THE INFERRED REFERENDUM — A RULE FOR COMMITTEE DECISIONS

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ABSTRACT

We present a new method of social choice. The result of our
method coincides with that of majority voting when it does
not produce an intransitivity among the alternatives under
consideration. When majority voting would produce an intran-
sitivity, our method orders the alternatives in the same way
as the transitive constituency would whom the committee
members are most likely to represent. Analysis of the appli-
cation of our method to three alternatives shows that a) the
resulting order depends only on the committee members' votes
between pairs of alternatives b) the resulting order is less
sensitive to irrelevant alternatives than the orders provided
by other schemes c) when majority voting provides an intran-
sitivity, the hypothesis that, in fact, the committee's
constituency is as we assume it to be is almost as likely as
the hypothesis that it precisely mirrors the committee.

I. INTRODUCTION

This paper considers a problem that has interested a substantial number of economists and political scientists. It is this: given rankings of three or more alternatives by each of several voters, find a single "satisfactory" ranking. The problem is known as that of the amalgamation of preferences or social choice or the design of constitutions. However, it is known, since 1951 all work on the problem has been influenced by Kenneth Arrow's (Arrow 1951, 1963). His proof that there is no method of ranking that simultaneously satisfies four natural and apparently easily satisfied criteria has been the inspiration for the style and direction of subsequent work by others. This work has advanced our understanding of the problem by constructing an impressive series of elaborations and alternative formulations of Arrow's conditions and by demonstrating a concomitant series of impossibility theorems. We believe that it is now appropriate to consider the design of "second best" amalgamation procedures; that is procedures that do not *unnecessarily* conflict with the usual desirable principles or that do so only in order to implement another desirable principle. The principle that motivates the amalgamation procedure described in this paper is honored and old. It is that the voters (e.g. legislators) represent a much larger constituency and it is from this that the voters derive their authority.

Before we describe our procedure, we hope to invite others to such efforts by sharing the reasons that persuade us that the design of amalgamation procedures may become as interesting to society at large as other prescriptive efforts by economists and political scientists. First, technical advances in portable electronic data processing and its tremendous decline in

price (e.g. programmable pocket calculators, computer terminals) have reduced
the cost and actual time of performing sophisticated mathematical operations
below that previously required to simply count votes between two alternatives.
Thus, practical amalgamation procedures need no longer be restricted to
addition and subtraction. Second, radio and television provide the means by
which a single individual or institution can rapidly convey a series of
alternatives to its constituency and telephone lines provide the means for a
prompt response by the constitutency. Thus, the combination of inexpensive
and rapid communication and data processing make the timely amalgamation of
opinion economically feasible. Third, there is already a substantial market
for the information gathered by opinion polls of various sorts. Amalgamation
procedures can be viewed as summary indicators of their results and as such
may be considerable interest to the polls' users. The design of useful
indicators involves the "correct interpretation" of the poll and this issue
will influence and can be influenced by formal work on amalgamation procedures.
Fourth, the management of some organizations (e.g. Common Cause) already feels
obliged to consult its members about their priorities. It is not unreasonable
to imagine that other organizations such as political parties, honorific
bodies (e.g. National Academy of Sciences) and labor unions will feel some
pressure to take account a their members' or constituencies' priorities. It
is also interesting to speculate on the mechanism and impact of stockholders
reclaiming some of the power of ownership from the management of their firms
by requiring the management to take some account of stockholder priorities.
Of course, each of these cases is somewhat different from the others. The
formal articulation of these differences and the design of appropriate amalga-
mation procedures for each can provide a stimulus for finding what is possible
as well as impossible in the amalgamation of diverse priorities.

As we have said, the situation we consider in this paper is that of representative government. In the second section, we sketch the considerations that led us to the amalgamation scheme we present. The third section is devoted to its mathematical formulation. A closed form result for three alternatives and a comparison of this result with Borda voting is given in the fourth section. The last section of the paper contains a summary and some as yet unanswered questions.

