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Non-Static Vacuum Strings : Exterior and Interior Solutions

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ABSTRACT

New non-static cylindrically symmetric solutions of Einsteins's equations are presented. Some of these solutions represent string-like objects. An exterior vacuum solution is matched to a non-vacuum interior solution for different forms of the energy-momentum tensor. They generalize the standard static string. *

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I. Introduction

Grand Unified Theories were first suggested as a possible scenario for the unification of all forces except gravity. It was soon realized that for these theories to work, the grand symmetry presented in the early universe had to be somehow broken in order to account for the electroweak and nuclear forces as different fields in the low energy universe that we observe today. One of the immediate and inescapable consequences of this breaking of the symmetry, is the formation of topological defects in the Higgs fields. These "knots" are simply the consequence of mismatches in the Higgs field orientation between regions that where not causally connected initially but come into contact later on, trapping regions of the false vacuum. These defects can be classified into three disjoint classes [1] according to the topological properties of the group describing them. They can be magnetic monopoles, cosmic string or domain walls.

Production and the cosmological consequences of these objects have been thoroughly investigated by many authors[2]. It has been shown [3] that the existence of walls is incompatible with present day observations of the microwave background radiation. Monopole production has been calculated in several simple models [4] giving an unacceptable number density. However, when more complicated cosmological models are used then overproduction of monopoles can be reduced by a considerable amount[5]. Furthermore, if inflation takes place in the early universe then the monopole problem disappears completely[6]. Vacuum strings, on the other hand, are much more difficult to rule out. In fact it has been shown[1] that it is possible to accomodate a network of stable strings and still explain observations of the microwave background radiation. Furthermore, cosmic strings might even explain galaxy formation and the large scale structure of the universe.

There is, at least in principle, one observational consequence of their existance. Strings act as gravitational lenses, producing focussing and even multiple images of luminous sources in the Universe. This effect was first noted by Vilenkin [7] and Gott and Alpert [8], and later used by Gott [9] to explain some observations made by Paczynski and Gorki [10] of what is belived to be a triple QSO, as just being the split image of a single object.

All these discussions have been based on the model of a string as a vacuum, static exterior and non-vacuum interior solution of Einstein's equations with cylindrical symmetry. The interior solution descibes a fluid of constant density ρ and constant pressure P_z along the direction of the string, with the property that $\rho = -P_z$. The two solutions are then matched at the boundary of the string. The purpose of this paper is to present a new 'stringy' solution and some byproducts of this. In our case the string energy density is a dynamical variable $\rho = \rho(r,t)$ with r the cylindrical radius. In section II we will give the metric, the field equations, their solutions and the matching conditions. Finally, some brief conclusions will be drawn.

II. The Field Equations

We will start with a metric describing a cylindrically symmetric space-time, with two killing vectors ∂_z and ∂_ϕ,

$$ds^2 = A^2(r)(dt^2 - B^2(t)dz^2) - D^2(t)(dr^2 + G^2(r)d\phi^2) \tag{1}$$

where r, ϕ and z are cylindrical coordinates defined in the range $0 \leq r \leq \infty$; $0 \leq \phi \leq 2\pi$; $-\infty \leq z \leq \infty$; $t \geq 0$. The Einstein field equations will be given (in units where $G = c = \hbar = 1$) by,

$$R_\mu^\nu = T_\mu^\nu - \frac{1}{2}\delta_\mu^\nu T \tag{2}$$

The energy-momentum tensor for the exterior solution will be $T^{\nu}_{\mu(ext)} = 0$, while for the interior solution it will have the form prescribed in ref.(7) (also see ref. (11) and (12)),

$$T^{\nu}_{\mu(int)} = \mu(t) f(r) diag(1,0,0,1) \tag{3}$$

The energy-momentum tensor reduces to the standard one if we take $\mu = \mu_0$. Now we can write the field equations as,

$$\frac{\ddot{B}}{A^2 B} - \frac{A'^2}{A^2 D^2} + 2\frac{\ddot{D}}{A^2 D} - \frac{A''}{AD^2} - \frac{A'G'}{AGD^2} = 0 \tag{4}$$

$$\frac{A'\dot{D}}{A^3 D} = 0 \tag{5}$$

$$\frac{\ddot{D}}{A^2 D} + \frac{\dot{D}^2}{D^2 A^2} + \frac{\dot{D}\dot{B}}{DBA^2} - 2\frac{A''}{AD^2} - \frac{G''}{GD^2} = T^0_0 \tag{6}$$

$$\frac{\ddot{D}}{A^2 D} + \frac{\dot{D}^2}{D^2 A^2} + \frac{\dot{D}\dot{B}}{DBA^2} - 2\frac{A'G'}{AGD^2} - \frac{G''}{GD^2} = T^0_0 \tag{7}$$

$$\frac{\ddot{B}}{A^2 B} + 2\frac{\dot{B}\dot{D}}{BDA^2} - \frac{A'^2}{A^2 D^2} - \frac{A''}{AD^2} - \frac{A'G'}{AGD^2} = 0 \tag{8}$$

