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Application of Catastrophe Theory to Nuclear Structure

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Abstract

Three two-parameter models, one describing an A-body system (the atomic nucleus) and two describing many-body systems (the van der Waals gas and the ferroelectric (perovskite) system) are compared within the framework of catastrophe theory. It is shown that each has a critical point (second order phase transition) when the two counteracting forces controlling it are in balance; further, each undergoes a first order phase transition when one of the forces vanishes (the deforming force for the nucleus, the attractive force for the van der Waals gas, and the dielectric constant for the perovskite). Finally, when both parameters are kept constant, a kind of phase transition may occur at a "critical" angular momentum, "critical" pressure, and "critical" electric field.

It is a great pleasure to dedicate the following considerations to Maurice Goldhaber, whose elegant and imaginative approach to physics is an inspiration and a source of constant pleasure to the whole physics community.

1) Background

It is obvious that nuclei are extremely complex systems consisting as they do of a finite but "large" number of protons and neutrons, interacting via strong forces which are only incompletely known. Even if it were known that all the relevant forces were two body forces, described by relatively simple interactions, the resulting many body problem would be so intractable that it would be difficult to draw a reliable conclusion from the necessarily approximate calculations. In addition, it is likely (as is the situation in solid state physics and statistical physics) that in different physical circumstances different aspects of the many body problem dominate the physical description. Consequently, even if the possible importance of meson degrees of freedom, relativistic effects, and internal nucleon structure is ignored, the statement that a sufficiently detailed knowledge of the nucleon-nucleon interactions is sufficient to deduce the nuclear properties would be more in the nature of a scientific exhortation than of an implementable principle.

It is important to recall that in spite of the intrinsic complexity of nuclei, they exhibit a wealth of remarkable empirical regularities. It is in the best tradition of physics to first exploit these regularities by making a physical model whose physical and mathematical behavior

simulates the nuclear properties in that particular domain. Later, the regularities codified by the model can be fit into a more general scheme, which starts from a more fundamental viewpoint. The original shell model, or individual particle model, exploits the many nuclear properties which can be described in terms of a few valence nucleons, while the collective model is in first instance based on the collective features of rotational spectra. The shell model works well for nuclei near closed shells, whereas the collective description is most efficient away from magic number nuclei. In almost all of these models a certain amount of experimental information is used, as for example in obtaining the matrix elements of an effective interaction.

In view of the complexity of atomic nuclei, it is clear that quite distinct approaches might be useful in obtaining new insights. One approach would be to deepen the understanding by investigating what concepts and rules of the models could be obtained from a more detailed examination of the many body problem. For example, in the interacting Boson model (IBA),[1] a specific microscopic dynamical proposal is made to account for the complex band structure exhibited in nuclear level schemes. In this model certain states of the "fermion" system consisting of an even number of both valence neutrons (N_n) and protons (N_p) is approximately described in terms of an equivalent set of N interacting Bosons $\left(N = \frac{N_n + N_p}{2}\right)$.

The other approach, which is dealt with in this paper, is to derive from the experimental regularities and the empirical rules "directly" a phenomenological model, and to investigate the consequences and conclusions following from it. Once an effective model of some generality

has been obtained, one can attempt a fundamental justification of the concepts used. It often happens that the model suggested by the experiments yields new information via mathematical analysis. In this connection it is interesting to explore the mathematical organization which catastrophe theory[2] imposes on a general class of models. In investigating the nature of the mathematical constraints which follow via catastrophe theory from the parameterized experimental data, one follows the opposite direction of studying the return to a more microscopic theory. In this approach the detailed nature of the interaction is not important, one just investigates the possible frameworks in which the theory is required to fit by the present organization of the data. The particular class of phenomena to be explored here in a somewhat tentative fashion are the regularities observed in the excited states of even-even nuclei.

2) The excited states of even-even nuclei. The variable moment of inertia model.

Very striking empirical regularities were observed in the ground state bands of even-even nuclei i.e. bands with level spins (J) and parities (π): 0+, 2+, 4+, ... In the introduction of a recent review article[3] entitled "The Variable Moment of Inertia Model and Theories of Nuclear Collective Motion," the sequence of discoveries underlying the model has been described, ranging from rotational bands, near-harmonic bands (interpreted by a surface vibration model, the first of the interacting boson models), bands founds in "transitional" (between vibrational and rotational) nuclei (in osmium and platinum), to bands found in near-

magic and finally in magic nuclei. While it was thought that all these bands in particular the first two, correspond to quite different modes of collective motions, the scaling law first observed by Mallmann[4] suggested that all these bands result from one and the same mechanism: Mallmann found that the ratio of energies E_J (where J = 6 or 8) to the 2+ state excitation energy are universal functions of one ratio, E_4/E_2. One can therefore infer that the ground state band spectra are functions depending on just two parameters.[5] Mallmann also showed that the observed behavior deviates already very close to the rotational limit from that proposed by Bohr and Mottelson, who expected a correction term for the energy $\alpha(J(J+1))^2$ due to rotation-vibration mixing.

