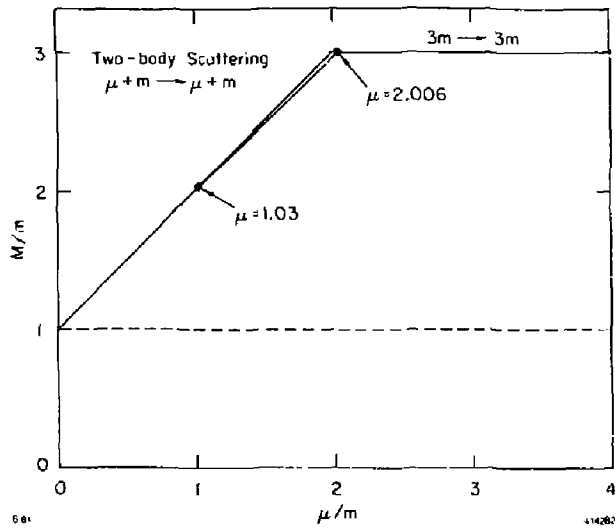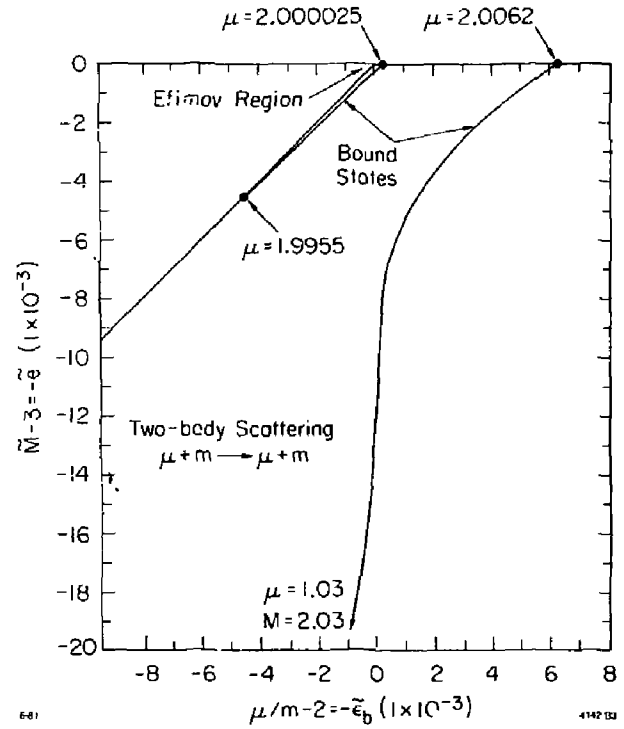
FIG. III-2a

FIG. III-2b

next be examined. With a given initial condition, there are three possible outcomes for the final state, as indicated in Fig. III-3. The first situation represents elastic scattering, and the others represent rearrangement.

The case of particles with identical kinematic and dynamical parameters will be examined in detail.

1. Form of Equations

Below three-body breakup threshold, the kernels $R_{ab}^{J=0}$ can be expressed in the form given in Eq. (3.2). The scattering equations for the amplitude $\hat{W}_{ab}^{J=0}$ from Eq. (2.72b) can be expressed

$$\hat{W}_{ab}^{J=0}(k_a|k_{b0}; M) = -2\pi \bar{T}_{ab} R(k_a, k_{b0}; M)$$

(3.18)

$$-2\pi \sum_c \bar{T}_{ac} \int_c \frac{M^2-m^2}{2M} dk_c' \frac{k_c'^2 \sqrt{s_c'} R(k_a, k_c'; M)}{r_c' \left[\sqrt{m^2 - \frac{\mu^2}{4}} - \sqrt{m^2 - \frac{s_c'}{4}} - io \right]} \hat{W}_{cb}^{J=0}(k_c|k_{b0}; M)$$

Since the masses m and μ are the same for all channels, there will only be two amplitudes; a direct (or elastic) amplitude, and a rearrangement (or reaction) amplitude:

$$\hat{W}_{aa}^{J=0}(k_a|k_{ao}; M) \equiv W_D(k_a|k_{ao}; M)$$

(3.19)

$$\hat{W}_{a+a}(k|k_{ao}; M) = \hat{W}_{a-a}(k|k_{ao}; M) \equiv W_R(k|k_{ao}; M)$$

