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NONLINEAR GYROKINETIC EQUATIONS FOR LOW-FREQUENCY ELECTROMAGNETIC WAVES IN GENERAL PLASMA EQUILIBRIA

BY

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Nonlinear Gyrokinetic Equations for Low-Frequency Electromagnetic Waves in General Plasma Equilibria

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ABSTRACT

A nonlinear gyrokinetic formalism for low-frequency (less than the cyclotron frequency) microscopic electromagnetic perturbations in general magnetic field configurations is developed. The nonlinear equations thus derived are valid in the strong-turbulence regime and contain effects due to finite Larmor radius, plasma inhomogeneities, and magnetic field geometries. The specific case of axisymmetric tokamaks is then considered, and a model nonlinear equation is derived for electrostatic drift waves. Also, applying the formalism to the shear Alfvén wave heating scheme, it is found that nonlinear ion Landau damping of kinetic shear-Alfvén waves is modified, both qualitatively and quantitatively, by the diamagnetic drift effects. In particular, wave energy is found to cascade in wavenumber instead of frequency.

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I. INTRODUCTION

Electromagnetic instabilities with frequencies lower than the ion-cyclotron frequency and perpendicular (to the magnetic field) wavelengths comparable to the ion Larmor radius are believed to be important for the transport processes in magnetically confined plasmas. One complicating factor in analyzing these low-frequency microscopic instabilities, which are driven by plasma inhomogeneities and shall be loosely termed kinetic drift-Alfvén waves, is that the destabilizing mechanisms are sensitive to effects associated with magnetic field configurations such as magnetic shear, magnetic gradient and curvature drifts, and trapped particles. To overcome this difficulty, Rutherford and Frieman[1] as well as Taylor and Hastie[2] have developed, independently, a formalism, now known as the gyrokinetic formalism, to treat the linear aspects of kinetic drift waves in general magnetic configurations. Recently, the linear gyrokinetic formalism has been extended to include electromagnetic perturbations[3,4] associated with shear and compressional Alfvén waves. Since the transport induced by the instabilities is ultimately determined by the nonlinear processes, it is, therefore, desirable to further extend the gyrokinetic formalism into the nonlinear regime while retaining crucial features such as finite Larmor radius and arbitrary magnetic field configurations. This constitutes the principal motivation of the present research.

In this work we develop the nonlinear gyrokinetic formalism based on a multiple-scale expansion.[5] That is, microscopic fluctuations vary on the fast (linear) time scale (i.e., typically, the diamagnetic drift frequency time scale), while macroscopic quantities are assumed to vary on the slow transport time scale. Here the cyclotron frequency time scale is the fastest time scale. Furthermore, consistent with experimental observations,[6,7] our formulation allows nonlinear time scales to be comparable to the linear ones and, hence, the results are valid in the strong-turbulence regime.

Section II contains the theoretical formulation and derivations of the nonlinear gyrokinetic equations. The specific case of axisymmetric tokamaks is further considered in Sec. III using the ballooning-mode representation[8-10] and a nonlinear drift-wave equation is derived in a limiting case. Noting that our results are also applicable to nonlinear heating via externally launched low-frequency waves, we also consider, in Sec. IV, nonlinear ion Landau damping (ion induced scattering) of the mode-converted kinetic (shear) Alfvén waves[11] and find that the inclusion of diamagnetic drift effects modifies, both qualitatively and quantitatively, the decay processes. Final conclusions and discussion are given in Sec. V.

II. THEORETICAL ANALYSES

(1) Guiding-center transformation and the two spatial scales. As in the linear formalism,[1,2,12] it is more convenient to carry out the analysis in the guiding-center phase space ($\underset{\sim}{X}$, $\underset{\sim}{V}$) which is related to the particle phase space ($\underset{\sim}{x}$, $\underset{\sim}{v}$) via the following guiding-center transformation

$$\underset{\sim}{X} = \underset{\sim}{x} + \underset{\sim}{v} \times \underset{\sim}{e}_\parallel/\Omega \quad , \tag{1}$$

$$\underset{\sim}{V} = \underset{\sim}{V}(\varepsilon, \mu, \alpha) \quad , \tag{2}$$

where $\varepsilon = v^2/2 + q\Phi_0/m$, $\mu = v_\perp^2/2B$, Φ_0 and $\underset{\sim}{B}$ are, respectively, the macroscopic (equilibrium) electric potential and magnetic field, $\Omega = qB/mc$, $\underset{\sim}{e}_\parallel = \underset{\sim}{B}/B$, α is the gyrophase, and

$$\underset{\sim}{v}_\perp = v_\perp (\underset{\sim}{e}_1 \cos\alpha + \underset{\sim}{e}_2 \sin\alpha) \quad , \tag{3}$$

with $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, and $\underset{\sim}{e}_\parallel$ being the local orthogonal unit vectors. Furthermore, noting that perturbations of interest here have perpendicular (to $\underset{\sim}{B}$) wavelengths of the order of the Larmor radius, ρ, which is much smaller than the macroscopic scale length, L_o, (i.e., $\lambda \equiv \rho/L_o$ is a small parameter), we may separate physical quantities into microscopic and macroscopic parts by averaging over the microscopic spatial variations. Thus

$$\underset{\sim}{p} = \underset{\sim}{P} + \delta\underset{\sim}{P} \quad , \tag{4}$$

where

$$\underset{\sim}{P} = \int d^2\underset{\sim}{X}_{\perp o}\, \underset{\sim}{p} / \int d^2\underset{\sim}{X}_{\perp o} \equiv \langle \underset{\sim}{p} \rangle_{\underset{\sim}{X}_{\perp o}} \quad , \tag{5}$$

$\langle \delta\underset{\sim}{P} \rangle_{\underset{\sim}{X}_{\perp o}} = 0$ and $X_{\perp o}$ corresponds to fast perpendicular spatial variations.

In the $(\underset{\sim}{X}, \underset{\sim}{V})$ phase space, the Vlasov equation then becomes

$$L_g f = -(q/m)(\delta R f) \quad , \tag{6}$$

where

$$L_g = \partial/\partial t + \partial\underset{\sim}{V}/\partial t \cdot \underset{\sim}{\nabla}_V + \partial\underset{\sim}{X}/\partial t \cdot \underset{\sim}{\nabla}_X + v_\parallel \underset{\sim}{e}_\parallel \cdot \underset{\sim}{\nabla}_X$$

$$+ \underset{\sim}{v} \cdot (\underset{\sim}{\lambda}_{B1} + \underset{\sim}{\lambda}_{B2}) + (q/m)(\underset{\sim}{E} - \underset{\sim}{E}_o) \cdot \underset{\sim}{v}\, \partial/\partial\varepsilon$$

$$+ (q/m)\underset{\sim}{E}\cdot[(\underset{\sim}{v}_\perp/B)(\partial/\partial\mu) + (\underset{\sim}{e}_\alpha/v_\perp)(\partial/\partial\alpha)]$$

$$+ \underset{\sim}{v}_E\cdot\underset{\sim}{\nabla}_X - \Omega\ \partial/\partial\alpha \quad , \tag{7}$$