Eq.(4) can immediately be integrated. It contains three different cases ($\cdot \equiv \frac{\partial}{\partial t}$,$' \equiv \frac{\partial}{\partial r}$),

$$i) \quad A' = 0 \Longrightarrow A = const.$$

$$ii) \quad \dot{D} = 0 \Longrightarrow D = const.$$

$$iii) \quad A' = 0 = \dot{D} \Longrightarrow A, D = const.$$

We will study each case separatly.

i) In this case we will take, without lose of generality $A = 1$. Then the field equations become,

$$\frac{\ddot{B}}{B} + 2\frac{\ddot{D}}{D} = 0 \tag{9}$$

$$\frac{\ddot{D}}{D} + \frac{\dot{D}^2}{D^2} + \frac{\dot{D}\dot{B}}{DB} - \frac{G''}{GD^2} = T_0^0 \tag{10}$$

$$\frac{\ddot{B}}{B} + 2\frac{\dot{B}\dot{D}}{BD} = 0 \tag{11}$$

Eqs.(9) and (11) can be integrated simultaneously for the exterior and interior cases as they are independent of the energy-momentum tensor. The solution is

$$B = B_0\dot{D} \tag{12}$$

$$\dot{D}^2 = \kappa_1 - \frac{2}{D} \tag{13}$$

It is clear from (13) that for $\kappa_1 > 0$ the solution is given by

$$D = \frac{1}{\kappa_1}(1 + cosh\psi) \tag{14.1}$$

$$t = t_0 + \frac{1}{\kappa_1^{\frac{3}{2}}}(\psi + sinh\psi) \tag{14.2}$$

For the exterior solution we can take $T_0^0 = 0$ in which case we get from (10)

$$D\ddot{D} + \dot{D}^2 + \frac{\dot{B}}{B} D\dot{D} = \kappa_2 = \frac{G''}{G} \tag{15}$$

where κ_2 is an arbitrary constant. The left hand side of this triple equality becomes a constraint on the solutions for D and B, which can be easily satisfied provided we choose $\kappa_1 = \kappa_2$, while the right hand side just yields

$$G'' - \kappa_1 G = 0 \tag{16}$$

Since $\kappa_1 > 0$ the solution of this equation is given by,

$$G(r) = C_1 e^{\sqrt{\kappa_1} r} + C_2 e^{-\sqrt{\kappa_1} r} \tag{17}$$

With C_1 and C_2 integration constants. In order to find an interior solution we have to specify the form of ${T_0}^0$. The continuity equation $T^{\mu\nu}_{;\nu} = 0$ together with eq.(10) suggest

$$T^0_0 = \frac{f(r)}{D^2}$$

Then eq.(10) becomes a separable equation in its radial and time dependent parts, so obtaining,

$$D\ddot{D} + \dot{D}^2 + \frac{\dot{B}}{B} D\dot{D} = \kappa_3 = \frac{G''}{G} + f(r) \tag{18}$$

where $f(r)$ is an arbitrary function of r. The first equality is satisfied by the same B and D found previously provided we take $\kappa_3 = \kappa_1$. However, the right hand side now gives a different equation, namely;

$$G'' + (f(r) - \kappa_1) G = 0 \tag{19}$$

This equation is identical to a Schrődinger equation describing a free particle moving in a potential $f(r)$ with energy κ_1. Clearly, to solve this equation we need to specify the form of $f(r)$. For the solution of (19) to be valid we require $G(r)$ and $G'(r)$ to be continuous at the boundary of the string, i.e.

$$G_{ext}(r_*) = G_{int}(r_*) \tag{20}$$

$$G'_{ext}(r_*) = G'_{int}(r_*) \tag{21}$$

This implies that any solution of the one-dimensional Schrődinger equation will be a valid solution of (19) (we are not establishing any physical relation between these two systems but merely pointing out the mathematical symilarity). Before attempting to solve (19) for a specific potential, let us examine the form of the metric and the matching conditions.