Using these amplitudes, the integral equation (3.18) can be discretized into a matrix relation and inverted using elementary linear algebraic techniques
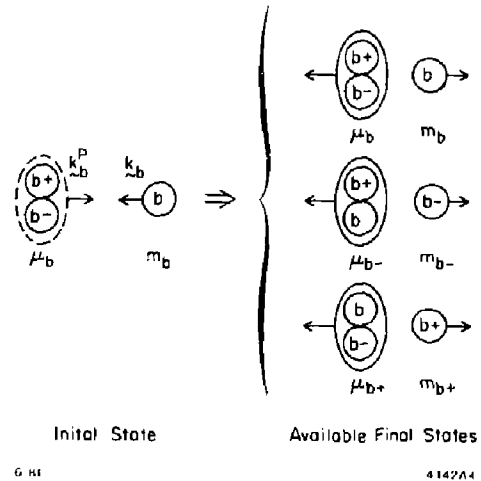

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FIG. III-3

$$\hat{W}_{ab} = \bar{\delta}_{ab} W_{ab}^{(s)} + \sum_c \bar{\delta}_{ac} K_{ac} \hat{W}_{cb}$$

$$W_D = 2K(1 - K - 2K^2)^{-1} W^{(s)} \qquad (3.20)$$

$$W_R = (1 - K - 2K^2)^{-1} W^{(s)}$$

After inversion, the amplitudes W_D and W_R can be related to physical observables as developed in Sec. II-B. The following relations pertain to the specific problem being developed:

$$\lim_{Z \to c_{ao} + c_{\mu_a o} + io} (Z - c_{ao} - c_{\mu_a o})$$

$$\times \langle k_1 \epsilon_1; \ k_2 \epsilon_2; \ k_3 \epsilon_3 | \delta^3(P) t_a(Z,0) | k_{10} \epsilon_{10}; \ k_{20} \epsilon_{20}; \ k_{30} \epsilon_{30} \rangle$$

$$= \epsilon_a \delta^3(k_a - k_{ao}) \epsilon_{\mu_a} \delta^3(k_{a+} + k_{a-} + k_{ao}) \frac{4|g|^2 \mu \sqrt{m^2 - \frac{\mu^2}{4}}}{\epsilon_\mu^2}$$

$$\lim_{Z \to c_a + c_{\mu_a} + io} (Z - c_a - c_\mu) \frac{g_a(k_{a+} k_{a-})}{D_a(s_a)} = \frac{4 g \nu \sqrt{m^2 - \frac{\mu^2}{4}}}{\epsilon_\mu} \qquad (3.21)$$

With these relations, one refers to Eq. (2.52b), (2.55b), and (2.51) to relate the calculated quantities to observables.

$$\langle \psi_a; \ k_a \epsilon_a; \ \psi_a(-k_a \epsilon_{\mu_a}) | A_{ab}^{(+)} (\epsilon_{bo} + \epsilon_{\mu_b o}, 0) | \psi_b; \ k_{bo} \epsilon_{bo}; \ \psi_b(-k_{bo} \epsilon_{\mu_b o}) \rangle$$

$$= \left[4\mu \sqrt{m^2 - \frac{\mu^2}{4}} \right]^{1/2} \hat{W}_{ab}(k_a | k_b; M) \left[4\mu \sqrt{m^2 - \frac{\mu^2}{4}} \right]^{1/2} \qquad (3.22)$$

$$\equiv A_{ab}^{(+)} (k_a | k_{bo}; \ \epsilon_{bo} + \epsilon_{\mu_b o}, \mu)$$

In addition, the on-shell unitarity of the operator S in Eq. (2.17) and (2.18) allows the amplitudes to be directly related to cross sections. Written in terms of $A_{\alpha\beta}$, the unitarity condition can be expressed

$$A_{\alpha\beta}^{(+)} (\vec{P}_{(0)}) - \left[A_{\alpha\beta}^{(+)} (\vec{P}_{(0)}) \right]^\dagger - \sum_\gamma \left[A_{\gamma\alpha}^{(+)} (\vec{P}_{(0)}) \right]^\dagger 2\pi i \delta^4 (\vec{P}_\gamma - \vec{P}_{(0)}) A_{\gamma\beta}^{(+)} (\vec{P}_{(0)}) \qquad (3.23)$$