$$\underset{\sim}{\lambda}_{B1} = \underset{\sim}{v} \times \underset{\sim}{\nabla}_X(\underset{\sim}{e}_\parallel/\Omega)\cdot\underset{\sim}{\nabla}_X \quad , \tag{8}$$

$$\underset{\sim}{\lambda}_{B2} = (\underset{\sim}{\nabla}_X\mu)(\partial/\partial\mu) + (\underset{\sim}{\nabla}_X\alpha)(\partial/\partial\alpha) \quad , \tag{9}$$

$$\underset{\sim}{\nabla}_X\mu = -[\mu\underset{\sim}{\nabla}_X B + v_\parallel\underset{\sim}{\nabla}_X\underset{\sim}{e}_\parallel\cdot\underset{\sim}{v}_\perp]/B \quad , \tag{10}$$

$$\underset{\sim}{\nabla}_X\alpha = (\underset{\sim}{\nabla}_X\underset{\sim}{e}_2)\cdot\underset{\sim}{e}_1 + (v_\parallel/v_\perp^2)\underset{\sim}{\nabla}_X\underset{\sim}{e}_\parallel\cdot(\underset{\sim}{v}_\perp \times \underset{\sim}{e}_\parallel) \quad , \tag{11}$$

$$\delta R = \delta\underset{\sim}{a}\cdot\underset{\sim}{\nabla}_v = \delta\underset{\sim}{a}\cdot\underset{\sim}{\nabla}_V + \delta\underset{\sim}{a} \times \underset{\sim}{e}_\parallel/\Omega\cdot\underset{\sim}{\nabla}_X \quad , \tag{12}$$

$$\delta\underset{\sim}{a} = \delta\underset{\sim}{E} + \underset{\sim}{v} \times \delta\underset{\sim}{B}/c \quad , \tag{13}$$

$\underset{\sim}{E}_o = -\underset{\sim}{\nabla}_X\Phi_o$, $\underset{\sim}{e}_\alpha = -\underset{\sim}{e}_1 \sin\alpha + \underset{\sim}{e}_2\cos\alpha$, and $\underset{\sim}{v}_E = c\underset{\sim}{E} \times \underset{\sim}{e}_\parallel/B$. Here L_g contains $\partial\underset{\sim}{V}/\partial t$ and $\partial\underset{\sim}{X}/\partial t$ because the macroscopic quantities are in general time-dependent. Performing spatial averaging on Eq. (6), we obtain

$$L_g F = -(q/m)\langle\delta R\delta F\rangle_{\underset{\sim}{X}_{\perp o}} \quad , \tag{14}$$

and

$$L_g\delta F = -(q/m)[\delta RF + \delta R\delta F - \langle\delta R\delta F\rangle_{\underset{\sim}{X}_{\perp o}}] \quad . \tag{15}$$

(ii) <u>Ordering</u>. To proceed further with Eqs. (14) and (15), we adopt the following orderings for the microscopic fluctuations

$$|\partial/\partial t|/|\Omega| \sim |\rho\, \underset{\sim}{e}_{\parallel}\cdot\underset{\sim}{\nabla}_X| \sim |\delta F/F| \sim |q\delta\phi/T|$$

$$\sim |\delta B/B| \sim |v_E/v_t| \sim O(\lambda) \quad , \tag{16}$$

v_t is the charactersitic (thermal) velocity, and

$$|\rho\underset{\sim}{\nabla}_{X_{\perp o}}| \sim O(1) \quad . \tag{17}$$

For macroscopic quantities, however, since they evolve on the transport (including that induced by turbulence) time scale, we take

$$|\partial/\partial t|/|\Omega| \sim O(\lambda^3) \quad , \tag{18}$$

in addition to

$$|\rho\underset{\sim}{\nabla}_X| \sim O(\lambda) \quad . \tag{19}$$

We remark that the ordering adopted here is consistent with experimental observations as well as most of the proposed phenomenological anomalous transport mechanisms.[6,7] Furthermore, since the nonlinear term, $\delta R\delta F$, is of $O(\lambda^2)$, i.e., comparable to the linear term, $L_g\delta F$, our ordering in principle contains strong-turbulence effects. Also, as will be shown later, in Eq. (47) the ordering of the transport time scale, Eq. (18), is consistent with that of the fluctuations, Eq. (16).

(iii) Solution of F. Using the small parameter λ, we have with $F =$

$F_o + F_1 + \ldots,$[1,2]

$$F_o = F_o(\varepsilon,\ \mu,\ \underset{\sim}{X}_\perp) \quad ; \tag{20}$$

that is, $\underset{\sim}{e}_\parallel \cdot \underset{\sim}{\nabla}_X F_o = 0$,

$$F_1 = (\tilde{\beta}/B)(\partial F_o/\partial\mu) \quad , \tag{21}$$

$$\tilde{\beta} = -\{\underset{\sim}{v}_\perp \cdot v_D + \int^\alpha (d\alpha'/\Omega)[v_\parallel(\underset{\sim}{v}_\perp \cdot \underset{\sim}{\nabla}_x \underset{\sim}{e}_\parallel \cdot \underset{\sim}{v}_\perp) - v_\perp^2 \underset{\sim}{\nabla}_x \cdot \underset{\sim}{e}_\parallel/2]\} \quad , \tag{22}$$

$$\underset{\sim}{v}_D = \underset{\sim}{v}_d + \underset{\sim}{v}_{Eo} \quad , \tag{23}$$

$$\underset{\sim}{v}_d = \underset{\sim}{e}_\parallel \times [(v_\perp^2/2)\underset{\sim}{\nabla}_x \ln B + v_\parallel^2 \underset{\sim}{e}_\parallel \cdot \nabla \underset{\sim}{e}_\parallel]/\Omega \quad , \tag{24}$$

and $\tilde{F}_1$ is the α-dependent part of F_1. To determine the α-independent part of F_1, one needs to go to $O(\lambda^2)$, where turbulence effects, $\langle\delta R\delta F\rangle_{\underset{\sim}{X}_{\perp o}}$, would also enter. In fact, following the procedures of neoclassical theory,[13] a formal transport theory including turbulence could be developed; this is, however, beyond the scope of the present work and will be left for future investigations. For the present purpose of obtaining a nonlinear gyrokinetic equation from Eq. (15), knowing F and $\tilde{F}_1$ is sufficient.