Since an earlier, more fundamental approach, namely to find a general expression for the energy spectrum of the form $E_J(N_p,N_n)$ did not lead to success, the new challenge was to guess at the dynamics underlying the Mallmann curves, whose number could soon be extended from two to six (J = 16) or even higher thanks to new data. The two main avenues available were to postulate either a rotational or a vibrational mechanism. If rotation is assumed, one has to conclude that since in general the energy spacings empirically found for increasing J are smaller than those given by the expression for a rigid rotor, $\frac{J(J+1)}{2I}$, it follows that the moment of inertia I increases with increasing J. This increase can be accounted for by adding to the rotational term a term $\frac{C}{2}(I - I_0)^2$ (accounting for the increase of centrifugal and Coriolis forces with increasing rotational velocity),

together with a variational principle required to determine I: The basic equations for the Variable Moment of Inertia Model are then

$$E_J(I) = \frac{1}{2} C(I - I_0)^2 + \frac{J(J+1)}{2I} \qquad (1)$$

I_0 and C are parameters characteristic for a given nucleus. The moment of inertia I itself is a dynamical variable; its dependence on the model parameters is given by

$$\left.\frac{\partial E}{\partial I}\right|_J = 0 \rightarrow I^3 - I_0 I^2 = \frac{J(J+1)}{2C} \qquad (2)$$

The second approach possible is the assumption that the dynamics underlying the Mallmann curves is that of the anharmonic vibrator. However, a comparison of the two approaches showed[6] that the rotational interpretation is in considerably better agreement with the data. Recent experimental results for higher spin states in near-magic nuclei[7] strengthen this conclusion. (It may be worth mentioning here that there is almost no correlation between the basic variable E_4/E_2 of the VMI model and the boson number N up to $N \tilde{<} 11$.)

Finite positive values of I_0 and C yield one real root for I (which is determined from equation (2)). For $I_0 > 0$, I_0 can be interpreted as the ground state moment of inertia. One finds empirically that I_0 rises as more proton and neutron pairs are added to closed shells. C, the stiffness parameter, decreases by about 5 orders of magnitude between the nucleus ^{12}C and the heaviest actinides. Within a nuclear species, C rises as one approaches a stable isotope.[3]

It is most important that the model defined by equations (1) and (2) can be extended to negative values of I_0;[5,3] this extension was suggested by the observation that I_0 decreases sharply as magic numbers are approached and appears to become negative before N = 82 (magic number) is reached. One finds good experimental agreement with (1) and (2) also for negative values of I_0. This means that while for $I_0 > 0$, $I(J = 0) = I_0$, for $I_0 \leq 0$, $I(J = 0) = 0$, i.e. the ground state moment of inertia vanishes and the nucleus actively resists being rotated. This resistance is measured by the threshold energy $\frac{C}{2} I_0^2$. However, in contrast to Mallmann's findings (based on very scarce data), bands in magic nuclei lie below the limit of validity of the model ($E_4/E_2 = 1.82$), for which $I_0 = -\infty$. Below this limit, an abrupt change in band structure occurs; in particular, values for 4+, 6+, 8+ etc. become almost degenerate. As we shall see, this limit corresponds to a first order phase transition. Moreover, it is easy to show that as I_0 changes sign, $\left.\frac{\partial I}{\partial J}\right|_{J=0}$ has a discontinuity reminiscent of a second order phase transition. We will show later that $I_0 = 0$ precisely corresponds to the critical point.

A phenomenon resembling a phase transition is further observed for the ground state band of each ever-even nucleus at a state J_c, where J_c denotes the "critical spin": above J_c a more or less abrupt deviation from the VMI prediction occurs, usually corresponding to a sudden increase in moment of inertia. For nuclei with many valence nucleon pairs J_c is high (between 10 and 16 in the rare earth region, > 20 in the actinides.) For near-magic nuclei[7] J_c is usually ≤ 8. For "pseudomagic nuclei," i.e. nuclei possessing 2(4) proton holes and 2(4) neutrons, or

vice versa, $J_c \leq 4$. However, in contrast to the large variation for J_c, the rotational velocity ω_c at which the deviation occurs, is approximately the same for all bands. (ω is related to I and J by the semiclassical equation $I\omega = \sqrt{J(J+1)}$). Phenomenologically, this effect can be described by band crossing, caused by one of several possible physical mechanisms. In many cases this mechanism consists in the decoupling of a single neutron or proton pair in a high j orbital under the influence of the Coriolis force. Subsequent alignment of this pair along the axis of symmetry brings about a large contribution to the moment of inertia.

