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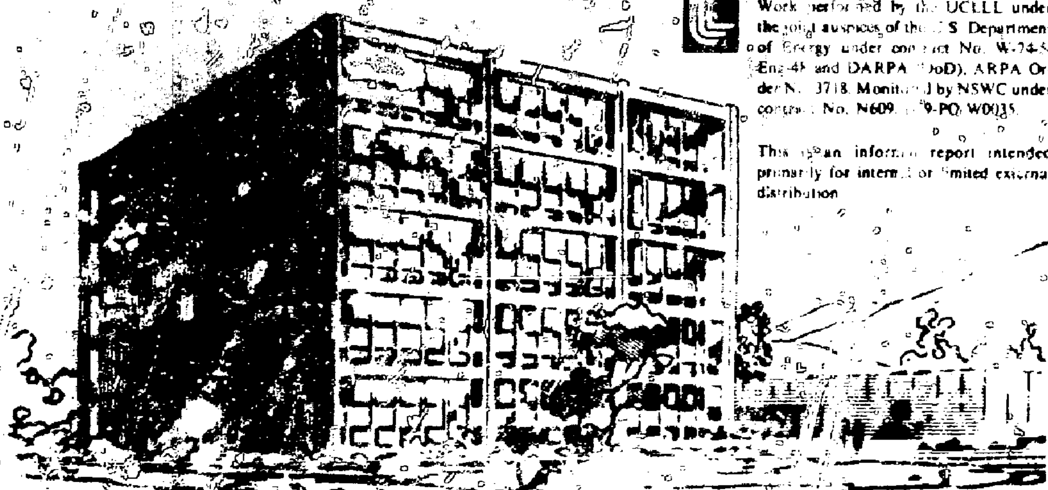

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Ohmic Dissipation During Vacuum Transport

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Abstract

The energy loss of a highly relativistic beam transported in an evacuated pipe of finite conductivity is calculated.

I. INTRODUCTION AND SUMMARY

We consider an axisymmetric beam transported in a vacuum pipe whose walls have large but finite electrical conductivity. The required focusing elements do not enter the present calculation, which is concerned only with the induced current in the pipe wall and its reaction on the beam. It is found that the rate of change of pulse energy (U) is accurately given by the formula*

$$\frac{\partial U}{\partial z} = -\frac{2}{\pi}\left(\frac{I}{c}\right)^2 \sqrt{\frac{cL}{R^2\sigma}} \quad , \tag{1}$$

where I is pulse current, L is pulse length, R is inside pipe radius and σ is wall conductivity. This loss is generally small. However the axial electric field varies along the pulse causing a loss of monoenergeticity; this field is calculated for a general current pulse form.

The derivation of the energy loss and longitudinal electric field only requires a solution of Maxwell's equations. This solution is obtained using several good approximations:

*Gaussian units are used.

1. The beam propagates in the z direction speed of light with unchanging pulse form. All functions (except energy) depend only on the independent variables $r = \sqrt{|\underset{\sim}{r}_\perp|^2}$ and the "retarded time"

$$x = ct - z . \tag{2}$$

2. Conductivity is large enough that the induced currents form a layer inside the pipe which is thin compared with the pipe thickness and radius.

3. If L_r is the scale rise length of the current pulse, then we assume

$$\frac{\sigma L_r}{c} \gg 1 , \tag{3a}$$

$$\frac{\sigma L_r}{c} \gg \frac{R^2}{L_r^2} . \tag{3b}$$

II. REDUCTION OF MAXWELL'S EQUATIONS

The relevant Maxwell equations are

$$\frac{\partial E_z}{\partial r} = \frac{\partial B_\theta}{\partial ct} + \frac{\partial E_r}{\partial z} , \tag{4}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r\, E_r = 4\pi\rho - \frac{\partial E_z}{\partial z} , \tag{5}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r\, B_\theta = 4\pi\frac{J_z}{c} + \frac{\partial E_z}{\partial ct} . \tag{6}$$

