Invariant Metrics for Hamiltonian Systems*

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Abstract

In this paper, invariant metrics are constructed for Hamiltonian systems. These metrics give rise to norms on the space of homeogeneous polynomials of phase-space variables. For an accelerator lattice described by a Hamiltonian, these norms characterize the nonlinear content of the lattice. Therefore, the performance of the lattice can be improved by minimizing the norm as a function of parameters describing the beam-line elements in the lattice. A four-fold increase in the dynamic aperture of a model FODO cell is obtained using this procedure.

I. INTRODUCTION

Given an accelerator lattice, various correction schemes (lumped correctors, shuffling of magnets etc.) can be used to improve its performance. However, to be able to implement these schemes, it is essential to have a "merit function" (depending on the parameters describing the beam-line elements) that can be minimized to produce the optimal lattice. This merit function should be a reliable measure of the nonlinearity of the lattice since it is nonlinear effects that degrade the performance of the lattice and it is these effects that have to be minimized. In this paper, we propose a merit function satisfying the above criteria. This function will turn out to be a positive definite symmetric bilinear form invariant under the action of the unitary group U(3).

We restrict ourselves to accelerator lattices described by a (nonlinear) Hamiltonian in a six-dimensional phase space. Given such a system, an equivalent description is provided by the following one-pass or one-period symplectic map[1]

$$\mathcal{M} = Me^{:f_3:}e^{:f_4:}\ldots e^{:f_m:}\ldots \tag{1}$$

Here, the 6×6 matrix M characterizes the linear part of the map and the Lie transformations $e^{:f_m:}$ characterize the nonlinear part. The operator $:f_m:$ is the Lie operator corresponding to the homogeneous polynomial $f_m(z)$ of degree m in the phase-space variables z_i $(i = 1, 2, \ldots 6)$.

The polynomial $f_m(z)$ can be expanded as follows:

$$f_m(z) = a_\alpha^{(m)} P_\alpha^{(m)}(z) , \tag{2}$$

where we have used Einstein's summation convention. Here $P_\alpha^{(m)}(z)$ denotes the following mth degree basis monomial

$$P_\alpha^{(m)}(z) = q_1^{r_1} p_1^{r_2} \ldots q_n^{r_5} p_i^{r} \quad r_1 + \cdots + r_6 = m. \tag{3}$$

The monomials are ordered using the index α[2]. The summation over α in Eq. (2) extends from 1 to $N(m)$ where $N(m)$ is given by the relation[3]

$$N(m) = \binom{m+5}{m}. \tag{4}$$

In this paper, we will construct a symmetric positive definite bilinear form on the space spanned by homogeneous polynomials of degree m in phase-space variables. This will enable us to define a norm on this space. This norm will also be invariant under the action of the unitary group U(3). Since the nonlinear part of a symplectic map is specified by homogeneous polynomials, a norm defined on the space of homogeneous polynomials can be used to quantify the nonlinear content of the map (or equivalently the lattice). Moreover, the norm is a function of parameters specifying the beam-line elements of the accelerator lattice under consideration. Therefore, one can vary these parameters so as to minimize this norm. This should lead to improvements in performance of the lattice. This is shown to be true for a model FODO cell later in the paper.

II. CONSTRUCTION OF INVARIANT METRICS

We start by defining a bilinear form $g_{\alpha\beta}^{(m)}$ as follows

$$g_{\alpha\beta}^{(m)} \equiv (P_\alpha^{(m)}(z) , P_\beta^{(m)}(z)) \tag{5}$$

where $(P_\alpha^{(m)}(z) , P_\beta^{(m)}(z))$ denotes a bilinear form defined on the space of basis monomials of degree m. We require the bilinear form to be symmetric and positive definite so that it can be used to define a norm on this space.

Ideally, we would also require $g_{\alpha\beta}^{(m)}$ to be invariant under the action of all symplectic maps. Then, we would obtain a metric as unique as possible. However, this turns out to be impossible. The set of all symplectic maps forms a non-compact Lie group[1]. It can be shown that such groups can not have invariant metrics[4]. Only compact groups can have such metrics. Therefore, we are forced to impose a more modest requirement that the metric be invariant under the action of the largest compact subgroup of the symplectic group. We will require that $g_{\alpha\beta}^{(m)}$ be invariant under the action of the compact unitary group U(3).