By examining the "forward" amplitudes and the expression for the total cross section (2.60), the "optical theorem" follows immediately:

$$\sigma_{\text{total}} = \frac{(2\pi)^3 2 \text{Im} \langle \phi_\beta; \text{ initial} | A_{\beta\beta}^{(+)} (\vec{P}_{(0)}) | \phi_\beta; \text{ initial} \rangle}{\left[\left(\vec{k}_{(1)0} \cdot \vec{k}_{(2)0} \right)^2 - \left(\vec{k}_{(1)0} \cdot \vec{k}_{(1)0} \right) \left(\vec{k}_{(2)0} \cdot \vec{k}_{(2)0} \right) \right]^{1/2}} \qquad (3.24)$$

In 3-CMS, this can explicitly be expressed:

$$\sigma_{\text{total}} = (2\pi)^3 \frac{2}{k_{bo}^2 M}$$

$$\times \text{Im} \langle \phi_b; \ k_{bo} \epsilon_{bo}; \ \psi_b(-k_{bo} \epsilon_{\mu_b o}) | A_{bb}^{(+)} (\epsilon_{bo} + \epsilon_{\mu_b o}, 0) | \phi_b; \ k_{bo} \epsilon_{bo}; \ \psi_b(-k_{bo} \epsilon_{\mu_b o}) \rangle \qquad (3.25)$$

Consider the angular momentum decomposition of the amplitudes. The "partial wave amplitudes" can be related to the calculated quantities using

$$A_{ab}^{(+)} (k_a | k_{bo}; \ \epsilon_{bo} + \epsilon_{\mu_b o}, \mu) \equiv \sum_J \frac{2J+1}{4\pi} P_J(\hat{k}_a \cdot \hat{k}_{bo}) A_{ab}^J (k_a | k_{bo}; \ \epsilon_{\mu_b o} + \epsilon_{bo}, \mu)$$

$$A_{ab}^J (k_a | k_b; \ \epsilon_{bo} + \epsilon_{\mu_b o}, \mu) = -4\nu \sqrt{m^2 - \frac{\mu^2}{4}} \ \hat{W}_{ab}^J (k_a | k_{bo}; \ \epsilon_{bo} + \epsilon_{\mu_b o}) \qquad (3.26)$$

The condition (3.23) can be expressed in terms of these angular momentum components

$$\text{Im}\, A^J_{bb}(k_b|k_b;\ M) = \sum_\gamma \frac{\pi k_\gamma}{M}\ |A^J_{\gamma b}(k'_\gamma|k_b;\ M)|^2 \tag{3.27}$$

where

$$k_\gamma^2 = \frac{\left[M^2 - (\mu_\gamma + m_\gamma)^2\right]\left[M^2 - (\mu_\gamma - m_\gamma)^2\right]}{4M^2}$$

This allows the definition of the standard phase shift and absorption parameter in the elastic channel by

$$A^J_{bb}(k_b|k_b;\ M) = \frac{M}{2\pi i k_b}\left[\eta^J_b(M,\mu_b)e^{2i\delta^J_b(M,\mu_b)} - 1\right] \equiv \frac{M}{\pi k_b}\ f^J_b(M,\mu_b) \tag{3.28}$$

To calculate these parameters, the singularity structure of the two-body input $D^{-1}_a(s_a)$ must be properly understood. By taking advantage of the relation

$$\frac{1}{x - i\eta}\ \xrightarrow[\eta\to 0]{}\ \frac{\mathscr{P}}{x} + i\pi\delta(x) \tag{3.29}$$

where the symbol $\mathscr{P}$ represents the principal value, the function can be written as follows

$$\tilde{D}_a\frac{1}{(s_a)} = \tilde{D}_a\frac{\mathscr{P}}{(s_a)} - i\pi\left[\frac{4\mu\sqrt{m^2 - \frac{\mu^2}{4}}\ (M^2 + m^2 - \mu^2)}{M\sqrt{(M^2 + m^2 - \mu^2)^2 - 4m^2 M^2}}\right]\delta(k_a - K_a) \tag{3.30}$$

where

$$K_a = \sqrt{\frac{(M^2 + m^2 - \mu^2)^2 - 4m^2 M^2}{4M^2}}$$

The numerical solutions for some of the observables will be examined for various values of μ and M.