In vacuum the only component of current is

$$\frac{J_z}{c} = \rho = \text{beam charge density} , \tag{7}$$

while in the pipe wall

$$\underset{\sim}{J} = \sigma \underset{\sim}{E} \,. \tag{8}$$

We note that the conducting wall will not support an internal charge density; charge continuity gives

$$0 = \frac{\partial\rho}{\partial t} + \underset{\sim}{\nabla} \cdot \underset{\sim}{J} = \frac{\partial\rho}{\partial t} + \underset{\sim}{\nabla} \cdot \sigma\underset{\sim}{E} = \frac{\partial\rho}{\partial t} + 4\pi\sigma\rho \,. \tag{9}$$

If there is no initial charge, ρ always vanishes. There is, however, a charge <u>layer</u> at R.

Using the assumed dependence on (z, t) (see Eq. 2), equations (4)-(6) become

$$\frac{\partial E_z}{\partial r} = \frac{\partial}{\partial x}\,(B_\theta - E_r) \,, \tag{10}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\, r\, E_r = 4\pi\rho + \frac{\partial E_z}{\partial x} \,, \tag{11}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\, r\, B_\theta = 4\pi\,\frac{J_z}{c} + \frac{\partial E_z}{\partial x} \,. \tag{12}$$

Taking the difference of equations (11) and (12), we have

$$\frac{1}{r}\frac{\partial}{\partial r}\, r\left(B_\theta - E_r\right) = 4\pi\left(\frac{J_z}{c} - \rho\right) . \tag{13}$$

Differentiating both sides of Eq. (13) with respect to x and substituting from Eq. (10) then gives

$$\frac{1}{r}\frac{\partial}{\partial r}\, r\,\frac{\partial E_z}{\partial r} = 4\pi\,\frac{\partial}{\partial x}\left(\frac{J_z}{c} - \rho\right). \tag{14}$$

In vacuum the rhs of Eq. (14) vanishes and (rejecting the solution which is singular at r = 0) one has

$$\frac{\partial E_z}{\partial r} = 0 \ . \tag{15}$$

Hence for $r < R$,

$$E_z = E_z(R, x) \ . \tag{16}$$

Equations (11) - (13) yield

$$E_r = B_\theta = \frac{2I_r}{rc} + \frac{r}{2}\frac{\partial E_z}{\partial x} \quad , \tag{17}$$

where I_r is the current inside radius r:

$$I_r = 2\pi \int_0^r dr' \, r' \, J_z(r', x) \ , \tag{18}$$

Inside the pipe wall ($r > R$) Eqs. (14), (8) and (9) give

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial E_z}{\partial r} = \frac{4\pi\sigma}{c}\frac{\partial E_z}{\partial x} \quad , \tag{19}$$

which is the only differential equation that needs to be solved. This is deferred to Section III.

To complete the formal development, it remains to specify boundary conditions and a jump condition at R. Clearly we want fields to vanish as $r \to \infty$. Since the beam velocity equals c, the fields also vanish for $x < x_0$, where x_0 marks the beam front. The most convenient jump condition is the continuity in r of E_z and B_θ (the current layer near R cannot be of infinitesimal thickness when $\sigma \neq \infty$):

$$B_\theta(R_+) = B_\theta(R_-) \quad , \quad E_z(R_+) = E_z(R_-) \ . \tag{20}$$

To relate B_θ to E_z at R_- we use Eq. (17):

$$B_\theta(R_-) = \frac{2I}{Rc} + \frac{R}{2}\frac{\partial E_z(R)}{\partial x} \quad . \tag{21}$$

An expression for B_θ (R_+) is found by integrating Eq. (12) over $R \leq r \leq \infty$. Using $J_z = \sigma E_z$ we have

$$B_\theta(R_+) = -\frac{1}{R}\int_R^\infty dr'\ r'\left[\frac{4\pi}{c}\ \sigma E_z + \frac{\partial E_z}{\partial x}\right]. \tag{22}$$