(iv) Nonlinear gyrokinetic equations. We now concentrate on Eq. (15) for the fluctuating δF. Since only terms up to $O(\lambda^2)$ are of interest here, the macroscopic background can be treated as frozen, and L_g becomes accurate to $O(\lambda)$,

$$L_g \simeq L_{gf} = \partial/\partial t + v_\parallel \underset{\sim}{e}_\parallel \cdot \underset{\sim}{\nabla}_X + \underset{\sim}{v} \cdot (\underset{\sim}{\lambda}_{B1} + \underset{\sim}{\lambda}_{B2})$$

$$+ (q/m)\underset{\sim}{E}_o[(\underset{\sim}{v}_\perp/B)(\partial/\partial\mu) + (\underset{\sim}{e}_\alpha/v_\perp)(\partial/\partial\alpha)]$$

$$+ \underset{\sim}{v}_{Eo}\cdot\underset{\sim}{\nabla}_X - \Omega\partial/\partial\alpha \quad . \tag{25}$$

Following the linear formalism,[3,4] we let

$$\delta F = (q/m)\delta F_a + \delta G \quad , \tag{26}$$

where

$$\delta F_a = [\delta\Phi\partial/\partial\varepsilon + (\delta\Phi - v_\parallel\delta A_\parallel/c)\ \partial/B\partial\mu]F_o \quad , \tag{27}$$

in order to remove the $O(\lambda)$ term, δRF_o, in Eq. (15). We note that we have adopted $\delta\Phi$ and $\delta\underset{\sim}{A}$ as field variables. Thus, $\delta\underset{\sim}{B} = \underset{\sim}{\nabla}_X \times \delta\underset{\sim}{A}$ and $\delta\underset{\sim}{E} = -(\underset{\sim}{\nabla}_X\delta\Phi + \partial\delta\underset{\sim}{A}/c\partial t)$. Substituting Eqs. (25) and (26) into Eq. (15), we obtain

$$L_{gf}\delta G = -(q/m)(R_\ell + R_{n\ell}) \quad , \tag{28}$$

where R_ℓ is the linear term[3,4] given by

$$R_\ell = R_{\ell 1} + R_{\ell 2} + R_{\ell 3} + R_{\ell 4} \quad , \tag{29}$$

$$R_{\ell 1} = (\partial\delta L/\partial t)(\partial F_o/\partial\varepsilon) - \underset{\sim}{\nabla}_o\delta L \times \underset{\sim}{e}_\parallel/\Omega\cdot\underset{\sim}{\nabla}_X F_o \quad , \tag{30}$$

$$\delta L \equiv \delta\Phi - \underset{\sim}{v}\cdot\delta\underset{\sim}{A}/c \quad , \tag{31}$$

$$\nabla_{\mathbf{o}} \equiv \partial/\partial \mathbf{X}_{\perp o} \quad , \tag{32}$$

$$R_{\ell 2} = (\partial F_o/B\partial\mu)(\partial/\partial t + v_\parallel \mathbf{e}_\parallel \cdot \nabla_{\mathbf{x}})\delta L \quad , \tag{33}$$

$$R_{\ell 3} = [\delta\Phi(v_\parallel \mathbf{e}_\parallel + \mathbf{v}_D)\cdot\nabla_{\mathbf{X}} - \delta\Phi\ \mathbf{v}_\perp \cdot \nabla_{\mathbf{x}} \ln B$$

$$-(\nabla_{\mathbf{v}}\tilde{\beta})\cdot\nabla_{\mathbf{X}}\delta\Phi + \Omega(\delta\Phi\tilde{\beta})'_\alpha(\partial/\partial\varepsilon + \partial/B\partial\mu)](\partial F_c/B\partial\mu) \quad , \tag{34}$$

$$R_{\ell 4} = -(1/c)\{v_\parallel(\mathbf{v}\cdot\nabla_{\mathbf{x}}\mathbf{e}_\parallel)\cdot\delta\mathbf{A} + \delta A_\parallel(\mathbf{v}\cdot\nabla\mathbf{e}_\parallel)\cdot\mathbf{v}$$

$$+ (q/m)\mathbf{e}_\parallel\cdot\mathbf{E}_o\delta A_\parallel - v_\parallel\delta A_\parallel \mathbf{v}\cdot\nabla_{\mathbf{x}}\ln B$$

$$- (\nabla_{\mathbf{v}}\tilde{\beta})\cdot[\nabla_{\mathbf{o}}(\mathbf{v}\cdot\delta\mathbf{A}) - (\mathbf{v}\cdot\nabla_{\mathbf{o}})\delta\mathbf{A}]$$

$$+ \Omega(\tilde{\beta}v_\parallel\delta A_\parallel)'_\alpha(\partial/B\partial\mu)\}(\partial F_o/B\partial\mu) \quad , \tag{35}$$

$$R_{n\ell} = \delta R_o\delta F - \langle\delta R_o\delta F\rangle_{\mathbf{X}_{\perp o}} \quad , \tag{36}$$

$$\delta\mathbf{R}_{\mathbf{o}} = \delta\mathbf{a}_{\mathbf{o}}\cdot\nabla_{\mathbf{v}} + \delta\mathbf{a}_{\mathbf{o}} \times \mathbf{e}_\parallel/\Omega\cdot\nabla_{\mathbf{o}} \quad , \tag{37}$$

$$\delta\mathbf{a}_{\mathbf{o}} = -\nabla_{\mathbf{o}}\delta\Phi + [\nabla_{\mathbf{o}}(\mathbf{v}\cdot\delta\mathbf{A}) - (\mathbf{v}_\perp\cdot\nabla_{\mathbf{o}})\delta\mathbf{A}]/c \quad , \tag{38}$$

and $(a)'_\alpha \equiv \partial a/\partial\alpha$. Expanding $\delta G = \delta G_o + \delta G_1 + \ldots$ and noting that the right-hand side of Eq. (28) is of $0(\lambda^2)$, we have, for $0(\lambda)$,

$$\partial\delta G_o/\partial\alpha = 0 \quad . \tag{39}$$

Gyrophase averaging the $O(\lambda^2)$ equation then yields

$$\langle L_{gf}\rangle_\alpha \, \delta G_o = -(q/m)\langle R_\ell + R_{n\ell}\rangle_\alpha \quad , \tag{40}$$

where

$$\langle L_{gf}\rangle_\alpha = \partial/\partial t + v_\parallel \mathbf{e}_\parallel \cdot \nabla_X + \mathbf{v}_D \cdot \nabla_o \quad , \tag{41}$$

and $\langle \ldots \rangle_\alpha \equiv (1/2\pi) \int_0^{2\pi} d\alpha(\ldots)$. If we further let

$$\delta G_o = -(q/m)\langle \delta L\rangle_\alpha \, \partial F_o/B\partial\mu + \delta H_o \quad , \tag{42}$$

Eq. (40) becomes

$$\langle L_{gf}\rangle_\alpha \delta H_o = (-q/m)(S_{\ell 1} + S_{\ell 2} + \langle R_{n\ell}\rangle_\alpha) \quad , \tag{43}$$

where

$$S_{\ell 1} = (\partial\langle \delta L\rangle_\alpha/\partial t)(\partial F_o/\partial \varepsilon) - \nabla_o \langle \delta L\rangle_\alpha \times \mathbf{e}_\parallel/\Omega \cdot \nabla_X F_o \quad , \tag{44}$$

$$S_{\ell 2} = (\partial F_o/B\partial\mu)\{v_\parallel \langle \mathbf{e}_\parallel \cdot \nabla_x \delta L\rangle_\alpha - v_\parallel \mathbf{e}_\parallel \cdot \nabla_X \langle \delta L\rangle_\alpha$$

$$- \mathbf{v}_D \cdot \nabla_X \langle \delta L\rangle_\alpha - \langle \mathbf{v}_\perp \delta\Phi\rangle_\alpha \cdot \nabla_x \ln B$$