2. Numerical Results

The phase shifts and inelasticity parameters consistent with Eq. (3.28) have been calculated. The integral equations were discretized using

Gaussian quadratures, with Legendre polynomials as weight functions. The functions $f^{J=0}_b(M,\mu_b)$ are plotted (as Argand diagrams) in Fig. III-4. The rest energy of the system corresponds to $M = \mu + m$, so that the relative kinetic energy is given by

$$e_K = M - (\mu + m) \tag{3.31}$$

Figure III-5 illustrates the total cross section for $J = 0$ in units of m^2, such that

$$\sigma_{total} = \sum_J \sigma^J_{tot} \tag{3.32}$$

It can be noted that the lowest resonance is more sharply peaked as the peak energy approaches the rest energy of the system ($e_K = 0$). In addition, in the regions where the resonance is well defined, it follows a path which is a reflection of the lowest energy three-body bound state about the line $e_K = 0$. From the diagrams, it is apparent that the resonance structure for scattering from the ultra-relativistic bound pairs is influenced considerably by the inelastic (i.e., rearrangement) processes. This structure will be extended into the breakup region ($M > 3m$) in Section III-C.

C. Break-up Scattering

The process of breakup can occur if the available center-of-momentum energy is greater than the sum of the rest masses of the constituent particle ($M \geq 3m$). The initial system will be described by a particle b scattering from a bound pair. Figure III-6 depicts the possible asymptotic states.

As viewed from the elastic scattering "channel," the possibility of breakup will open an additional inelastic "channel."
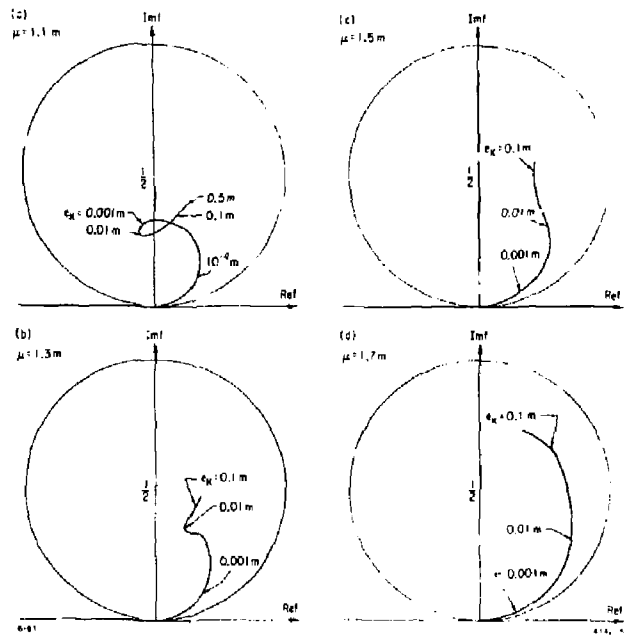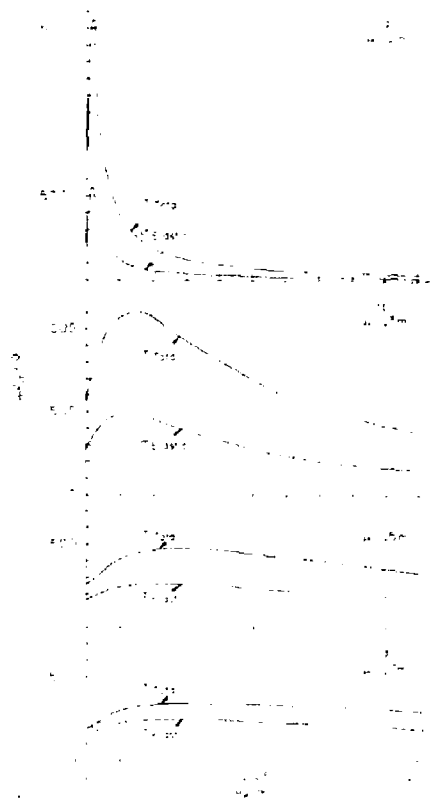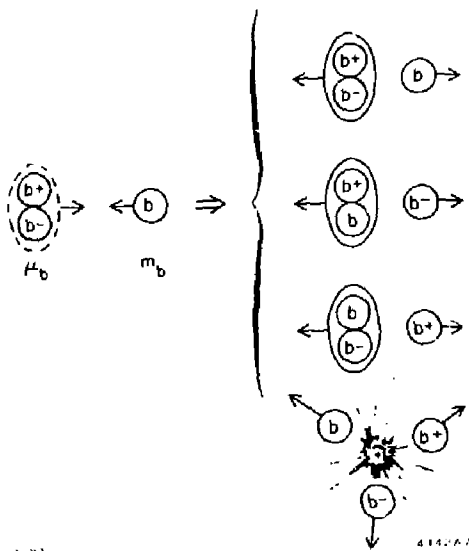
FIG. III-4