Before we proceed to the analysis of the VMI model, let us briefly address the question of what can be learned from the VMI model about the organization of the rotating assembly of nucleons? Unfortunately the answer is that in addition to a number of previous efforts, a recent attempt[8] to derive the model from a fundamental approach to collective motions of a nuclear many-body system has so far not succeeded, mainly because of the difficulties of taking the effects of the Pauli principle properly into account. However, it was possible to infer the configurations of the rotating assembly of nucleons by correlating the measured electric quadrupole transition moments with the moments of inertia derived via VMI from the spectrum:[9] for nuclei with not more than four valence nucleons of one kind (neutrons or protons), a rotating "alpha - dumbbell" model suggests itself, whereas for nuclei with ≥ 6 valence neutrons and protons, a two-fluid model consisting of a superfluid and an inertial fluid gives excellent agreement with the data.

3) Ideas of catastrophe theory

Thom and Mather proved a beautiful abstract mathematical classification theorem, which for the present purposes may be paraphrased in this way. Suppose one has a system described by n variables, $x_1 \ldots x_n$, and s parameters $\xi_1 \ldots \xi_s$. Suppose the physical states are given by

$$\left(\frac{\partial V}{\partial x_i}\right) = 0 \qquad i = 1 \ldots n. \tag{3}$$

This is a set of points, of s dimensions, call it V_s. To each point in V_s corresponds a physical state. Now consider a projection of V_s on the space $\xi_1 \ldots \xi_s$. If the projection is one to one, a knowledge of the ξ's will define the physical state uniquely. If the projection is not one to one, a single set of ξ's will define a number of physical states. If one moves in the ξ space from a region where one state in V_s corresponds to one set of ξ's, to a region where more than one state in V_s corresponds to one ξ, one has gone from a one phase region to a region of two or more phases. As the ξ variables are changed in this manner, the system has undergone a discontinuous transition. Stated differently as one varies ξ's, the number and type of critical points (i.e., points where $\left(\frac{\partial V}{\partial x_i}\right) = 0$), changes. The mathematical results referred to give an exhaustive classification of the changes in the number and type of critical points for certain values of s. The result of Thom and Mather now asserts that near such a region in V_s, where these changes occur, one can obtain a canonical form for V_i, the effective potential. (Note this is only true near the boundaries). What the form is depends only on the number of ξ

variables in question. For a 4 dimensional ξ space there are 7 such basic forms. Consequently the potential can be written

$$V_s\ (x_1 \ldots x_n,\ \xi_1 \ldots \xi_s) \cong V_0 + \sum_{i=1}^{7} \alpha_i V_i(x,\xi) \qquad (4)$$

Here the α_i are constants, the V_i are specific functions. In (4) V_0 is regular near the boundaries. The power of the theorem lies in the fact that the functions V_i all of which satisfy $\frac{\partial V_i}{\partial x_i} = 0$, have a universal form and can be determined once and for all. The Thom theorem determines these functions V_i uniquely, up to a diffeomorphism which is a differentiable homeomorphism. (A homeomorphism is a one-to one mapping, which is continuous and has a continuous inverse.) For differen numbers of parameters ξ the functions V_i have characteristically different forms. For s = 1,2, the form of the corresponding potential is unique

$$\begin{aligned} &s = 1 \quad V = x^3 + \xi_1 x && \xi_1 \\ &s = 2 \quad V = x^4 + \tfrac{1}{2}\xi_1 x^2 + \xi_2 x && \xi_1,\ \xi_2 \end{aligned} \qquad (5a)$$

If one has three ξ parameters, the effective potential can have a number of forms; one is

$$s = 3 \quad V = x^5 + \tfrac{1}{3}\xi_1 x^3 + \tfrac{1}{2}\xi_2 x^2 + \xi_3 x \qquad \xi_1,\ \xi_2,\ \xi_3\ . \qquad (5b)$$

There are <u>two</u> other possibilities for s = 3

$$V_3 = x_1^2 x_2 \pm x_2^3 + \xi_1 x_2^2 + \xi_2 x_1 + \xi_3 x_2. \qquad (5c)$$

The states are now given by

$$\frac{\partial V_3}{\partial x_1} = 0 \quad \underline{\text{and}} \quad \frac{\partial V_3}{\partial x_2} = 0 \tag{5d}$$

These cases are called the swallow's tail, the elliptic and hyperbolic umbilic respectively.[10] For $s > 3$ there are more complicated forms for V. One of the forms for $s = 4$ for the effective potential is

$$V = \frac{1}{6} x^6 + \frac{1}{4} \xi_1 x^4 + \frac{1}{3} \xi_2 x^3 + \frac{1}{2} \xi_3 x^2 + \xi_4 x \tag{5e}$$

The states now must satisfy the equation $\frac{\partial V}{\partial x} = 0$, which is a 5th degree equation. It will be noted later that the best experimental fit for a VMI model with additional parameters allowed leads just to such 5th degree equations.