$$- \langle \nabla_v \tilde{\beta} \cdot \nabla_o \delta\Phi\rangle_\alpha - \langle \mathbf{v} \cdot \delta \mathbf{A}/c\rangle_\alpha - \langle \mathbf{v} \cdot \delta \mathbf{A}/c\rangle_\alpha v_\parallel \mathbf{e}_\parallel \cdot \nabla_X \ln B$$

$$+ v_\parallel \langle \mathbf{v} \delta A_\parallel/c\rangle_\alpha \cdot \nabla_x \ln B + \langle (\nabla_v \tilde{\beta}) \cdot [\nabla_o(\mathbf{v} \cdot \delta \mathbf{A})$$

$$- (\underset{\sim}{v}\cdot\underset{\sim}{\nabla}_o)\delta\underset{\sim}{A}]/c\rangle_\alpha - v_\parallel\langle(\underset{\sim}{v}\cdot\underset{\sim}{\nabla}_x\underset{\sim}{e}_\parallel)\cdot\delta\underset{\sim}{A}/c\rangle_\alpha$$

$$- \langle\delta A_\parallel(\underset{\sim}{v}\cdot\underset{\sim}{\nabla}_x)\underset{\sim}{e}_\parallel\cdot\underset{\sim}{v}/c\rangle_\alpha - (q/m)\underset{\sim}{e}_\parallel\cdot\underset{\sim}{E}_o\langle\delta A_\parallel/c\rangle_\alpha\} \quad , \tag{45}$$

and, after some algebraic manipulations,

$$\langle R_{n\ell}\rangle_\alpha = \langle\delta R_o\delta F_o\rangle_\alpha = \langle\delta R_o[(q/m)\delta F_a + \delta G_o]\rangle_\alpha$$

$$= -\underset{\sim}{\nabla}_o\langle\delta L\rangle_\alpha \times \underset{\sim}{e}_\parallel/\Omega\cdot\underset{\sim}{\nabla}_o\delta H_o \quad . \tag{46}$$

Here we have noted that

$$\langle\langle\delta R_o\delta F_o\rangle_\alpha\rangle_{\underset{\sim}{X}_{\perp o}} = 0 \quad . \tag{47}$$

Equation (47) indicates that effects of turbulence on the α-independent part of F is of $O(\lambda^3)$ and, therefore, consistent with the ordering of the transport time scale, Eq. (18). Combining Eqs. (43) and (46) gives

$$\bar{L}_g\delta H_o = \{\partial/\partial t + v_\parallel\underset{\sim}{e}_\parallel\cdot\underset{\sim}{\nabla}_X + [\underset{\sim}{v}_D + (c/B)\underset{\sim}{e}_\parallel \times \underset{\sim}{\nabla}_o\langle\delta L\rangle_\alpha]\cdot\underset{\sim}{\nabla}_o\}\delta H_o$$

$$= -(q/m)(S_{\ell 1} + S_{\ell 2}) \quad . \tag{48}$$

Equation (48) is the desired nonlinear gyrokinetic equation. Combining Maxwell's equations and δF, given by Eqs. (26), (42), and (48), the microscopic dynamics is then, in principle, completely determined.

Since the theoretical analyses employ two spatial scales, it is instructive to proceed further with the following WKB ansatz

$$\delta F(\underset{\sim}{x}, \underset{\sim}{v}) = \sum_{\underset{\sim}{k}_\perp} \delta\bar{F}(\underset{\sim}{x}, \underset{\sim}{v}; \underset{\sim}{k}_\perp) \exp[i \int^{\underset{\sim}{x}_{\perp o}} \underset{\sim}{k}_\perp \cdot d\underset{\sim}{x}_\perp]$$

$$= \sum_{\underset{\sim}{k}_\perp} \delta\bar{F}(\underset{\sim}{X}, \underset{\sim}{V}; \underset{\sim}{k}_\perp) \exp[i \int^{\underset{\sim}{X}_{\perp o}} \underset{\sim}{k}_\perp \cdot d\underset{\sim}{X}_\perp - iL(\underset{\sim}{k}_\perp)] \quad , \tag{49}$$

where $L(\underset{\sim}{k}_\perp) = \underset{\sim}{k}_\perp \cdot \underset{\sim}{v} \times \underset{\sim}{e}_\parallel/\Omega$, and $\delta\bar{F}$ as well as $\underset{\sim}{k}_\perp$ contain slow spatial variations. Equation (48) then reduces to

$$[\partial/\partial t + v_\parallel \underset{\sim}{e}_\parallel \cdot \underset{\sim}{\nabla}_X + i\underset{\sim}{k}_\perp \cdot \underset{\sim}{v}_D]\delta\bar{H}_o(\underset{\sim}{k}_\perp) + (c/B) \sum_{\underset{\sim}{k}'_\perp + \underset{\sim}{k}''_\perp = \underset{\sim}{k}_\perp} [\underset{\sim}{e}_\parallel \cdot (\underset{\sim}{k}_\perp \times \underset{\sim}{k}'_\perp)]$$

$$\langle\delta\bar{L}\rangle_\alpha(\underset{\sim}{k}'_\perp)\delta\bar{H}_o(\underset{\sim}{k}''_\perp) = -(q/m)S_{\ell 1}(\underset{\sim}{k}_\perp) \quad , \tag{50}$$

where

$$\langle\delta\bar{L}\rangle_\alpha(\underset{\sim}{k}_\perp) = [\delta\bar{\Phi} - v_\parallel \delta\bar{A}_\parallel/c] \; J_o(\gamma) + v_\perp J_1(\gamma)\delta\bar{b}_\parallel/k_\perp c \quad , \tag{51}$$

$$S_{\ell 1}(\underset{\sim}{k}_\perp) = [(\partial F_o/\partial\varepsilon)(\partial/\partial t) + i\underset{\sim}{e}_\parallel \times \underset{\sim}{k}_\perp/\Omega \cdot \underset{\sim}{\nabla}_X F_o]\langle\delta\bar{L}\rangle_\alpha(\underset{\sim}{k}_\perp) \quad , \tag{52}$$

$\gamma = k_\perp v_\perp/\Omega$, and $S_{\ell 2}(\underset{\sim}{k}_\perp)$ is of higher order and ignorable here. Equation (50) and, hence, Eq. (48) show that the nonlinearity arises from a gyrophase-averaged <u>effective</u> $\delta\underset{\sim}{E}_{eff} \times \underset{\sim}{B} \cdot \nabla_X$ coupling. Here,

$$\delta\underset{\sim}{E}_{eff} = -\underset{\sim}{\nabla}_o(\delta\Phi - \underset{\sim}{v} \cdot \delta\underset{\sim}{A}/c) \quad . \tag{53}$$

It is interesting to note that the nonlinear polarization drift is contained, not very obviously, within the finite Larmor radius corrections in J_0 and J_1. In fact, the electrostatic nonlinear drift-wave equation first obtained by Hasegawa and Mima[14] can be readily derived from Eq. (50) in the appropriate limits, i.e., adiabatic electrons and cold fluid ions (Sec. III). Equation (48) [or Eq. (50)], of course, is much more general and also is not easy to solve. In the next two sections, we consider more specific applications of the general results obtained so far.