II. MOTIVATION

In this section, we describe the motivation that leads to the amalgamation procedure that is presented in the next section. Let us consider a committee of several members that must rank three or more alternatives on behalf of a much larger constituency. The committee is supposed to derive its authority from the belief that it represents its constituency. Our problem is to design an amalgamation procedure by which the committee can combine the possibly diverse rankings of its members.

From the infinity of possible procedures we consider only those that satisfy the principles of unanimity and anonymity. The principle of unanimity is this: if each committee member ranks the alternatives in the same way as every other committee member then the committee ranking is the one that is unanimously held. There are, of course, no practical situations in which one would disregard this principle. The principle of anonymity is stronger. It requires that the committee's ranking depends only on the number and not the names of committee members who hold each of the possible rankings of the alternatives. This is the formal statement that each of the committee members has an equal influence on the decision and in particular that none is a dictator.

We come now to a different kind of principle, one that following Arrow everyone has called the independence of irrelevant alternatives. This principle requires that, no matter what rankings are held by committee members, the committee ranking among any subset of alternatives shall be independent of the presence or ranks of the remaining alternatives. In particular, whether one alternative is to be ranked above another should depend only on the relative ranks of the two alternatives by each of the committee members. Adherence to

this principle reduces the determination of the committee's ranking to a series of comparisons between pairs of alternatives. The principle of independence of irrelevant alternatives, unlike the principles of unanimity and anonymity which are formalizations of ethical beliefs, draws its strong appeal from practicalities because if it is not required one must ask which or how many other alternatives the committee must consider before it can make a judgement between any two alternatives. For this question, there is no á priori answer. Unfortunately, as is well known, Arrow's work showed that there is no amalgamation procedure that depends only on the committee members' rankings and that always satisfies the principles of independence of irrelevant alternatives, unanimity and non-dictatorship (á fortiori anonymity). The question of what to do in the absence of such a procedure remains.

In order to capture as much of the independence of irrelevant alternatives as possible while still retaining the principles of unanimity and anonymity the amalgamation procedure that we consider coincides with the method of majority decision (MMD) when this does provide an unambiguous order. MMD is simply the procedure that has the committee rank one alternative above another when the majority of its members do so. Of course, MMD does not always provide an ordering of the alternatives (if it did so, Arrow's Thm would be false). Instead, it sometimes produces an intransitivity (or circularity).

When we consider those situations in which the appplication of MMD to the committee members rankings would produce an intransitivity, we are motivated by the following thoughts. Since the committee derives its authority from the belief that it represents its constituency, one might hold a referendum among the constituency, using MMD, on the alternatives in question. While it is quite possible that the constituency itself will be intransitive, it is also quite possible that the constituency will not be so. Small samples do not always

faithfully reflect the populations from which they are drawn. Thus, the following two questions arise. What is the most likely transitive constituency (population) from which this intransitive committee (sample) was drawn? What is the difference between the likelihood that the committee represents this transitive constituency and the likelihood that the constituency is faithfully reflected in the committee's intransitivity? After some refinement, both of these questions can be mathematically posed and answered. In this way we can infer the results of a hypothetical referendum. If the answer to the second question is not too large, we suggest that the committee's ranking should be that of the transitive constituency which it most likely represents. How large is "too large"? This is a question that cannot be answered without a knowledge of the importance of the alternatives in question to the constituency that will be affected by the committee's ranking and thus it is a question that we will not attempt to answer abstractly. Some readers may feel that any intransitivity by the committee should be resolved by a real referendum among the constituency. We cannot agree because one of the most important reasons for the formation of any committee and the delegation of authority to it is the opportunity to pursue other matters that is thereby afforded to those who do not serve on the committee. To judge by the unwillingness of many to be actively involved in politics, this opportunity is highly valued.

Thus, the amalgamation procedure we propose is this: the committee's ranking of the alternatives is that of the most likely transitive constituency that it represents.