A glance at (5) shows that if a system is parametrized by two variables ξ_1, ξ_2, and the states of the system can be obtained as the minimum of some effective potential, this potential must be (locally)

$$V = x^4 + \frac{1}{2} \xi_1 x^2 + \xi_2 x, \tag{6}$$

while the states must satisfy

$$\frac{\partial V}{\partial X} = 0 \Longrightarrow \quad x^3 + \xi_1 x + \xi_2 = 0. \tag{7}$$

Similarly if one wants to contemplate a description in which more than two parameters occur, one is compelled in this framework to go to (5c), (5d), or (5e).

It is clear that depending on ξ_1 and ξ_2, (7) either has one or three

real solutions for x. Thus corresponding to ξ_1 ξ_2, there is either one state (one minimum of V) or two states (+ 1 unstable state) (two minima + one maximum.) The region in the ξ_1 ξ_2 plane, where (7) has three real solutions for x, is separated from the region with one real solution by a cusp, whose equation is

$$4\ \xi_1^3 + 27\ \xi_2^2 = 0, \tag{8}$$

(See figure 1)

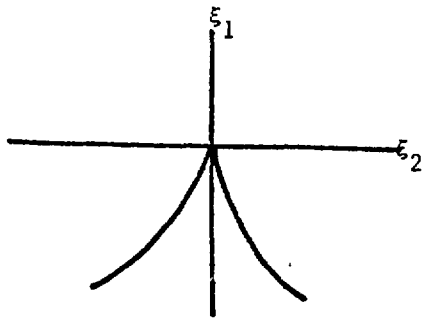


Figure 1

Eq. (8) can be obtained directly from the formula for the discriminant of a cubic equation. In principle, crossing the cusp in the ξ_1 ξ_2 plane corresponds to a discontinuous transition from a one-phase to a two-phase region. The Thom theorem is an abstract mathematical theorem. For its use in physics, the relation between the abstract parameters and physical quantities must be established. This must be done by casting the equations which characterize the physical states in one of the standard forms (5). In this process the relationship between the physical parameters and the mathematical ξ parameters is not always so direct. It is presumed that such an identification is unique. In the cases treated here that is certainly true.

It is simple for the case of the van der Waals gas, where experiments give as equation of state

(p = pressure, v = volume, T absolute temperature):

$$(p + \frac{a}{v^2})\ (v - b) = RT \tag{9}$$

(R is the gas constant, a and b, depend on the gas)

It is well known that there is a critical point, with temperature, volume, pressure given by

$$p_c = \frac{1}{27}\ \frac{a}{b^2} \qquad RT_c = \frac{8}{27}\ \frac{a}{b} \qquad v_c = 3b \tag{10}$$

For $T > T_c$ there is only a gas phase, for $T < T_c$ there are two possible phases. In that case, ($T < T_c$) moving along a constant p line the system will undergo a gas liquid phase transition.

For $T > T_c$ there is no phase transition. The van der Waals case fits exactly in the category (6) of Thom[11]. This can be seen explicitly by introducing new variables

$$\pi = \frac{p}{p_c} \qquad w = \frac{v}{v_c} \qquad t = \frac{T}{T_c} \quad \text{in (9)}, \tag{11}$$

which leads to

$$(\pi + \frac{3}{w^2})\ (w - \frac{1}{3}) = 8t \tag{12}$$

To recover a cubic of the form (6) write

$$x' = \frac{1}{w}\ , \text{ and introduce } x = x' - 1 \tag{13}$$

$$p' = \pi - 1 = \frac{p - p_c}{p_c} \tag{13a}$$

$$t' = t - 1 = \frac{T - T_c}{T_c} \tag{13b}$$

Substituting in (12) gives

$$x^3 + \frac{1}{3}(8t' + p')\, x + \frac{1}{3}(8t' - 2p') = 0 \tag{14}$$

or

$$x^3 + \xi_1 x + \xi_2 = 0 \qquad \text{with} \tag{15}$$

$$\xi_1 = \frac{1}{3}(8t' + p') \tag{16}$$

$$\xi_2 = \frac{1}{3}(8t' - 2p') \tag{17}$$

Thus the equation of state of the van der Waals gas liquid system fits perfectly in the simplest cusp catastrope scheme of Thom. Observe that the critical point, eq.(10) in terms of ξ_1, ξ_2, x, is given by