We define the bilinear form $(P_\alpha^{(m)}(z), P_\beta^{(m)}(z))$ as follows[3]:

$$(P_\alpha^{(m)}(z), P_\beta^{(m)}(z)) \equiv \frac{1}{r^{2m}} \int d\Omega_5 \, P_\alpha^{(m)}(z) \, P_\beta^{(m)}(z) , \tag{6}$$

where $d\Omega_5$ is the solid angle for the 5-sphere and $r^2 = q_1^2 + p_1^2 + \cdots + p_3^2$. We show that it is invariant under the action of U(3). Consider the following expression

$$(\dot{U}P_\alpha^{(m)}(z), \dot{U}P_\beta^{(m)}(z)) = \frac{1}{r^{2m}} \int d\Omega_5 \, P_\alpha^{(m)}(Uz) P_\beta^{(m)}(Uz). \tag{7}$$

Here, $\dot{U}$ is the Lie transformation corresponding to the element U belonging to U(3)[3]. We change to a new variable z' defined to be equal to Uz. Since the solid angle $d\Omega_5$ is invariant under the action of U(3), we obtain the relation

$$(\dot{U}P_\alpha^{(m)}(z), \dot{U}P_\beta^{(m)}(z)) = \frac{1}{r^{2m}} \int d\Omega_5 \, P_\alpha^{(m)}(z') P_\beta^{(m)}(z'). \tag{8}$$

Since U was an arbitrary element of U(3), we get the desired result

$$(\dot{U}P_\alpha^{(m)}(z), \dot{U}P_\beta^{(m)}(z)) = (P_\alpha^{(m)}(z), P_\beta^{(m)}(z)) \tag{9}$$
$$\forall \, U \, \in \, \text{U(3)}. \tag{10}$$

It is easily seen from the definition that this bilinear form is symmetric. Obviously, it is also positive definite. Hence, Eq. (6) gives a valid invariant metric.

This invariant metric can be evaluated as follows. Consider the following equation

$$\int d^6 z \, e^{-r^2} P_\alpha^{(m)}(z) \, P_\beta^{(m)}(z) = \int dr \, r^{2m+5} e^{-r^2}$$
$$\times \frac{1}{r^{2m}} \int d\Omega_5 \, P_\alpha^{(m)}(z) \, P_\beta^{(m)}(z) . \tag{11}$$

This is seen to be correct since we have merely reexpressed the infinitesimal volume element $d^6 z$ in terms of the radius vector r and solid angle $d\Omega_5$. Inserting Eq. (6) in Eq. (11), we obtain the relation

$$(P_\alpha^{(m)}(z), P_\beta^{(m)}(z)) = \frac{\int d^6 z \, e^{-r^2} P_\alpha^{(m)}(z) \, P_\beta^{(m)}(z)}{\int dr \, e^{-r^2} r^{2m+5}} . \tag{12}$$

Both the numerator and the denominator can now evaluated easily[3].

Using the above construction, we obtain the following expression for $g_{\alpha\beta}^{(3)}$ (we do not list entries below the diagonal; we also restrict ourselves to the four-dimensional case due to lack of space):

$$g_{i\,i}^{(3)} = c_1 \quad i = 1, 11, 17, 20,$$
$$g_{i\,i}^{(3)} = c_2 \quad i = 2, 3, 4, 5, 8, 10, 12, 13, 14, 16, 18, 19,$$
$$g_{i\,i}^{(3)} = c_3 \quad i = 6, 7, 9, 15,$$
$$g_{1\,5}^{(3)} = g_{1\,8}^{(3)} = g_{1\,10}^{(3)} = g_{2\,11}^{(3)} = c_2,$$
$$g_{3\,17}^{(3)} = g_{4\,20}^{(3)} = g_{11\,14}^{(3)} = g_{11\,16}^{(3)} = c_2,$$
$$g_{12\,17}^{(3)} = g_{13\,20}^{(3)} = g_{17\,19}^{(3)} = g_{18\,20}^{(3)} = c_2,$$
$$g_{2\,14}^{(3)} = g_{2\,16}^{(3)} = g_{3\,12}^{(3)} = g_{3\,19}^{(3)} = c_3,$$
$$g_{4\,13}^{(3)} = g_{4\,18}^{(3)} = g_{5\,8}^{(3)} = g_{5\,10}^{(3)} = c_3,$$
$$g_{6\,10}^{(3)} = g_{12\,19}^{(3)} = g_{13\,18}^{(3)} = g_{14\,16}^{(3)} = c_3.$$

Here the indices $1, 2, \ldots 20$ represent monomials q_1^3, $q_1^2 p_1$, $q_1^2 q_2$, $q_1^2 p_2$, $q_1 p_1^2$, $q_1 p_1 q_2$, $q_1 p_1 p_2$, $q_1 q_2^2$, $q_1 q_2 p_2$, $q_1 p_2^2$, p_1^3, $p_1^2 q_2$, $p_1^2 p_2$, $p_1 q_2^2$, $p_1 q_2 p_2$, $p_1 p_2^2$, q_2^3, $q_2^2 p_2$, $q_2 p_2^2$, and p_2^3 respectively. And the constants c_1, c_2, and c_3 have the following values

$$c_1 = 5/64, \quad c_2 = c_1/5, \quad c_3 = c_1/15. \tag{13}$$

III. CONSTRUCTION OF NORMS

Using the metric defined above, we now define a norm on the space of homogeneous polynomials of degree m. This norm can then serve as a merit function that can be used to minimize nonlinearities of degree m.