FIG. III-5

FIG. III-6

1. Form of Equations

The particular equations for this process can be obtained in a straightforward way as described in Section III-B. The amplitudes for breakup from an initial channel b, as well as for three particle-to-three-particle scattering, are summarized below.

$$\langle \phi_0 : (\underline{k}_1 c_1; \underline{k}_2 c_2; \underline{k}_3 c_3), (M_b, 0) | A_{ob}^{(+)} (M_b, 0) | \phi_b \cdot \underline{k}_{bo} c_{bo}; \psi_b (-\underline{k}_{bo} \varepsilon_{bho}) \rangle$$

$$= \sum_a - \frac{g_a(\underline{k}_a + \underline{k}_a)}{D_a(s_a)} \hat{W}_{ab}(\underline{k}_a | \underline{k}_{bo}; M_b) \left[4\mu \sqrt{m^2 - \frac{\mu^2}{4}} \right]^{1/2} \quad \text{where } N_R = \varepsilon_{bo} + \varepsilon_{\mu b o}$$

$$\langle \phi_o : (\underline{k}_1 c_1; \underline{k}_2 c_2; \underline{k}_3 c_3), (M,0) | A_{oo}^{(+)} (M, \ell) | \phi_o : (\underline{k}_{10} c_{10}; \underline{k}_{20} c_{20}; \underline{k}_{30} \varepsilon_{30}), (M,0) \rangle$$

$$= \sum_{ab} - \frac{g_a(\underline{k}_a - \underline{k}_a)}{D_a(s_a)} \hat{W}_{ab}(\underline{k}_a | \underline{k}_{bo}; M) \frac{g_b(\underline{k}_{bo} \underline{k}_{b-o})}{D_b(s_b)} \tag{3.33}$$

where the function g_a and D_a are described by Eq. (2.77).

Above breakup threshold, the singularity structure of the non-interacting resolvant must be properly handled. The singularity occurs only for $M > 3$, and within a limited range of the parameters k_a and k_b. This range is given by Eq. (3.34).

$$0 \le k_a \le \sqrt{\frac{(M^2 - 9m^2)(M^2 - m^2)}{4M^2}} \equiv k_{amax}$$

$$k_{bmin} \le k_b \le k_{bmax} \tag{3.34}$$

where

$$k_{\substack{bmax \\ min}} \equiv \frac{1}{2} \left| k_a \pm (M - \varepsilon_o) \sqrt{\frac{M^2 - 3m^2 - 2\varepsilon_a M}{M^2 + m^2 - 2\varepsilon_a M}} \right|$$

The singularity takes the form of Eq. (3.29). Thus the kernel $R_{ab}^{J=0}$ in Eq. (2.72a) can be expressed

$$R_{ab}^{J=0}(k_a,k_b; M) = \frac{g_a^* g_b}{k_a k_b} \left[\pi i \Theta(k_{amax} - k_a)\Theta(k_{bmax} - k_b)\Theta(k_b - k_{bmin}) \right.$$

$$\left. + \mathscr{G} \log \left| \frac{\sqrt{m_{ab}^2 + (k_a + k_b)^2} + \varepsilon_a + \varepsilon_b - M}{\sqrt{m_{ab}^2 + (k_a - k_b)^2} + \varepsilon_a + \varepsilon_b - M} \right| \right] \qquad (3.35)$$

The solutions of (2.72b) using the kernel (3.35) will be examined for
$J = 0$.