$$x = 0 \quad \xi_1 = \xi_2 = 0 \tag{18}$$

Note finally that since the van der Waals equation is cubic in the density, the identification of x with the density was forced by the equation defining the stable states (7). Also in the van der Waals model, the potential V itself has no especially transparent physical meaning. In the van der Waals case, the parameters "ξ" of Thom are therefore by (16) and (17) linear combinations of pressure and temperature. Note finally that crossing the cusp

produces a first order phase transition. More precisely: crossing the cusp, together with the Maxwell rule, yields a first order phase transition. (The Maxwell rule asserts that the stable isotherm, p = constant, bisects the area of the van der Waals loop.) At the critical point $\left(\frac{\partial p}{\partial v}\right)$ and $\left(\frac{\partial^2 p}{\partial v^2}\right)$ both vanish. This fixes p_c, v_c and T_c.

4) Catastrophe interpretation of the variable moment of inertia model

It is clear that the Thom classification can be directly applied to the VMI equations (1) and (2). The states are defined via a minimum condition on the energy by (2). The two parameters I_0 and C determine the effective moment of inertia by the equation of state. This equation is again a cubic equation, so it can be put in the standard form (7). If one introduces in the equation (2) as a new state variable

$$x = (I - \frac{1}{3} I_0) \tag{19}$$

one finds that the equation which determines this "difference moment of inertia" is again of the form

$$x^3 + \xi_1 x + \xi_2 = 0 \qquad \text{with} \tag{20}$$

$$\xi_1 = - \frac{1}{3} I_0^2 \tag{21}$$

$$\xi_2 = - \left(\frac{2}{27} I_0^3 + \frac{J(J+1)}{2C} \right) \tag{22}$$

This formal identity implies that in principle all the phenomena occurring in the van der Waals gas have their counterpart in the variable moment of inertia model. There is a critical point given by $x = 0$, $\xi_1 = \xi_2 = 0$, $I_0 = 0$, $J = 0$.

The remark made earlier that $\left(\frac{\partial I}{\partial J}\right)$ at $J = 0$ has a discontinuity for $I_0 = 0$ and corresponds to a phase transition of the second kind, is in precise harmony with the observation that $I_0 = 0$, $J = 0$, corresponds to the critical point. It is further interesting to observe that the cusp given by equation (8) can by the identifications (21) and (22) be transcribed to the form

$$I_0^3 = -\frac{27}{8}\,\frac{J(J+1)}{C}\,. \qquad (23)$$

This very simple relation shows the <u>formal</u> analogy between the van der Waals description of a gas and the Variable Moment of Inertia description of a nucleus. Both fit in the cusp catastrophe scheme of Thom, both can be expected to exhibit phase transition type behavior. Although there is formal identity between the two descriptions, there are, as will be shown below, characteristic physical differences which preclude a direct identification of the phenomena.

5) <u>Discussion and Interpretation</u>

a) It is easy to see by inspection that in the van der Waals case ξ_1 and ξ_2 can both be positive and negative. Any straight line in the ξ_1 ξ_2 plane in the van der Waals case can represent a sequence of physically possible temperatures and pressures. For example the line $\xi_1 - \xi_2 =$ constant represents a constant pressure line. In general, lines could intersect the cusp twice, each intersection (except at $\xi_1 = \xi_2 = 0$) corresponding to a first

order phase transition. The situation is different for the VMI model; because of (21) ξ_1 can only assume negative values, so only the lower part of Figure 2 has a physical significance. One can deduce from the basic formulae that for given $I_0 > 0$, a sequence of different spin states (varying J) are given by a straight line parallel to the ξ_2 axis which ends at the cusp. That state is the J = 0 state.