III.   MATHEMATICAL FORMULATION

In order to precisely formulate the method of the inferred referendum, we will introduce some notation and make some preliminary comments. Let A denote the number of alternatives. The number of possible permutations or rankings of the alternatives is A!. Let N_p denote the number of committee members whose ranking is the pth permutation of the alternatives. Let V denote the number of committee members. Since each committee member ranks the alternatives one knows that $\sum_{p=1}^{A!} N_p = V$. Let π_p denote the probability that a member of the constituency whom the committee represents would choose the pth permutation of the alternatives as his ranking. The probability that a committee of V members which represents its constituency will have the values N_1, N_2, . . . $N_{A!}$ is given by $V! \prod_{p=1}^{A!} \left\{ \dfrac{\pi_p^{N_p}}{N_p!} \right\}$.

The probability that any member of the constituency will prefer alternative x to alternative y minus the probability that his preference is the reverse is denoted by $\Delta\pi_{xy}$. This is a linear function of the π_ps,

$\Delta\pi_{xy} = \sum_{p=1}^{A!} R_{xy,p} \pi_p$. The matrix R is rectangular with ½ A(A-1) rows, one for each pair of alternatives, and A! columns, one for each possible ranking. Each matrix element of R is either +1 or -1 and the sum of the matrix elements in each row is zero. The relation between the number of committee members who hold the pth ranking of the alternatives, N_p, and the committee's pairwise vote between the alternatives x and y, V_{xy}, is given by $V_{xy} = \sum_{p=1}^{A!} R_{xy,p} N_p$.

For each ranking of the alternatives, there is a set of $\frac{1}{2}A(A-1)$ inequalities that is satisfied by the π_ps of a constituency whose probability of selecting that ranking grows to one as its size grows to infinity. This association is demonstrated by the following example for A=4. For the permutation (a, b, c, d) the associated set of inequalities is

$$\Delta\pi_{ab} > 0$$

$$\Delta\pi_{ac} > 0 \qquad \Delta\pi_{bc} > 0$$

$$\Delta\pi_{ad} > 0 \qquad \Delta\pi_{bd} > 0 \qquad \Delta\pi_{cd} > 0 \quad .$$

We call the constituency transitive when all the inequalities are satisfied and practically transitive when one or more inequalities are replaced by equalities. Of course, when there are three alternatives, x, y, z, such that $\Delta\pi_{xy} > 0$, $\Delta\pi_{yz} > 0$ and $\Delta\pi_{xz} < 0$, we call the constituency intransitive. It should be understood that, when A $\geq$ 4, there are constituencies that are not members of any of these classes. Examples of these constituencies are those for which $\Delta\pi_{ab} > 0$, $\Delta\pi_{bc} > 0$, $\Delta\pi_{ac} = 0$, $\Delta\pi_{ad} < 0$, $\Delta\pi_{cd} > 0$ and $\Delta\pi_{bd} = 0$. We call such constituencies practically intransitive. Later, when our practical task is to construct the π_p from the committee members' preferences, we will refer to equalities of the form $\Delta\pi_{xy} = 0$ as taut constraints.

With these conventions established, the central problem can be addressed. Given the committee's N_1, N_2, . . . $N_{A!}$, infer constituency that the committee most probably represents from among the transitive or practically transitive constituencies To solve this problem, we use the method of maximum likelihood which is this: for given N_1, N_2, . . . $N_{A!}$ define the likelihood, L, of π_1, π_2, . . . , $\pi_{A!}$ thus

$$L(N,\pi) \equiv V! \prod_{i=1}^{A!} \left\{ \frac{\pi_p^{N_p}}{N_p!} \right\}$$

and from among the collection of all constituencies that are transitive or practically transitive identify those that make the likelihood a maximum. The same answers are obtained if, instead of L, lnL is maximized. We prefer the latter because with it sums and differences replace products and ratios in calculations. Thus the problem becomes: from among the transitive or practically transitive constituencies, find the one(s) for which

$$MAX = lnL(N, \pi) = \sum_{p=1}^{A!} N_p ln\pi_p + \left[lnV! - \sum_{p=1}^{A!} lnN_p! \right] .$$

Our amalgamation procedure is now precisely stated and could be implemented on a computer when the number of alternatives, A, is not too large.