Equations (20) - (22) give the relation between I (in vacuum) and E_z in the wall

$$-\frac{1}{R}\int_R^\infty dr'\ r'\left[\frac{4\pi\sigma}{c}\ E_z + \frac{\partial E_z}{\partial x}\right] - \frac{R}{2}\ \frac{\partial E_z(R)}{\partial x} = +\frac{2I}{Rc}\ . \tag{23}$$

III. SOLUTION FOR E_z

Equation (19) is solved subject to the condition eq. (23) and the boundary condition $E_z(r \to \infty) = 0$. We adopt the thin layer approximation

$$\frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} \longrightarrow \frac{\partial^2}{\partial r^2} \quad , \tag{24a}$$

$$dr\ r \longrightarrow dr\ R \ , \tag{24b}$$

so the problem has slab geometry.

It is convenient at this point to employ the Fourier transform of I and E_z:

$$I_\omega = \int_{-\infty}^\infty dx\ e^{i\omega x}\ I(x)\ , \tag{25a}$$

$$I(x) = \frac{1}{2\pi}\int_{-\infty}^\infty d\omega\ e^{-i\omega x}\ I_\omega\ , \tag{25b}$$

with $\mathrm{Im}(\omega) > 0$ in the definition of I_ω (recall $I = 0$ for $x < x_0$); E_z is defined similarily. Equations (19) and (23) give

$$\frac{\partial^2 E_{z\omega}}{\partial r^2} = -\frac{4\pi\sigma i\omega}{c} E_{z\omega} , \tag{26}$$

$$-\frac{1}{R}\int_R^\infty dr'\, R\left[\frac{4\pi\sigma}{c} E_{z\omega} - i\omega\, E_{z\omega}\right] + \frac{R}{2}\, i\omega\, E_{z\omega} = +\frac{2I_\omega}{Rc} . \tag{27}$$

The decaying solution of Eq. (26) is

$$E_{z\omega} = E_{z\omega}(R)\exp\left[-\sqrt{\frac{-4\pi\sigma i\omega}{c}}\,(r - R)\right] . \tag{28}$$

Inserting this expression into Eq. (27) we find

$$E_z\,(R)\left(\frac{1}{i\omega}\sqrt{\frac{-4\pi\sigma i\omega}{c}} + i\omega\sqrt{\frac{c}{-4\pi\sigma i\omega}} + \frac{Ri\omega}{2}\right) = \frac{2I_\omega}{Rc} . \tag{29}$$

The high σ approximation is now employed to justify dropping the second and third terms on the lhs of Eq. (29). Note that $\omega \leq 1$, so the three terms to be compared are in the ratios

$$\sqrt{\frac{\sigma L_r}{c}} \;;\; \sqrt{\frac{c}{L_r\sigma}} \;;\; \frac{R}{L_r} \tag{30}$$

Only the first term is appreciable by the orders of Eq. (3), and we have

$$E_{z\omega} = i\omega\sqrt{\frac{c}{-4\pi\sigma i\omega}}\,\frac{2I_\omega}{Rc} . \tag{31}$$

On transformation back to the x-domain, Eqs. (31) and (25b) yield the formal solution

$$E_z(r < R) = E_z(R) = \frac{1}{2\pi}\int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} d\omega\, e^{-i\omega x}\; i\omega\sqrt{\frac{c}{-4\pi\sigma i\omega}}\,\frac{2I_\omega}{Rc} . \tag{32}$$

The transform variable (ω) can be completely eliminated as follows. First we integrate by parts in the definition of I_ω to obtain

$$I_\omega = \int_{-\infty}^{+\infty} dx' \, e^{i\omega x'} \, I(x') = - \int_{-\infty}^{+\infty} dx' \, \frac{e^{i\omega x'}}{i\omega} \frac{\partial I}{\partial x'} \, . \tag{33}$$