Figure 2

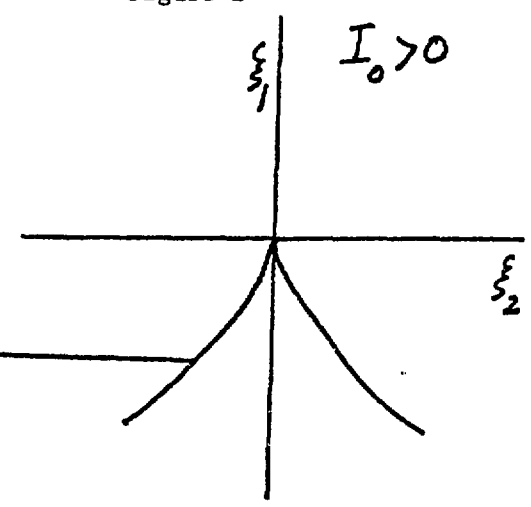
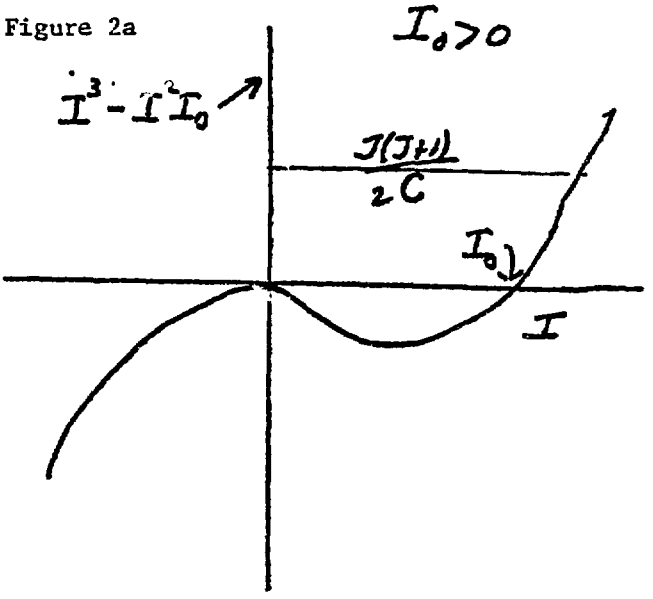


Figure 2a



In that case (for $I_0 > 0$) there is for each I_0, C, J, only one stable moment of inertia I of the nucleus (Figure 2). For higher spins ξ_2 decreases monotonically. As can be seen from (2) the basic relation between I, I_0, J and C may be written as

$$I^3 - I^2 I_0 = \frac{J(J+1)}{2C} \quad .$$

In figure (2a) we have graphed $I^3 - I^2 I_0$ for $I_0 > 0$. Since $\frac{J(J+1)}{2C}$ is always positive, equation (2) (as is evident from Figure 2a) for given

I_0 has always one positive solution for I.

On the other hand, if $I_0 < 0$ (Figure 3), the sequence of nuclear states for different J values starts at the (ξ_2) positive branch of the cusp for J = 0, and proceeds for larger J to negative values

Figure 3

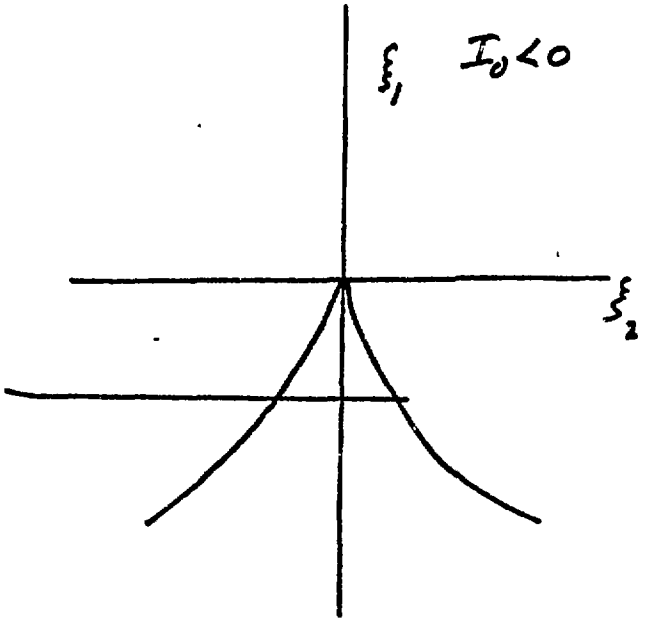
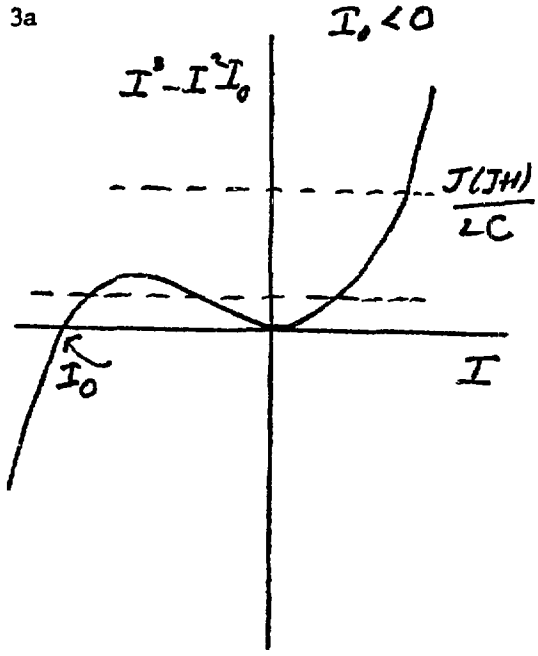