III. AXISYMMETRIC TOKAMAKS

(1) General formulation. Here we consider the specific case of axisymmetric tokamaks and, using the ballooning-mode representation,[8-10,15] further explore the properties of the nonlinear gyrokinetic equation, Eq. (48). Thus, employing the ψ (poloidal flux), ξ (toroidal angle), and χ (poloidal angle-like) coordinates, we have

$$\underset{\sim}{B} = \underset{\sim}{\nabla}_X \xi \times \underset{\sim}{\nabla}_X \psi + I(\psi, \chi) \underset{\sim}{\nabla}_X \xi \quad , \tag{54}$$

$$F_o = F_o(\psi, \mu, \varepsilon) \quad , \tag{55}$$

$$\delta H_o = \sum_n \sum_m \delta h_{n,m}(\psi, \underset{\sim}{v}) \exp[i(n\xi - m\chi + n \int^{\psi} k \, d\psi] \quad , \tag{56}$$

$$\delta h_{n,m}(\psi, \underset{\sim}{v}) = \int_{-\infty}^{\infty} d\hat{\theta}_n \, \delta\bar{h}_n(\psi, \hat{\theta}_n, \underset{\sim}{v}) g_{n,m}(\hat{\theta}_n) \quad . \tag{57}$$

$$g_{n,m}(\hat{\theta}_n) = \exp[i(m\hat{\theta}_n - n \int^{\hat{\theta}_n} \nu \, d\hat{\theta})] \quad , \tag{58}$$

$\nu = IJ/R^2$, and $J = (\underset{\sim}{\nabla}_X \psi \times \underset{\sim}{\nabla}_X \xi \cdot \underset{\sim}{\nabla}_X \chi)^{-1}$ is the Jacobian. Equation (48) or, equivalently, Eq. (43) then becomes

$$\int_{-\infty}^{\infty} d\hat{\theta}_n\, g_{n,m}(\hat{\theta}_n)\{[\partial/\partial t + (v_\parallel/JB)(\partial/\partial\hat{\theta}_n) + i\underset{\sim}{k}_\perp\cdot\underset{\sim}{v}_D]\delta\bar{h}_n$$

$$+ (q/m)\bar{S}_{\ell n}\} = -(q/m)\langle R_{n\ell}\rangle_\alpha(m,n) \quad , \tag{59}$$

where

$$\underset{\sim}{k}_\perp = (nB/RB_\chi)(\underset{\sim}{e}_\parallel \times \underset{\sim}{e}_\phi) - nRB_\chi[\int_0^{\hat{\theta}_n} (\partial\nu/\partial\psi)\, d\hat{\theta} - k] \quad , \tag{60}$$

$$\bar{S}_{\ell n} = [(\partial F_o/\partial\varepsilon)(\partial/\partial t) - i(nB/\Omega)(\partial F_o/\partial\psi)]\delta\bar{L}_n \quad , \tag{61}$$

$$\delta\bar{L}_n = J_o(\gamma)(\delta\bar{\Phi} - v_\parallel\delta\bar{A}_\parallel/c)_n + J_1(\gamma)v_\perp\delta\bar{b}_{\parallel n}/k_\perp c \quad , \tag{62}$$

$$\langle R_{n\ell}\rangle_\alpha(m,n) = -\frac{1}{2}\sum_{\substack{n'+n''\\=n}}\ \sum_{\substack{m'+m''\\=m}}\ \sum_{\substack{n'k'+n''k''\\=nk}} \int_{-\infty}^{\infty} d\hat{\theta}_{n'} \int_{-\infty}^{\infty} d\hat{\theta}_{n''}\, g_{n',m'}$$

$$g_{n'',m''}\, C_{n',n''}(\delta\bar{L}_{n'}\delta\bar{h}_{n''} - \delta\bar{L}_{n''}\delta\bar{h}_{n'}) \quad , \tag{63}$$

$$C_{n',n''} = (\underset{\sim}{e}_\parallel/\Omega)\cdot(\underset{\sim}{k}'_\perp \times \underset{\sim}{k}''_\perp) = (Bn'/\Omega)[n'' \int_{\hat{\theta}_{n'}}^{\hat{\theta}_{n''}} (\partial\nu/\partial\psi)\, d\hat{\theta}$$

$$+ n(k' - k)] \quad , \tag{64}$$

and, again, $\gamma = k_\perp v_\perp/\Omega$. After some manipulations, Eq. (63) can be shown to be

$$\langle R_{n\ell}\rangle_\alpha(m,n) = \int_{-\infty}^{\infty} d\hat{\theta}_n\, g_{n,m}(\hat{\theta}_n)(\bar{R}_{n\ell})_n \quad , \tag{65}$$

where

$$(\tilde{R}_{n\ell})_n = -\pi \sum_{\substack{n'+n''\\=n}} \sum_{\substack{n'k'+n''k''\\=nk}} \sum_p \exp(-in''2\pi pQ)$$

$$\tilde{C}_{n',n''} \int_{-\infty}^{\infty} d\hat{\theta}_{n'}\, \delta(\hat{\theta}_n - \hat{\theta}_{n'}) \int_{-\infty}^{\infty} d\hat{\theta}_{n''}\, \delta(\hat{\theta}_{n''} - \hat{\theta}_n - 2\pi p)$$

$$(\delta\bar{L}_{n'}\delta\bar{h}_{n''} - \delta\bar{L}_{n''}\delta\bar{h}_{n'}) \quad , \tag{66}$$

$$Q \equiv (1/2\pi) \int_0^{2\pi} \nu\, d\hat{\theta} \quad , \tag{67}$$

and

$$\tilde{C}_{n',n''} = (Bn'/\Omega)[n''2\pi p(\partial Q/\partial\psi) + n(k' - k)] \quad . \tag{68}$$

Combining Eqs. (59) and (65), we finally derive the nonlinear gyrokinetic equation for axisymmetric tokamaks

$$[\partial/\partial t + (v_\parallel/JB)(\partial/\partial\hat{\theta}_n) + i\underset{\sim}{k}_\perp\cdot\underset{\sim}{v}_D]\delta\bar{h}_n = -(q/m)[\bar{S}_{\ell n} + (\tilde{R}_{n\ell})_n] \quad . \tag{69}$$

We now make some qualitative remarks. For simplicity, we shall ignore global amplitude modulations by letting $k = k' = k'' = 0$ and, hence, $\tilde{C}_{n',n''} \propto p$. For flute-like modes, which are highly localized about the mode-rational surfaces, $\exp(-in''2\pi pQ) \simeq 1$ and $\delta\bar{h}$ as well as $\delta\bar{L}$ are weakly dependent on $\hat{\theta}$. Therefore, we have, roughly, $(\tilde{R}_{n\ell})_n \propto \sum_p p = 0$; i.e., nonlinear coupling is much reduced, which may be expected because there is little overlap between the modes in the ψ (radial) coordinate. On the other hand, for modes which are strongly ballooning, and hence, $(\delta\bar{\Phi}, \delta\bar{L})(\hat{\theta} + p2\pi) \simeq 0$ for $p \neq 0$; the nonlinear coupling, again, is small.

This is because, in this limit, radial structures are rather broad, and hence the $\delta\hat{E} \times \underset{\sim}{B}\cdot\nabla_X$ coupling is rendered ineffective. Thus qualitatively speaking, Eq. (66) appears to indicate that the most effective nonlinear coupling occurs among modes which are moderately ballooning.