However, in order to anticipate the result of applying this procedure, and also to reduce the cost of implementing it on a computer, it is desirable to have theorems that provide guidance to the search for a solution. Several such results are easily obtained and are described below.

The first and single most important general result is ThmI.

ThmI   If the committee members' rankings provide an order
            when amalgamated by MMD then our procedure provides
            the same order.

The proof of this turns on two observations. The first is that the absolute maximum of the likelihood is achieved when $\pi_p = N_p/V$. The second is that

$\sum_{p=1}^{A!} R_{xy,p} N_p/V = V_{xy}/V$. The next result is of great technical importance.

ThmII   If the committee members' rankings do not provide
            an order when amalgamated by MMD then there is no
            transitive constituency that maximizes the likeli-
            hood. Thus, one must search among practically
            transitive constituencies in order to identify
            our order.

This is proved by invoking the Kuhn-Tucker Thm which is to say that since the maximum is by hypothesis not in the interior of the allowed region it must

occur somewhere on the boundary. The third theorem compares the likelihood of two different practically transitive constituencies.

ThmIII   If the set of taut constraints for a practically
          transitive constituency is a subset of the taut
          constraints for another practically transitive
          constituency then the first cannot be less likely
          than the second.

Although this result just follows from the definition of a constrained maximum, it is important because it tells us to examine practically transitive constituencies with few taut contraints before examining those with many. Therefore, we have examined π_ps that satisfy one taut constraint. For these cases, the result is given by the next theorem.

ThmIV   If the following five conditions hold:

1) $|V_{ab}| < |V_{cd}|$

2) $0 = \sum_{p=1}^{A!} R_{ab,p}\, \hat{\pi}_p$

3) subject to (2) $\sum_{p=1}^{A!} N_p \ln\hat{\pi}_p = MAX$

4) $0 = \sum_{p=1}^{A!} R_{cd,p}\, \hat{\hat{\pi}}_p$

5) subject to (4) $\sum_{p=1}^{A!} N_p \ln\hat{\hat{\pi}}_p = MAX$

then

$$\ln L(N,\hat{\pi}) > \ln L(N,\hat{\hat{\pi}}) \ .$$

The proof consists of the appropriate calculation. From this, we learn that one should examine taut constraints between alternatives with a close vote before examining those pairs whose vote is not close.

Next we present our estimate of $\Delta\pi_{xy}$ when $\Delta\pi_{ab}=0$. The calculation from which this result derives has a number of steps and we omit them.

ThmV   If the following conditions hold

1) $0 = \sum\limits_{p=1}^{A!} R_{ab,p}\pi_p$

2) subject to (1) $\sum\limits_{p=1}^{A!} N_p \ln\pi_p = MAX$

then

$$\Delta\pi_{xy} = \frac{V_{xy}}{V} \cdot \frac{1 + \left(1-2\tilde{\theta}_{xy,ab}\right)\dfrac{V_{ab}}{V_{xy}}}{1 - \left(\dfrac{V_{ab}}{V}\right)^2}$$

where $\tilde{\theta}_{xy,ab} \equiv \dfrac{1}{V} \sum\limits_{p=1}^{A!} \dfrac{1}{2}\left(1 + R_{xy,p}R_{ab,p}\right) N_p$.

The quantity $\tilde{\theta}_{xy,ab}$ is the ratio of the number of committee members who rank a over b and rank x over y plus those who rank b over a and y over x to the total number of committee members. By establishing bounds, which we omit, on $\tilde{\theta}_{xy,ab}$ we have obtained Thm VI.

ThmVI   If the conditions of ThmV hold and $|V_{ab}| < |V_{xy}|$
            then $\Delta\pi_{xy}$ has the same sign as V_{xy}.

Results that deal with the joint application of two or more constraints can also be derived. However, the necessary calculations are considerably more involved than those required to obtain the results displayed above.