This expression is inserted in Eq. (32) to get, after some rearrangement of terms,

$$E_z = - \frac{1}{2\pi Rc} \sqrt{\frac{c}{\pi\sigma}} \int_{-\infty}^{+\infty} dx' \, \frac{\partial I}{\partial x'} \int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} d\omega \, \frac{e^{-i\omega(x-x')}}{\sqrt{-i\omega}} \, . \tag{34}$$

The operator

$$F(x - x') \equiv \int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} d\omega \, \frac{e^{-i\omega(x-x')}}{\sqrt{-i\omega}} \tag{35}$$

is to be evaluated. Note that for $x < x'$ the inversion contour may be closed in the upper ω half plane, which contains no poles or cuts, hence

$$F(x < x') = 0 \, . \tag{36}$$

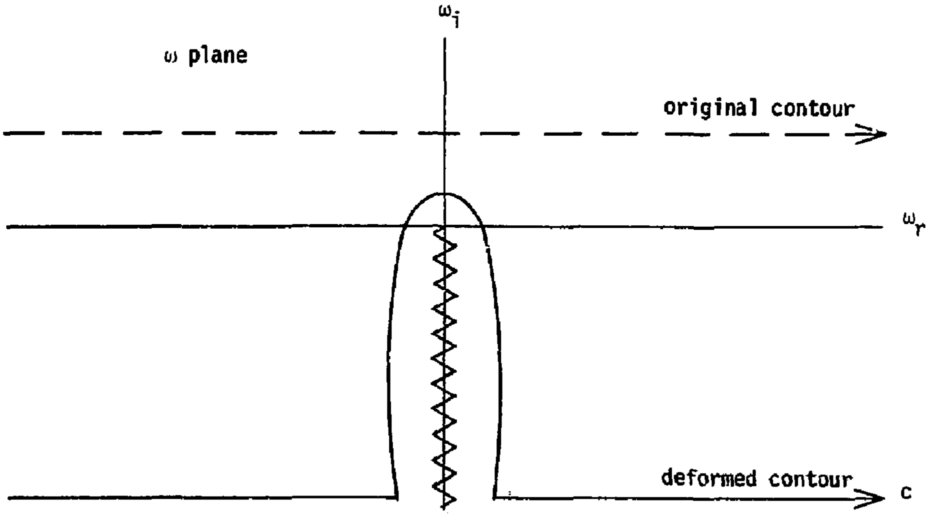
This result should be expected on casual grounds; we write

$$F = H(x - x') \int d\omega \, \frac{e^{-i\omega|x-x'|}}{\sqrt{-i\omega}} \quad , \tag{37}$$

where H is the unit step function.

The only non-analytic feature of the integrand in the lower half-plane is a branch cut along the negative imaginary axis. The original contour of integration may be deformed to the contour C shown in figure 1.

figure 1



Since $\sqrt{\omega}$ changes sign across the cut, we have

$$F = H(x - x')\, 2 \int_{-i\infty-\varepsilon}^{-\varepsilon} d\omega\, \frac{e^{-i\omega|x-x'|}}{\sqrt{-i\omega}} . \tag{38}$$

This form is essentially a Fresnel integral, which can be evaluated by the substitution $i\omega = u^2$:

$$F = H(x - x')\, 2 \int_{\infty}^{0} \left(\frac{2u\,du}{i}\right) \frac{e^{-u^2|x-x'|}}{\sqrt{-u^2}} \tag{39}$$

$$= H(x - x')\, 4 \int_{0}^{\infty} du\; e^{-u^2|x-x'|}$$

$$= H(x - x')\, 4\, \frac{1}{\sqrt{|x - x'|}} \int_{0}^{\infty} dv\; e^{-v^2}$$

$$F = H(x - x') \; 2 \sqrt{\frac{\pi}{|x - x'|}} \quad . \tag{40}$$

Substitution of Eq. (40) into Eq. (34) yields

$$E_z = -\frac{1}{\pi Rc}\sqrt{\frac{c}{\sigma}} \int_{-\infty}^{+\infty} dx' \, \frac{\partial I}{\partial x'} \frac{H(x - x')}{\sqrt{|x - x'|}} \quad . \tag{41}$$