(ii) Electrostatic drift waves with adiabatic electrons and cold fluid ions. We assume, for simplicity, F_o to be a local Maxwellian; i.e.,

$$F_o = F_M(\psi, \varepsilon) \quad , \tag{70}$$

and neglect any equilibrium electric potential Φ_o. Since electrons are adiabatic, their nonlinear contributions are negligible, and the quasi-neutrality condition becomes

$$(1 + \tau)\delta\psi_n = 2\pi \int B d\mu \int dv_\parallel \; \delta\bar{h}_{in} J_o(\gamma_i) \equiv \delta\tilde{n}_{in} \quad . \tag{71}$$

Here, $\delta\psi_n \equiv e\delta\hat{\Phi}_n/T_e$, $\tau = T_e/T_i >> 1$, and $J_o(\gamma_i) \simeq 1 - \gamma_i^2/4$. Meanwhile, neglecting the ion-sound term, we multiply the ion Eq. (69) by $J_o(\gamma_i)$ and carry out the velocity integration to obtain

$$(\partial/\partial t + i\underset{\sim}{k}_\perp\cdot\bar{\underset{\sim}{v}}_{di})\delta\tilde{n}_{in} = \tau(1 - b_i)(\partial/\partial t - i\omega_{*in})\delta\psi_n$$

$$- (e/m_i)2\pi \int B d\mu \int dv_\parallel \; J_o(\gamma_i)(\bar{R}_{n\ell})_{in} \quad , \tag{72}$$

where, for $j = e, i$,

$$\bar{v}_{dj} = \underset{\sim}{e}_\parallel \times (v_j^2/2)(\nabla_X \ln B + \underset{\sim}{e}_\parallel\cdot\nabla\underset{\sim}{e}_\parallel)/\Omega_j \quad , \tag{73}$$

$$\omega_{*j} = (v_j^2 nB/2\Omega_j)|\partial \ln N_{oj}/\partial\psi| \quad , \tag{74}$$

and $b_i = k_\perp^2 \rho_i^2/2$. In deriving Eq. (72), we have noted that in the zeroth-order approximation, $\delta\bar{h}_{in} \simeq \delta\tilde{n}_{in} F_{Mi}$. Expanding the J_o's in $(\bar{R}_{n\ell})_{in}$ and using the quasi-neutrality condition, Eq. (71), it is then straightforward to derive

$$[(1 + k_\perp^2\rho_s^2)(\partial/\partial t) - i\underset{\sim}{k}_\perp\cdot\bar{\underset{\sim}{v}}_{de} + i\omega_{*e}]\delta\psi_n$$

$$= \pi C_s^2 \sum_{\substack{n'+n'' \\ =n}} \sum_{\substack{n'k'+n''k'' \\ =nk}} \sum_p \exp(-in''2\pi pQ)\widehat{C}_{n',n''} \int_{-\infty}^{\infty} d\hat{\theta}_{n'}\, \delta(\hat{\theta}_n - \hat{\theta}_{n'})$$

$$\int_{-\infty}^{\infty} d\hat{\theta}_{n''}\, \delta(\hat{\theta}_{n''} - \hat{\theta}_n - 2\pi p)\rho_s^2[(k_\perp'')^2 - (k_\perp')^2]\delta\phi_{n'}\delta\phi_{n''} \quad , \tag{75}$$

where $C_s^2 = \tau v_i^2/2$ and $\rho_s = C_s/\Omega_i$. Equation (75) may be regarded as the tokamak version of the nonlinear drift-wave equation of Hasegawa and Mima.[14] It contains the poloidal mode-coupling effects due to the $\nabla_{\sim x} B$ and curvature drifts and employs the ballooning-mode representation. The assumption of adiabatic electrons and cold fluid ions thus results in a single nonlinear equation instead of coupled ones, as in more general cases. To make Eq. (75) more tractable, we further assume concentric, circular magnetic surfaces and ignore the global wavenumber (k). Equation (75) then reduces to

$$[(1 + \tilde{k}_\perp^2\rho_s^2)(\partial/\partial t) - i\bar{\omega}_{de}T_c(\hat{\theta}) + i\omega_{*e}]\delta\phi(n, \hat{\theta})$$

$$= \pi\Omega_i\hat{S} \sum_{\substack{n'+n'' \\ =n}} \sum_p \exp(-in''2\pi pQ)\rho_s^4\, (k_n,k_{n''}2\pi p)$$

$$[(\tilde{k}_{\perp}'')^2 - (\tilde{k}_{\perp}')^2]\delta\phi(n', \hat{\theta})\delta\phi(n'', \hat{\theta} + 2\pi p) \quad , \tag{76}$$

where $\omega_{*e} = k_n \rho_s C_s / r_n$, $r_n^{-1} = |d\ln N/dr|$, $k_n = nQ/r$, $\bar{\omega}_{de} = 2r_n \omega_{*e}/R$, $\hat{S} = r(\partial Q/\partial r)/Q$,

$$T_c = \cos\hat{\theta} + \hat{S}\hat{\theta}\sin\hat{\theta} \quad , \tag{77}$$

$$\tilde{k}_{\perp}^2 = k_n^2(1 + \hat{S}^2\hat{\theta}^2) \quad , \tag{78}$$

$$(\tilde{k}_{\perp}')^2 = k_{n'}^2(1 + \hat{S}^2\hat{\theta}^2) \quad , \tag{79}$$

and

$$(\tilde{k}_{\perp}'')^2 = k_{n''}^2[1 + \hat{S}^2(\hat{\theta} + 2\pi p)^2] \quad . \tag{80}$$

We note that Eq. (76) is a two-dimensional $(n, \hat{\theta})$ nonlinear equation and may be viewed as the simplest model equation for electrostatic drift waves in axisymmetric tokamaks. Detailed studies of Eq. (76) will be reported in a future publication.

IV. NONLINEAR ION LANDAU DAMPING OF KINETIC ALFVÉN WAVES

As noted in Sec. I, the results obtained in Sec. II are rather general and, therefore, are also useful for describing nonlinear processes associated with low-frequency wave heating. To illustrate this potential application, we consider here nonlinear ion Landau damping of the mode-converted kinetic (shear) Alfvén waves.[11] The major difference between our work and Ref. 16 is that we include the diamagnetic drift effects. Another minor difference is that no small ion-Larmor-radius expansion is taken here.