Each metric $g_{\alpha\beta}^{(m)}$ gives rise to a norm on the space of homogeneous polynomials of degree m. Consider a general homogeneous polynomial of degree m denoted by f_m. We are interested in obtaining a norm for storage ring lattices. Since the emittances in the three degrees of freedom can be quite different, we normalize them by factoring out the betatron functions This is achieved by going to the so-called normal form[1] of the linear part M of the map $\mathcal{M}$. Let A be the symplectic transformation that takes M into its normal form N i.e.

$$N = AMA^{-1} \tag{14}$$

where N is a block-diagonal matrix with 2×2 blocks on the diagonal[5]. Applying the transformation A to the map $\mathcal{M}$, we obtain the result

$$\mathcal{N}_2 = A\mathcal{M}A^{-1} = AMA^{-1}Ae^{:f_3:}e^{:f_4:}\ldots e^{:f_m:}\ldots A^{-1}. \tag{15}$$

Using Eq. (14), we get the relation

$$\mathcal{N}_2 = Ne^{:f_3^{tr}:}e^{:f_4^{tr}:}\ldots e^{:f_m^{tr}:}\ldots \tag{16}$$

where[1]

$$f_m^{tr} = Af_m(z) = f_m(Az). \tag{17}$$

Since A depends on M, the f_m^{tr}'s also now depend on the linear part of the map. These transformed f_m's can be reexpressed in the original basis as follows

$$f_m^{tr}(z) = b_\alpha^{(m)} P_\alpha^{(m)}(z) . \tag{18}$$

We are now in a position to define a norm on the space of homogeneous polynomials

$$\|f_m\| \equiv (f_m^{\mathrm{tr}}, f_m^{\mathrm{tr}})^{\frac{1}{2}} = (b_\alpha^{(m)} P_\alpha^{(m)}(z), b_\beta^{(m)} P_\beta^{(m)}(z))^{\frac{1}{2}}. \tag{19}$$

Using Eq. (5), we get the following result

$$\|f_m\| = (g_{\alpha\beta}^{(m)} \, b_\alpha^{(m)} b_\beta^{(m)})^{\frac{1}{2}}. \tag{20}$$

From the above equation, we see that the norm $\|f_m\|$ is a function of parameters characterizing the beam-line elements in the accelerator lattice that we started out with (since the coefficients $b_\alpha^{(m)}$ are determined by these parameters). Therefore, one can think of varying these parameters so as to minimize this norm. Since the norm quantifies the nonlinear content of the lattice, this may lead to improvements in the performance of the system. We also note that $\|f_m\|^2$ is a positive definite quadratic function of the strengths of the m-th order multipoles (e.g. $\|f_3\|^2$ is such a function of the sextupole strengths). Hence $\|f_m\|^2$ is guaranteed to have an unique global minimum as a function of these multipole strengths.

IV. EXAMPLE

In this section, we study a model FODO cell with systematic sextupole errors to illustrate the utility of the invariant metric. The FODO cell consists of the following elements: a thin-sextupole corrector, a drift, a focusing-quadrupole with fringe fields and sextupole error, a drift, a thin-sextupole corrector, a drift, a defocusing-quadrupole with fringe fields and sextupole error, a drift, and finally, another thin-sextupole corrector.

First, we turn off the correctors and compute the norm $\|f_3\|^2$ in a four-dimensional phase space. It has a certain value (≈ 400 in our case). Next, we set the corrector strengths by minimizing the norm. The minimum is found to correspond roughly[6] to setting the corrector strengths according to Simpson's rule (i.e. the three strenghts are in the ratio 1:4:1)[7]. For this setting, the value of $\|f_3\|^2$ is reduced (from the uncorrected case) by almost two orders of magnitude. To verify that third order nonlinearities have actually been reduced in magnitude, the dynamic aperture of the FODO cell was computed for these two cases. The dynamic aperture for the corrected case was found to be larger by a factor of four.

V. SUMMARY

In this paper, we constructed invariant merit functions for accelerator lattices described by Hamiltonians. These metrics were used to define norms on the space of homogeneous polynomials of phase-space variables. These norms quantify the nonlinear content of the accelerator lattice. They can be minimized as a function of parameters describing the beam-line elements to improve the performance of the lattice. Finally, we considered a model FODO cell with sextupole errors. By minimizing the third degree norm using correctors, we obtained a four-fold increase in the dynamic aperture.

VI. REFERENCES

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[5] We note that in going to the normal form, we have "used up" the non-compact part of the symplectic group Sp(6,R) and we are left only with the compact subgroup.

[6] Due to lack of a proper optimizer routine, we did not make a global search for the minimum. In this case, we already knew what the approximate minimum was supposed to be from theoretical considerations. See Ref. [7] below.

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