IV. RESULTS FOR A=3

In this section, we consider the situation in which the committee must rank three alternatives. First, we show how the solution of the maximum likelihood problem can be obtained from inspection of the pairwise votes. Second, we find the difference between the logarithm of the likelihood of the constituency that precisely mirrors the committee when it is intransitive and the logarithm of the practically transitive constituency from which we obtain our order. Third, we prove that our amalgamation procedure conflicts with principle of the independence of irrelevant alternatives less often than does the well-known Borda amalgamation procedure or the mixture of MMD and Borda's procedure suggested by Duncan Black (Black, 1958).

Recall that ThmI asserts that if MMD provides a ranking then our procedure provides the same ranking. Hence, we consider the situations in which MMD does not provide a ranking. These occur when $V_{ab} > 0$, $V_{bc} > 0$, $V_{ac} < 0$ or $V_{ab} < 0$, $V_{bc} < 0$, $V_{ac} > 0$. For the sake of definiteness, we assume that V_{ac} is the "closest vote" or more precisely $|V_{ac}| < |V_{ab}|$ and $|V_{ac}| < |V_{bc}|$. It follows from ThmIV that the likelihood of the $\hat{\pi}_p$s that satisfy the single taut constraint $\Delta\pi_{ac} = 0$ and that maximize $\ln L$ is greater than the likelihood of the π_ps that satisfy either $\Delta\pi_{ab} = 0$ or $\Delta\pi_{bc} = 0$. Á fortiori, by ThmIII, π_ps that satisfy both constraints $\Delta\pi_{ab} = 0$ and $\Delta\pi_{bc} = 0$ are also less likely. The ranking generated by the $\hat{\pi}_p$s can be determined from ThmVI. Since $|V_{ac}| < |V_{ab}|$ and $|V_{ac}| < |V_{bc}|$, $\Delta\pi_{ab}$ and $\Delta\pi_{bc}$ have the same signs as V_{ab} and V_{bc}. Thus, we have proved ThmVII.

ThmVII   Let A=3 then the committee's pairwise votes, by themselves, determine the result of the inferred referendum. In particular, when $V_{ab} > V_{bc} >- V_{ac} > 0$ the result of our amalgamation procedure is the ranking (a, b, c) and similarly for other cases in which MMD does not provide a ranking.

Our recognition that in practice a committee's constituency might be transitive even though the committee was intransitive led to the formulation of the amalgamation procedure we have presented. Let us now be much more specific about this possibility. Since our procedure associates a practically transitive constituency with an intransitive committee, we can compare the likelihood of this constituency, $\hat{\pi}_p$, with the likelihood of the most likely constituency, $\tilde{\pi}_p$, which, of course, is intransitive. The result of that comparison will conform to ThmVIII.

ThmVIII   Let A=3 and suppose the committee members' rankings do not provide an order when amalgamated by MMD then with $\tilde{\pi}_p$ and $\hat{\pi}_p$ defined as in the text above and with

$$\sigma \equiv MIN \left\{ |V_{ab}|/V, \ |V_{ac}|/V, \ |V_{bc}|/V \right\}$$

one obtains

$$\ln \left[L(N,\tilde{\pi})/L(N,\hat{\pi}) \right] = \tfrac{1}{2}V \left[(1+\sigma)\ln(1+\sigma) + (1-\sigma)\ln(1-\sigma) \right]$$

$$\leq \tfrac{1}{2}V \left[0.113 \ . \ . \ . \right] \quad .$$

This theorem shows that the practically transitive constituency is "almost" as likely as the intransitive one and thus the procedure that we arrived at is compatible with our initial motivation. ThmVIII is proved by combining two different ideas. One proves that $\sigma \leq \frac{1}{3}$ and one finds $\tilde{\pi}_p$ and $\hat{\pi}_p$.