For the sake of simplicity, we adopt the WKB description, Eqs. (49) and (50), and ignore the geometrical effects. Furthermore, F_o is taken to be Maxwellian and for the present purpose we can assume a weak-turbulence subsidiary ordering as well as ignore the compressional Alfvén perturbations $\delta \underset{\sim}{A}_\perp = 0$ [i.e., $\beta \equiv$ (plasma pressure/magnetic pressure) << 1]. Thus, the relevant equations are

$$\delta\bar{F}(\underset{\sim}{k}) = -(q/T)\delta\bar{\Phi}(\underset{\sim}{k})F_o + \delta\bar{H}_o(\underset{\sim}{k}) \exp(i\underset{\sim}{k}\cdot\underset{\sim}{v} \times \underset{\sim}{e}_\parallel/\Omega) \quad , \tag{81}$$

and

$$(\omega_{\underset{\sim}{k}} - k_\parallel v_\parallel)\delta H_o(\underset{\sim}{k}) = (q/T)(\omega + \omega_*)_{\underset{\sim}{k}} F_o \delta\bar{L}(\underset{\sim}{k})J_o(\gamma)$$
$$- i(c/B) \sum_{\underset{\sim}{k}'+\underset{\sim}{k}''=\underset{\sim}{k}} [\delta\bar{L}(\underset{\sim}{k}')J_o(\gamma')\delta\bar{H}_o(\underset{\sim}{k}'')(\underset{\sim}{k} \times \underset{\sim}{k}')\cdot\underset{\sim}{e}_\parallel] \ , \tag{82}$$

where $\delta\bar{L}(\underset{\sim}{k}) = \delta\bar{\phi}(\underset{\sim}{k}) - v_\parallel \delta\bar{A}_\parallel(\underset{\sim}{k})/c$, $\omega_{*\underset{\sim}{k}} = k_y cT/eBL_n$, $L_n^{-1} = |d\ln N_o/dx|$, x is the nonuniformity direction, and temperature gradients are ignored. Expanding $\delta\bar{H}_o = \delta H_o^{(1)} + \delta H_o^{(2)} + \delta H_o^{(3)} + \ldots$, we have, in the linear order,

$$\delta H_o^{(1)}(\underset{\sim}{k}) = \frac{q}{T}\cdot\frac{(\omega + \omega_*)_{\underset{\sim}{k}}}{\omega_{\underset{\sim}{k}} - k_\parallel v_\parallel} F_o \delta\bar{L}(\underset{\sim}{k})J_o(\gamma) \quad . \tag{83}$$

To calculate $\delta H_o^{(2),(3)}$, we let $(\omega_{\underset{\sim}{k}}, \underset{\sim}{k})$ and $(\omega_{\underset{\sim}{k}'}, \underset{\sim}{k}')$ be the normal modes, while $(\omega_{\underset{\sim}{q}} = \omega_{\underset{\sim}{k}} - \omega_{\underset{\sim}{k}'}, \underset{\sim}{q} = \underset{\sim}{k} - \underset{\sim}{k}')$ is the virtual or quasi mode. Also, we have in mind

$$|\omega_{\underset{\sim}{q}}| \sim |q_\parallel v_i| \sim |k_\parallel, k_\parallel'|v_i \ll |\omega_{*\underset{\sim}{q}}| \sim |\omega_{\underset{\sim}{k},\underset{\sim}{k}'}| \ll |k_\parallel, k_\parallel'|v_e \quad . \tag{84}$$

Let us now calculate the $\underset{\sim}{q}$-mode response. For the ions we find, noting Eq. (84), that

$$\delta H_{oi}^{(2)}(\underset{\sim}{q}) = [\omega_{*qe}/(\omega_{\underset{\sim}{q}} - q_\parallel v_\parallel)]\delta\hat{P}_{\underset{\sim}{q}}\delta\tilde{\Phi}(\underset{\sim}{k})\delta\tilde{\Phi}(\underset{\sim}{k}')J_o(\gamma)J_o(\gamma')F_{oi} \quad , \tag{85}$$

where

$$\delta\hat{P}_{\underset{\sim}{q}} = i(\Omega_i/\omega_{\underset{\sim}{k}'})\rho_s^2[(\underset{\sim}{k} \times \underset{\sim}{k}')\cdot\underset{\sim}{e}_\parallel] \quad , \tag{86}$$

and $\delta\tilde{\Phi} \equiv e\delta\tilde{\Phi}/T_e$. In deriving Eq. (85), we have observed that $|\delta\bar{\Phi}| \sim |\omega\delta\bar{A}_\parallel/ck_\parallel|$ for the $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ modes. As to electron nonlinearity, it is $O(|\omega/\omega_{*e}|_{\underset{\sim}{q}})$ smaller and, hence, negligible. Furthermore, for $\beta \ll 1$, we have $|\omega_{\underset{\sim}{q}}| \ll |q_\parallel V_A|$, and the $\underset{\sim}{q}$-mode response is predominately electrostatic. Here V_A is the Alfven speed. The quasi-neutrality condition then yields

$$\delta\tilde{\Phi}_{\underset{\sim}{q}}^{(1)} = -(F_1/\Gamma_{o\underset{\sim}{q}})\delta\hat{P}_{\underset{\sim}{q}}\delta\tilde{\Phi}(\underset{\sim}{k})\delta\tilde{\Phi}(\underset{\sim}{k}') \quad , \tag{87}$$

where

$$F_1(\underset{\sim}{k}, \underset{\sim}{k}') = 2\pi \int_o^\infty v_\perp\, dv_\perp\, J_o(\gamma)J_o(\gamma')J_o(\gamma_{\underset{\sim}{q}})F_M(v_\perp) \equiv \langle J_o(\gamma)J_o(\gamma')J_o(\gamma_{\underset{\sim}{q}})\rangle_\perp \quad , \tag{88}$$

$$\Gamma_{oq} = I_o(b_q)\exp(-b_q) \quad , \tag{89}$$

$\gamma_q = q_\perp v_\perp/\Omega_i$ and $b_q = q_\perp^2\rho_i^2/2$. Equations (83) and (87) then give $\delta H_{oi}^{(1)}(q)$.

For the k-mode, the ion nonlinear response can be shown to be

$$\delta H_{oi}^{(3)}(k) = |\delta\widetilde{P}_q|^2|\delta\widetilde{\Phi}(k')|^2\delta\widehat{\Phi}(k)F_{oi}\cdot\left(\frac{\omega_{*e}}{\omega - q_\parallel v_\parallel}\right)_q$$

$$\cdot[J_o^2(\gamma')J_o(\gamma) - F_1 J_o(\gamma_q)J_o(\gamma')/\Gamma_{oq}] \quad . \tag{90}$$

From Eq. (90) we can readily calculate $\delta n_i^{(3)}(k)$. Meanwhile, the electron nonlinearity, again is found to be $O(|\omega/\omega_{*e}|_q)$ smaller. Substituting $\delta n_i^{(3)}(k)$ and the linear response into the quasi-neutrality condition and parallel Ampere's law, it is straightforward to derive the following nonlinear dispersion relation

$$D_\ell(k) = -\frac{\omega_{*eq}}{|q_\parallel|v_i} Z_{iq}\left[1 - \left(1 - \frac{\omega_{*e}}{\omega}\right)_k\left(\frac{\omega}{k_\parallel V_A k_\perp \rho_s}\right)_k^2\right]|\delta\widetilde{P}_q\delta\widetilde{\Phi}(k')|^2$$

$$\cdot G(k, k') \quad , \tag{91}$$

where $D_\ell(k)$ is the linear dispersion relation given by

$$D_\ell(k) = (\tau + \omega_{*e}/\omega)_k(1 - \Gamma_{ok}) + (1 - \omega_{*e}/\omega)_k$$

$$\cdot[1 - \omega_k^2(\tau + \omega_{*e}/\omega)_k(1 - \Gamma_{ok})/(k_\parallel V_A k_\perp \rho_s)^2] \quad , \tag{92}$$

$$G(k, k') = \langle J_o^2(\gamma)J_o^2(\gamma')\rangle_\perp - F_1^2/\Gamma_{oq} \quad , \tag{93}$$

$Z_{iq} \equiv Z(\omega_q/|q_\parallel|v_i)$ with Z being the standard plasma dispersion function.