2. Numerical Solutions

The numerical treatment of the equations was similar to that developed
in Section III-B. The solutions smoothly matched those below breakup, and
required increasing numerical work as the energy increased.

The behavior of the cross sections beyond the resonance regions is
demonstrated in Fig. III-7. The Argand diagrams exhibited minor variance
beyond the regions covered in Fig. III-4.

For completeness, the solution for the moderately relativistically
bound state μ = 1.9m is demonstrated in Fig. III-8. In this figure the
region above and below breakup threshold is demonstrated on the single
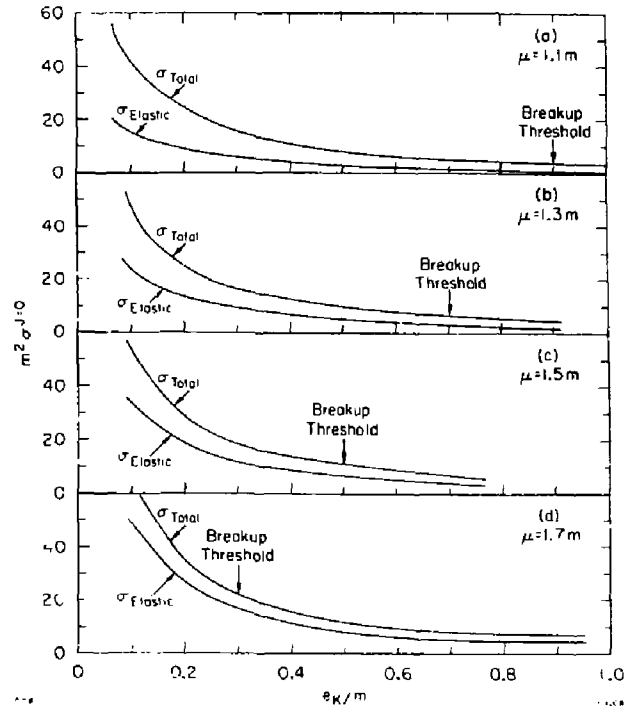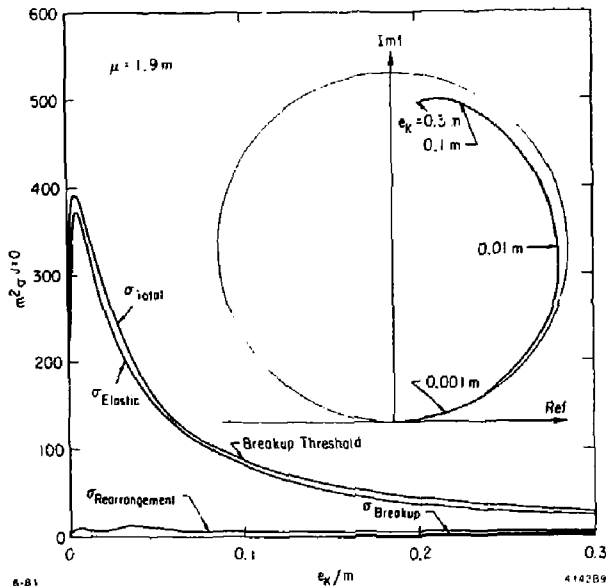graphs.

FIG. III-7

FIG. III-8

CHAPTER IV

CONCLUSIONS

The equations explored define a self-consistent, unitary set of scattering equations which give stable solutions in this model. It should be noted that the equations in the form given are most suited numerically to the relativistic regime, although the non-relativistic limit to the equations is well defined. In the form explored, the equations correctly describe results predicted from non-relativistic models if the parameters involved are related.

The formalism explored in Chapter II generates eigenstates of a fully interacting three-body system in terms of boundary states in a covariant way. These states satisfy a type of cluster form invariance if one of the particles does not interact. Internal angular momentum can be included in the formalism in a straightforward way.

Since in the model examined the equations reduce to a single parameter integral equation, the numerical methods involved in this exploration were straightforward. Advanced numerical techniques exist in the literature which allow exploration of the amplitudes involved in a more complex model. However, in order to more reasonably reproduce the high energy phenomenology, the inclusion of particle-antiparticle symmetries and multiparticle processes must be examined in the formalism.

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