Let us now consider how our procedure conflicts with the principle of the independence of irrelevant alternatives. As long as the committee members' rankings can be amalgamated by MMD the deletion of one of the three alternatives will leave the relative rank of the remaining pair unchanged. In the situation $V_{ab} > V_{bc} \succ V_{ac} > 0$, our amalgamation procedure produces the ranking (a, b, c). Note that if a (resp. c) is deleted then b is still ranked above c (resp. a is still ranked above b). However, if b is deleted then a is no longer ranked above c; instead c would be ranked above a. Thus, our procedure

does not adhere to the principle of the independence of irrelevant alternatives. In this regard, it is of interest to compare our procedure to the well-known one of Borda voting (Black, 1958). That procedure also satisfies the principles of unanimity and anonymity and violates the independence of irrelevant alternatives. The Borda method assigns to each alternative the sum of the ranks assigned to it by the voters. For example, if the committee has five members and a certain alternative is ranked first by two voters, second by two voters and third by one voter then this alternative is awarded the weighted sum of its ranks, $1\cdot2 + 2\cdot2 + 3\cdot1 = 9$. The alternatives are then ordered according to these numbers with low numbers ranked above high numbers. Although it is not immediately apparent, the final order depends only on the pairwise votes and in fact we have established the following result.

ThmIX   Let $R(a)$ be the sum of the ranks of alternative a
then

$$R(a) = \frac{1}{2} V(A+1) - \frac{1}{2} \sum_{\substack{c=1 \\ c \neq a}}^{A!} V_{ac}$$

and thus

$$R(a) - R(b) = - \left[V_{ab} + \frac{1}{2} \sum_{\substack{c=1 \\ c \neq a \\ c \neq b}}^{A!} \left(V_{ac} - V_{bc} \right) \right]$$

When $A=3$ one can infer from ThmVII and ThmVI that each instance in which our procedure conflicts with the independence of irrelevant alternatives is also an instance of conflict between Borda and the same principle. Moreover, it is easy to find intransitive committees for which Borda conflicts with this principle and our procedure does not. Examples of this situation occur when $V_{ab} > V_{bc} > -V_{ac} > 0$ and $V_{bc} < \frac{1}{2}(V_{ab} - V_{ac})$. The first set of inequalities

define an intransitive vote. The second set guarantees that the Borda ranking

is (a,c,b). In this case, the omission of either alternative a or alternative

b will change the relative ranks of the remaining pair of alternatives. However,

ThmVIII shows that our procedure provides the ranking (a,b,c). With this

ranking, the omission of alternative a will not change the relative ranks of b

and c though, of course, the omission of b will change the relative ranks of a

and c. It is also easy to find transitive committees for which the omission of

an alternative will change the relative Borda ranks of the remaining alternatives.

   We state these results in ThmX.

ThmX   For A=3, the situations in which our amalgamation
       procedure conflicts with the principle of the inde-
       pendence of irrelevant alternatives is a proper
       subset of those situations in which Black's proce-
       dure (replacing MMD with Borda when the former
       provides an intransitivity) conflicts with the same
       principle. A fortiori, the same is true when our
       procedure is compared to the Borda procedure.

In this sense, the amalgamation procedure we propose conflicts less with the

hypothesis of Arrow's Thm than does the Borda procedure.

V. SUMMARY

By considering how a committee that ranks alternatives on behalf of others should resolve an intransitivity resulting from the application of MMD to its members' rankings, we have been led to propose a new amalgamation procedure that we call the method of the inferred referendum. We have analyzed the application of the method to three alternatives and found the method to be in less conflict with the hypothesis of Arrow's Thm than two well known amalgamation procedures. We have also determined the difference between the likelihood that the constituency whom the committee represents would choose the order our amalgamation procedure provides and the likelihood that the constituency would vote precisely as the committee did.

Although the language used in this paper was chosen to suggest the application of this procedure to political processes, it also appears natural to apply our procedure to a group of consultants or experts charged with ranking alternative projects who were selected for the sake of economy and time from a larger pool of equally knowledgeable persons.

Two classes of issues remain for further investigation. The first is the derivation of theorems that will allow us to anticipate the results of our amalgamation procedure when there are four or more alternatives. The second is the sensitivity of our procedure to strategic voting or more explicitly the incentives for voters to change their preferences as they become aware of how others may vote. We plan to take these issues up in the future.

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