We note here that the value of E_z far behind the pulse is independent of the details of I(x). Integrating Eq. (41) by parts and taking the limit x >> x' gives

$$E_z \approx +\frac{1}{\pi Rc}\sqrt{\frac{c}{\sigma}} \int_{-\infty}^{+\infty} dx' \, \frac{I(x')}{2\sqrt{x}^3} = \frac{1}{2\pi R}\sqrt{\frac{c}{\sigma}} \; \frac{Q}{\sqrt{x}^3} \quad , \tag{42}$$

where

$$Q = \int_{-\infty}^{+\infty} dx \, \frac{I}{c} \tag{43}$$

is the total charge in the pulse.

IV. ENERGY LOSS DURING BEAM TRANSPORT

Having determined the form of the longitudinal electric field, we may compute the ohmic dissipation for specified current waveforms. To begin, we consider the simple pulse form

$$I(x) = \begin{cases} 0 & x < 0 \\ I & 0 < x < L \\ 0 & x > L \end{cases} \quad . \tag{43}$$

In this case, the source of E_z to be inserted in Eq. (41) is

$$\frac{\partial I}{\partial x} = I \left[\delta(x) - \delta(L - x) \right] \quad . \tag{44}$$

We have

$$E_z = -\frac{1}{\pi Rc}\sqrt{\frac{c}{\sigma}}\ I\left[\frac{H(x)}{\sqrt{x}} - \frac{H(x-L)}{\sqrt{x-L}}\right]. \tag{45}$$

The rate of energy loss from a pulse is in general given by

$$\frac{\partial U}{\partial z} = \int_{-\infty}^{+\infty} dx\ \frac{I(x)}{c}\ E_z(x)\ , \tag{46}$$

which for the specific case of (Eq. 43) becomes

$$\frac{\partial U}{\partial z} = \frac{1}{c}\int_0^L dx\ I\left(-\frac{1}{\pi Rc}\sqrt{\frac{c}{\sigma}}\ \frac{I}{\sqrt{x}}\right) = -\frac{2}{\pi}\left(\frac{I}{c}\right)^2\sqrt{\frac{cL}{R^2\sigma}}\ . \tag{47}$$

This expression is similar to the ohmic loss formula for a long beam pulse in a plasma

$$\left(\frac{\partial U}{\partial z}\right)_{\text{beam-plasma}} \approx -\left(\frac{I}{c}\right)^2\ . \tag{48}$$

The extra factor $\sqrt{cL/R^2\sigma}$ is usually small compared with unity and represents the shielding power of the wall against current penetration. A square root of σ appears because the eddy current is concentrated in a thin but expanding skin.

As a numerical example we consider a beam with the profile described by Eq. (43), transported in aluminum pipe:

$$I = 10^4\ \text{Amps} = 3\times 10^{13}\ \text{esu},\qquad \sigma = 10^{17}\ s^{-1} \approx \sigma_{Al} \tag{49a}$$
$$L = 3\ \text{m} = 300\ \text{cm}\quad ,\qquad R = 10\ \text{cm}$$

Inserting these numbers in Eq. (47) we obtain

$$\frac{\partial U}{\partial z} = -\ 604\ \text{erg/cm}\ . \tag{49b}$$

More generally, and in mixed units:

$$\frac{\partial U}{\partial z} = -\left(.349 \times 10^{-3}\ \frac{J}{m}\right) I_{kA}^{2}\ \frac{L_m^{1/2}}{R_{cm}}\left(\frac{\sigma_{Al}}{\sigma}\right)^{1/2} . \tag{50}$$