With the help of eq.(12) the exterior and interior metrics can be put in the following form,

$$ds^2 = dt^2 - D^2(t)(dr^2 + G^2(r)d\phi^2) - \dot{D}^2(t)dz^2) \tag{22}$$

where $D(t)$ is given by eq.(14) and $G(r)$ by eq.(17) for the exterior and by eq.(19) for the interior. Since the coordinate systems used to describe the exterior and the interior are the same, then the matching conditions are given by eqs.(20) and (21). $G(r)$ and $G'(r)$ must be continuous at the boundary. We only need to choosea sensible 'potential' for which eq.(19) has a solution consistent with the matching conditions.

We will now analyse in more detail a specific potential. If we take $f(r) = f_0$ for the interior solution, then (19) has the following solution,

$$G(r) = \tilde{C}_1 e^{\sqrt{\alpha} r} + \tilde{C}_2 e^{-\sqrt{\alpha} r} \; ; \; \alpha > 0 \tag{23.1}$$

$$G(r) = \bar{C}_1 cos(\sqrt{\alpha} r) + \bar{C}_2 sin(\sqrt{\alpha} r) \; ; \; \alpha < 0 \tag{23.2}$$

with $\alpha \equiv f_0 - \kappa_1$. We shall now concentrate on the second case and will take $\bar{C}_1 = 0$. Then the metric can be written as,

$$ds^2 = dt^2 - D^2(t)(dr^2 + \bar{C}_2 sin^2(\sqrt{\alpha} r) d\phi^2) - \dot{D}^2(t) dz^2 \tag{24}$$

For every $t = const.$ hypersurface it is possible to rescale the coordinates so as to get,

$$ds^2 = -\left[d\tilde{r}^2 + C_0^2 sin^2 \left(\frac{\tilde{r}}{r_0} \right) d\phi^2 + d\tilde{z}^2 \right] \tag{25}$$

This slice is identical to the equivalent slice of the standard string [7]. The main difference is that for each hypersurface we need to rescale differently in order to obtain the standard string. In this solution the interior is a non-static space-time

For the exterior solution let $C_1 = -C_2 \equiv \frac{1}{2} C$ then,

$$G(r) = C sinh(\sqrt{\kappa_1} r) \tag{26}$$

Near the string wall (r small) the metric can be approximated by

$$ds^2 = dt^2 - D^2(t)(dr^2 + C^2 r^2 d\phi^2) - \dot{D}^2(t) dz^2 \tag{27}$$

and again a $t = const.$ slice reduces to the equivalent slice of the static exterior solution.

The analysis of the stability of this solution, and those obtained for more complicated 'potentials' is underway and will be reported elsewhere. However, we might speculate that these types of solution will be unstable for any potential other than $f(r) = const.$ or

$f(r) = \delta(r)$, because if the density distribution is non-homogeneous then different parts of the string (cross-section) would feel different tensions and this would tend to disrupt the string. There might be some gravitational radiation coming from this configuration as it settles into a lower energy configuration (maybe the static case). There could even be a ' Birkhoff-like' theorem for cylindrically symmetric solutions.

ii) In this case we shall take D = 1. Now the field equations read

$$\frac{A''}{A} + \frac{A'^2}{A^2} + \frac{A'G'}{AG} - \frac{\ddot{B}}{A^2 B} = 0 \tag{29}$$

$$2\frac{A''}{A} + \frac{G''}{G} = -T_0^0 \tag{30}$$

$$2\frac{A'G'}{AG} + \frac{G''}{G} = -T_0^0 \tag{31}$$

If we now substitute (30) into (31) we get,

$$\frac{A''}{A} - \frac{A'G'}{AG} = 0 \tag{32}$$

which can be easily integrate to give

$$G = G_0 A' \tag{33}$$

For the exterior solution we take as before $T_0^0 = 0$ and using (30) and (33) we arrive at

$$\frac{A'''}{A} + 2\frac{A''}{A} + = 0 \tag{34}$$

Which immediately integrates to give

$$A(r) = \frac{1}{4}(2C_2 - C_1) + \frac{C_1}{2}r + C_3 e^{-2r} \tag{35}$$

$$G(r) = G_0(\frac{C_1}{2} - 2C_3 e^{-2r}) \tag{36}$$

Eq. (31) is identically satisfied, while (29) now yields the triple equality

$$\left(\frac{A''}{A} + \frac{A'G'}{AG} + \frac{A'^2}{A^2}\right) A^2 = \kappa_1 = \frac{\ddot{B}}{B} \tag{37}$$

The right hand side of (37) again becomes a constraint on the form of A(r), which can be satisfied provided we take $C_3 = 0$ and $C_1 = 2\kappa_1{}^{\frac{1}{2}}$. We can then write A(r) as

$$A(r) = A_0 + \kappa_1{}^{\frac{1}{2}} r \tag{38}$$

with $A_0 = \frac{1}{2}(C_2 - \kappa_1{}^{\frac{1}{2}})$. The left hand side of (37) yields the following equation for B(t),

$$\ddot{B} - \kappa_1 B = 0 \tag{39}$$

which has the same (formal) solution as (17) if we exchange G $\rightarrow$ B and r $\rightarrow$ t.