In the low-β limit of interest here, $|\omega_{*e}/k_{\parallel}V_A|_{\underset{\sim}{k}} < 1$, and assuming $\omega_{\underset{\sim}{k}} = \omega_{r\underset{\sim}{k}} + i\omega_{i\underset{\sim}{k}}$ with $|\omega_i/\omega_r| << 1$, Eq. (92) gives the known result that there exist two branches of waves; the drift waves with

$$\omega_{r\underset{\sim}{k},d} \simeq \omega_{*e\underset{\sim}{k}}\Gamma_{o\underset{\sim}{k}}/[1 + \tau(1 - \Gamma_{o\underset{\sim}{k}})] \quad , \tag{94}$$

and the kinetic shear Alfvén waves with

$$\omega^2_{r\underset{\sim}{k},A} \simeq (k_{\parallel}V_A)^2[1 + \tau(1 - \Gamma_{o\underset{\sim}{k}})]k_{\perp}^2\rho_s^2/\tau(1 - \Gamma_{o\underset{\sim}{k}}) \quad . \tag{95}$$

We note that nonlinear ion Landau damping of the drift waves, as described by Eq. (91), has previously been investigated.[17] Concentrating on the kinetic shear-Alfvén waves, we find

$$\left(\frac{\omega_i}{\omega_r}\right)_{\underset{\sim}{k},A} = \frac{(\omega_{*\underset{\sim}{k}'} - \omega_{*\underset{\sim}{k}})_e}{2\tau(1 - \Gamma_{o\underset{\sim}{k}})[1 + \tau(1 - \Gamma_{o\underset{\sim}{k}})]} G(\underset{\sim}{k}, \underset{\sim}{k}') \frac{\mathrm{Im}\, Z_{i\underset{\sim}{q}}}{|q_{\parallel}v_i|} |\delta\tilde{\Phi}(\underset{\sim}{k}')\delta\tilde{F}_{\underset{\sim}{q}}|^2 \quad . \tag{96}$$

Comparing the ω_i given by Eq. (96) to that of Ref. 16 (which, we recall, does not incude the ω_* effects), we find the present parametric-decay growth rate is larger by a factor of $O(|\omega_{*i\underset{\sim}{q}}|/|q_{\parallel}v_i|) > 1$. Furthermore, noting that $\mathrm{Im}\, Z_{i\underset{\sim}{q}} \simeq |q_{\parallel}|v_i\pi\delta(\omega_{\underset{\sim}{k}'} - \omega_{\underset{\sim}{k}})$, $G(\underset{\sim}{k}, \underset{\sim}{k}') \geq 0$, and $\omega_i \propto (\omega_{*\underset{\sim}{k}'} - \omega_{*\underset{\sim}{k}}) \propto (k_y' - k_y)$; hence, the daughter wave $(\omega_{\underset{\sim}{k}}, \underset{\sim}{k})$ has a smaller k_y but the same ω as the pump wave $(\omega_{\underset{\sim}{k}'}, \underset{\sim}{k}')$. This is qualitatively different from the results of Ref. 16, where $\omega_i \propto (\omega_{\underset{\sim}{k}'} - \omega_{\underset{\sim}{k}})$. In fact, as might be expected, one can show that the results of Ref. 16 are valid for $|\omega_{*i\underset{\sim}{q}}| < |q_{\parallel}v_i|$, i.e., in the opposite limit.

Finally, we briefly discuss the implication of the present results on the shear Alfvén wave heating scheme. Since the pump wave (i.e., the mode-converted kinetic shear Alfvén wave) has $k'_x\rho_i \sim O(1)$ and $k'_y\rho_i \sim O(\rho_i/a) \ll 1$ with a being the tokamak minor radius, we have $\omega_i \propto |\underset{\sim}{k} \times \underset{\sim}{k}' \cdot \hat{e}_\parallel|^2 = (k'_x k_y)^2$, and hence nonlinear coupling becomes appreciable for $k_y\rho_i \sim O(1) \gg k'_y\rho_i$; that is, $\omega_i < 0$. Thus, our results suggest that parametric decay through nonlinear ion Landau damping will be, in general, unlikely to occur.

V. CONCLUSIONS AND DISCUSSIONS

In this work, a systematic formalism for the nonlinear interactions of microscopic low-frequency electromagnetic waves has been developed. This formalism extends the linear gyrokinetic formalism of Refs. 1 and 2 into the nonlinear regime. Corresponding nonlinear gyrokinetic equations, valid for general magnetic field configurations as well as the strong-turbulence regime, are derived. Effects due to fully electromagnetic perturbations, finite Larmor radii, plasma inhomogeneities, magnetic drifts, and magnetic trapping are retained. The results are thus rather general and should have wide application. Note that it is straightforward to extend the treatment to include collisional effects by retaining the Fokker-Planck collision operator in Eqs. (7) and (15). As a specific example of possible applications, we consider axisymmetric tokamaks and explore in more detail the properties of the nonlinear gyrokinetic equations via the ballooning-mode representation. Furthermore, a single nonlinear equation is derived for electrostatic drift waves in the limit of adiabatic electrons and cold fluid ions which retains the crucial features of toroidal geometry and nonlinear coupling and, therefore, may be exploited as a useful model equation. On the other hand, we have also applied the results to the shear Alfvén wave heating scheme and considered the parametric decays of the mode-converted kinetic Alfvén waves via nonlinear ion Landau damping (ion-induced scattering). Here it is found that the diamagnetic drift effects not only enhance the parametric growth rate but also modify the decay process qualitatively. That is, the daughter waves tend to have smaller perpendicular (to $\underset{\sim}{B}$ and the density gradient) wavenumbers (i.e., smaller poloidal mode numbers for tokamak plasmas) instead of frequencies as suggested by uniform-plasma calculations. This property, therefore, suggests that the mode-converted kinetic Alfvén waves with small poloidal mode numbers will not, in general, parametrically decay via nonlinear ion Landau damping. Other possible channels, such as resonant decay to drift waves, of course, are not ruled out and need to be investigated.

Let us comment on some other possible applications of the general results obtained here. One possible application is the following: by taking the limit of adiabatic electrons and cold fluid ions, a sufficiently simple nonlinear equation may also be derived for the kinetic shear-Alfvén waves and hence could serve as a model equation for studying the nonlinear evolution and associated transport of kinetic ballooning-mode instabilities.[18] Another interesting application is to simulate plasmas employing the nonlinear gyrokinetic equation, Eq. (48) or, more transparently, Eq. (50). This application will further extend the gyrokinetic simulation scheme initially proposed by Lee[19] for electrostatic waves in simple (slab) geometries to fully electromagnetic pertubations in general plasma equilibria. The details of these applications, however, remain to be worked out.

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