A possible criticism of Eq. (47) is that it is derived from singular $E_z \propto x^{-1/2}$; it may therefore be a poor approximation for a pulse with finite rise length. For comparison, the energy loss associated with the Gaussian profile

$$I(x) = \sqrt{\frac{6}{\pi}}\ I\ e^{-6x^2/L^2} , \tag{51}$$

which has the same net charge and rms width as the flat profile, is reduced only 21%. Hence we regard Eq. (47) as representative unless the current is internally modulated within the pulse (say $I \propto \sin \omega_0 x$).

The energy loss formula for general $I(x)$, obtained by inserting the form Eq. (41) into Eq. (46) is

$$\frac{\partial U}{\partial z} = -\frac{1}{\pi}\sqrt{\frac{c}{R^2\sigma}} \int_{-\infty}^{\infty} \frac{dx}{c}\, I(x) \int_{-\infty}^{\infty} \frac{dx'}{c}\, \frac{\partial I}{\partial x'}\, \frac{H(x - x')}{\sqrt{|x - x'|}} . \tag{52}$$

Often a convenient alternative to the evaluation of Eq. (52) is the use of the Fourier transforms of I and E_z. An application of Parseval's Theorem to Eq.(46) gives

$$\frac{\partial U}{\partial z} = \frac{1}{c}\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, E_{z_\omega}\, I_{-\omega} = -\frac{1}{2\pi}\sqrt{\frac{c}{R^2\sigma}}\, \frac{1}{\sqrt{\pi}} \int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} d\omega\, \frac{I_\omega}{c}\, \frac{I_{-\omega}}{c}\sqrt{-i\omega} . \tag{53}$$

V. EVALUATION OF E_z

Expression (45) gives (inside the pulse)

$$E_z = -\frac{1}{\pi}\ \frac{I}{c}\sqrt{\frac{c}{R^2 L\sigma}}\sqrt{\frac{L}{x}}\ . \tag{54}$$

Recalling $I/c = 100\ I_{kA}$ and 3×10^4 volts/m equals one statvolt/cm, we have the numerical expression

$$E_z = -\left(52.3\ \frac{\text{volts}}{\text{m}}\right) I_{kA}\, R_{cm}^{-1}\, L_m^{-1/2} \left(\frac{\sigma_{Al}}{\sigma}\right)^{1/2} \left(\frac{L}{x}\right)^{1/2} \tag{55}$$

This field produces an averaged loss of beam energy in the transport system which is not significant for most applications. A more serious effect may be the variation in beam energy along the pulse caused by an x-dependent E_z. Equation (55) is actually a poor measure for this effect since the infinitisimal rise length has produced singular loss at $x = 0$. To obtain a better estimate of $E_z(x)$ we have evaluated the case of linear ramps at the head and tail of the pulse:

$$\frac{I(x)}{I_o} = \begin{cases} 0 & x < 0\ , \\ \frac{x}{L_r} & 0 < x < L_r\ , \\ 1 & L_r < x < L - L_r\ , \\ \frac{L-x}{L_r} & L - L_r < x < L\ . \end{cases} \tag{56}$$

we find

$$E_z = -\frac{2}{\pi}\sqrt{\frac{c}{R^2\sigma}}\ \frac{I_o}{cL_r}\Big\{H(x)\sqrt{x} - H(x - L_r)\sqrt{x - L_r} -$$

$$- H(x - L + L_r)\sqrt{x - L + L_r} + H(x - L)\sqrt{x - L}\Big\}\ . \tag{57}$$

The peak field, which occurs at $x = L_r$, is

$$E_z(L_r) = -\frac{2}{\pi}\sqrt{\frac{c}{R^2\sigma L_r}}\ \frac{I_o}{c} \tag{5[illegible]}$$

This value is larger than the mean field by a factor of $(L/L_r)^{1/2}$.

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