For the interior case, the solution described by eq.(33) is still valid, and when substituted into (29), for the case $f(r) = f_0$ we get,

$$\frac{A'''}{A} + 2\frac{A''}{A} = -T_0^0 = -f_0 \equiv -\frac{2g_0}{27} \tag{40}$$

with g_0 a new constant defined in terms of f_0. The general solution to eq.(27) is,

$$A(r) = C_1 e^{m_1 r} + e^{ar}\left[C_2 cos(br) + C_3 sin(br)\right] \tag{41}$$

with

$$b = \frac{1}{2\sqrt{3}}\left\{\left[-(g_0+1) + {g_0}^{\frac{1}{2}}(g_0+2)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[(g_0+1) + {g_0}^{\frac{1}{2}}(g_0+2)^{\frac{1}{2}}\right]^{\frac{1}{3}}\right\}$$

$$m_1 = \frac{1}{3}\left\{\left[(g_0+1) - {g_0}^{\frac{1}{2}}(g_0+2)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[(g_0+1) + {g_0}^{\frac{1}{2}}(g_0+2)^{\frac{1}{2}}\right]^{\frac{1}{3}} + 1\right\}$$

For small r the solution for A(r) can be approximated by

$$A(r) \simeq C_1 + bC_3 r \tag{42}$$

for which the matching conditions are satisfied provided we take $bC_3 = {\kappa_1}^{\frac{1}{2}}$ and $C_1 = A_0$. Now we can rewrite the metric in the form

$$ds^2 = A^2(r)(dt^2 - B^2(t)dz^2) - dr^2 - G_0^2 A'^2(r) d\phi^2 \tag{43}$$

This case does not reduce to the static string case, and even though it has the right Killing vectors and both exterior and interior solutions can be found and matched, it is extremely difficult to give any interpretation to this space-time.

iii) In this case we take A = 1 = D. For this case the field equations become

$$\frac{\ddot{B}}{B} = 0 \tag{44}$$

$$\frac{G''}{G} = -T_0^0 \tag{45}$$

For the exterior solution we get

$$B = B_0 t + B_1 \tag{46}$$

$$G = G_0 r + G_1 \tag{47}$$

where B_0, B_1, G_0 and G_1 are integration constants. We shall consider only one, namely, $G_1 = 0$. Then the metric becomes

$$ds^2 = dt^2 - (B_0 t + B_1)^2 dz^2 - (dr^2 + G_0^2 r^2 d\phi^2) \tag{48}$$

B(t) for the interior is given by eq (46), while G(r) is the solution to

$$G'' + f(r)G = 0 \tag{49}$$

For the special case we are interested, namely $f(r) = f_0$ we get

$$G(r) = G_1 e^{\sqrt{f_0} r} + G_2 e^{-\sqrt{f_0} r} \tag{50}$$

If we take $G_1 = G_2 \equiv \frac{1}{2} G_1$ then for small r we get

$$ds^2 = dt^2 - (B_0 t + B_1)^2 dz^2 - (dr^2 + G_1^2 f_0^2 r^2 d\phi^2) \tag{51}$$

The matching of these solutions is achived provided we take $G_1 f_0 = G_0$. Two particular cases are worth mentioning. If $B_0 = 0$ then the solution becomes the standard static string described in the literature. When $B_1 = 0$ we get a slightly different metric which describes a string being stretched by the horizon as t increases, and whose cross-section remains the same for every t-hypersurface.

III. Conclusions

In this paper we have presented some new cylindrically symmetric solutions of Einstein's field equations. Some of these solutions represent string-like objects for which an interior non-vacuum solution was matched to an exterior vacuum solution. We found that the solutions naturally falls into three classes depending on whether A(r) or D(t) or both are constants. The static string solution described by Vilenkin[7] and Gott[8] is a particular case of our solution. The non-static behaviour of the solution could changed some of the results obtained using static strings. The lensing property, for one, will probably be modified as the amplification or reduction of the amplitud due to focussing or defocussing of bundles of light rays coming from the source are changed.

As mentioned earlier, the analysis of the stability of these solutons is still underway. However, the solution where the density distribution is a delta function or a constant ouhgt to be stable.

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