Figure 3a



Then for negative I_0, there exists a sequence of nuclear states where I can have three real values. If one again graphs $I^3 - I^2 I_0$, but now for negative I_0, one obtains Figure 3a. It is now indeed clear (from the figure) that for a range of spin values there are three real solutions for I, for given I_0 and C. However only one of these has a positive moment of inertia I, so only one corresponds to a physical state. Furthermore, as J increases from the "3 root region" to the one root region, the positive root changes continuously. Consequently, when in Figure (3) one passes the cusp, there

will not be a discontinuous change in I; hence no phase transition type phenomenon occurs, in contrast to the van der Waals case.

b) There are other physical situations exhibiting a similar phase transition type behavior. A very instructive example is that of ferroelectricity. For certain substances called perovskites there can, at temperatures $T < T_c$, exist a spontaneous polarization, while for $T > T_c$, no such polarization exists; such a substance is $BaTiO_3$. In all three cases the physical phenomena result from the competition between two opposing tendencies. In the van der Waals case these tendencies are just the repulsive and attractive regions of the intermolecular forces (characterized schematically by the a and b terms respectively). The pressure plays the role of an external influence on the system. It is therefore not surprising that the pressure is related to the Thom ξ parameters. In the Variable Moment of Inertia model, the competition refers to the tendency to preserve the spherical shape of the closed shell which conflicts with the tendency to deform the nuclear shape. The Pauli principle tends to make the nucleus spherical, while the residual forces between valence nucleons, usually taken to be pairing forces and quadrupole - quadrupole interactions, cause deformation. The rotation of the nucleus, via the Coriolis force, brings about a decrease of the pairing interactions. The parameter C reflects the stability of the nucleus in question (i.e. the balance of the Coulomb repulsion and the symmetry term) as well as the degree of surface rigidity which is related to shell filling. I_0 decreases from a maximum value at the center of a given proton and neutron shell to zero at about two nucleon pairs away from magic. It reaches $(-\infty)$ at the magic numbers. In the ferroelectric case these tendencies

refer to the polarization P, which tends to pull the crystal apart and the elastic forces tending to restore the original shape. An external electric field E plays a role similar to the rotation. The basic formulae describing a ferroelectric crystal are[12]

$$P^3 + \gamma(T - T_0)\, P - \alpha\, E = 0 \tag{24}$$

E = external field, P = polarization, α = constant

γ is a constant, T the temperature, T_0 a constant.

The term $\gamma(T - T_0)$ is directly related to the dielectric constant ϵ_o of the material. A very good fit is obtained with

$$\epsilon_o = \frac{a}{\gamma(T - T_0)} \tag{25}$$

When ϵ_o passes through infinity, the linear term in (24) changes sign. This signals a phase transition (of the second order if E = 0, of the first order if $E \neq 0$). The three physically similar cases correspond to the same Thom category. It is obvious that (24) is already in the standard form $x^3 + \xi_1 x + \xi_2 = 0$. It is equally obvious that x has to be identified with the polarization. In Table I, the parameters ξ_1, ξ_2, and the variables x are collected for the three cases.

	van der Waals	ferroelectric	VMI
x	$\frac{v_c - v}{v} = \frac{\rho}{\rho_c} - 1$	P	$I - \frac{1}{3} I_0$
ξ_1	$\frac{1}{3}(8t' + p')$	$\gamma(T - T_0) = \frac{a}{\epsilon_o}$	$-\frac{1}{3} I_0^2$
ξ_2	$\frac{1}{3}(8t' - 2p')$	$-\alpha E$	$-\left(\frac{2}{27} I_0^3 + \frac{J(J+1)}{2C}\right)$

TABLE I

What do the correspondences shown in Table I mean in physical terms? A glance at Table I shows that $\frac{v_c - v}{v} = \frac{\rho}{\rho_c} - 1$ (ρ is the density) plays the same <u>role</u> as does $I - \frac{1}{3} I_0$. That is in harmony with an earlier suggestion[13] which relates I to the density. The suggestion[13] that J corresponds to the pressure and I_0 to the temperature, is essentially confirmed, but needs some modification. From Table I one sees that I_0 and J are determined by both pressure and temperature (t' and p').

$$I_0 \langle = \rangle \sqrt{-(8t' + p')}; \quad \frac{J(J+1)}{2C} \langle = \rangle -\frac{1}{3}(8t' - 2p') - \frac{2}{27}\left(\sqrt{-(8t' + p')}\right)^3 \tag{26}$$

Similarly, from Table I it is seen that the parameter I_0 is related to the dielectric constant and the spin J to the electric field E.

These three cases, all belonging to the same Thom category exhibit similar phase transition behavior. As $I_0 \to -\infty$, the VMI model undergoes a first order phase transition. Physically, as $I_0 \to -\infty$, the nucleus cannot be cranked, an external torque will not produce rotation. In order to produce rotation one or more nucleon pairs must be promoted to a higher state, thereby deforming the nucleus. This deformed nucleus can then be rotated. A very similar behavior occurs in the ferroelectric case. As $\epsilon_o \to 0$, no externally imposed field can produce an internal electric field in the solid. In the van der Waals case, if a = 0, i.e. there is no attractive force, the critical temperature $T_c = 0$. This implies that it would take infinite pressure to liquefy the gas.

Let us now consider the phenomenon referred to earlier, namely the deviation of nuclear ground state bands from the VMI predictions at critical spin values J_c ("critical" is used here in a sense which is related, but

not identical to the use in "critical" phenomena), so that for spins larger than J_c the moment of inertia increases dramatically. The comparisons given earlier allow an understanding of this phenomenon in terms of the ferroelectric model.

In the nuclear case the physical interpretation is that once the nucleus rotates so fast that the centrifugal and Coriolis forces overbalance the pairing forces, the moment of inertia will approach that of a rigid rotor. Similarly, in the ferroelectric case, once E becomes larger than the elastic forces, further increase in E does not yield a substantial increase in polarization. In the comparison with the van der Waals case J corresponds to p', implying a phase transition from gaseous to liquid.

It is of interest to note here that for the heaviest nuclei no backbending was observed up to high spin states. Indeed, for ^{238}U the ground state band energies[15] can be fitted[16] up to J = 28 (highest level known so far) within $\lesssim$ 1% by the two-parameter VMI model, i.e., $J_c > 28$. An even much better fit is achieved by an expression of the form[6, 14]

$$E_J(I) = \frac{1}{2} C(I - I_0)^2 + \frac{J(J+1)}{2I} + \alpha (I - I_0)^3 + \beta (I - I_0)^4; \qquad (27)$$

fifth and sixth order expressions for E_J (I) do not improve the quality of the fit. Together with $\frac{\partial E}{\partial I} = 0$, Eq. (27) yields a fifth degree equation for I, corresponding to the Thom category 5(e) which has as special cases all the qualitative features mentioned for the equations containing fewer parameters, but there are added refinements. Thus from a systematic and empirical viewpoint, this expression appears to be preferred.

The various correspondences one can derive from Table I essentially confirm those arrived at earlier[13] on more intuitive grounds. However

the analysis presented here provides a substantially deeper insight in the subtle relationship between variables and parameters of these three systems.

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References

1. F. Iachello and A. Arima, Phys. Lett. 53B, 309 (1974);
   A. Arima and F. Iachello, Ann. Phys. N. Y., 99, 253 (1976);
   J. B. McGrory, Phys. Rev. Lett. 41, 533 (1978).
2. R. Thom: Structural Stability and Morphogenesis, W. A. Benjamin (1975);
   M. Dresden: Fundamental Interactions in Physics, Plenum (1970).
3. G. Scharff-Goldhaber, C. Dover, and A. Goodman, Ann. Rev. Nucl. Sci. (1976) and references therein.
4. C. Mallmann, Phys. Rev. Lett. 2, 507 (1959).
5. A. S. Goldhaber and G. S. Goldhaber, Phys. Rev. C 17, 1171 (1978).
6. T. K. Das, R. M. Dreizler, and A. Klein, Phys. Rev. C 2, 632 (1970)
7. G. Scharff-Goldhaber, J. Phys. G: Nucl. Phys. No. 11 (1979); Corrigenda, No. 2 (1980).
8. B. Buck, L. C. Biedenharn and R. Y. Cusson, Nucl. Phys. A317, 205 (1979).
9. A. S. Goldhaber and G. Scharff-Goldhaber, Phys. Rev. C 17, 1171 (1978).
10. T. Poston and I. N. Stewart, Taylor Expansion and Catastrophe's, Pitman, London (1976).
11. D. Fowler, "Toward a Theoretical Biology," 26, 239, 317, Edinburgh University Press (1972).
12. C. Kittel: Introduction to Solid State Physics, John Wiley & Sons, 1977.
13. G. Scharff-Goldhaber, 4th Int. Conference on Atomic Masses and Fundamental Constants, September 6, 1971, p. 272, Teddington.
14. Saethre et al., Nucl. Phys. A207, 486-512 (1973).
15. E. Grosse et al., Phys. Rev. Lett. 35, 565 (1975);
    E. Grosse, priv. comm. to L. Peker.
16. L. Peker and S. Pearlstein, priv. comm.