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Anisotropic Jüttner (relativistic Boltzmann) distribution

Rudolf A. Treumann^{1,a} and Wolfgang Baumjohann²

¹Department of Geophysics and Environmental Sciences, Munich University, Munich, Germany

²Space Research Institute, Austrian Academy of Sciences, Graz, Austria

^acurrently at: International Space Science Institute, Bern, Switzerland

Correspondence to: Rudolf A. Treumann (art@geophysik.uni-muenchen.de)

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Abstract. A rigorous derivation of the Jüttner (covariant Boltzmann) distribution is provided for anisotropic pressure (or temperature) tensors. It was in similar form anticipated first by Gladd (1983). Its manifestly covariant version follows straightforwardly from its scalar property.

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The Jüttner distribution (Jüttner, 1911) is the relativistically generalized classical isotropic Maxwell–Boltzmann distribution, whether written in its dependence on relativistic particle energy $\epsilon_p = m\gamma(\mathbf{p})c^2$ or momentum \mathbf{p} , with $\gamma = \sqrt{1 + \mathbf{p}^2/m^2c^2}$. The phase-volume element $d\mathbf{x} d\mathbf{p}$ is covariant (a consequence of its scalar nature). Hence the Jüttner distribution is as well covariant but not manifestly covariant. In anisotropic relativistic gases or plasmas (for application to anisotropic plasmas cf. Yoon, 1989, 2007; Gladd, 1983) and in drifting plasmas (for recent examples cf. e.g. Swisdak, 2013; Lopez et al., 2014; Alves et al., 2015; Zenitani, 2015; DeVore et al., 2015) the form of the Jüttner distribution is usually assumed. Below we provide its simple analytical derivation and manifestly covariant version.

Thermally relativistic implies thermal speeds $v_e/c \gtrsim 10^{-2}$. Avoiding creation, annihilation, and Compton interactions requires $T < mc^2$, with T temperature in energy units, hence weakly relativistic thermal electrons of some $10 \text{ eV} < T < 0.5 \text{ MeV}$, covering most hot classical plasmas.

Maxwell–Boltzmann distributions are solutions of the stationary one-particle Boltzmann equation with the argument of the ratio of the single particle to average thermal energies, viz. ϵ_p/T . Properly normalized they give the probability at temperature T for finding all particles of given momentum \mathbf{p} (or energy ϵ_p) in the interval $d\mathbf{p}$ (or $d\epsilon_p$) in momentum-space volume $d\mathbf{p}$. With three-momentum vector

$\mathbf{p} = (p_\perp \cos\phi, p_\perp \sin\phi, p_\parallel)$ in index notation

$$\epsilon_p^2 = c^2 p_i \delta_j^i p^j + m^2 c^4 \quad (1)$$

suggests introduction of temperature anisotropy guided by the diagonal anisotropy of pressure $\mathbf{P} = N [T_\perp \delta_i^j + (T_\parallel - T_\perp) \delta_3^3]$ (as for instance in magnetized plasma), with anisotropy in direction 3 (in plasma the direction of the magnetic field $\mathbf{b} = \mathbf{B}/B$, for instance). The inverse pressure/temperature tensor is $\mathbf{P}^{-1} = (T_\perp N)^{-1} \Theta$,

$$\Theta = \Theta_j^i = \delta_i^j + (A - 1) \delta_3^3, \text{ with } A = T_\perp / T_\parallel. \quad (2)$$

Replacing δ_j^i in Eq. (1) with Θ_j^i , valid in the four-velocity frame $U_j = (\epsilon_p/mc, 0)$, defining $\beta_\perp = mc^2/T_\perp$, $\beta_\parallel = mc^2/T_\parallel$, putting $p_\perp = p \sin\theta$, $p_\parallel = p \cos\theta$, and defining $p/mc \rightarrow p$ yields

$$\frac{\epsilon_p^2}{T_\perp^2} = \beta_\perp^2 \left[1 + p^2 (\sin^2\theta + A \cos^2\theta) \right]. \quad (3)$$

The square root of Eq. (3) enters the Boltzmann factor. Up to normalization C , the anisotropic Jüttner distribution function of the ideal gas becomes

$$F_j(\mathbf{p}, A) = C \exp \left\{ -\beta_\perp \sqrt{1 + p_\perp^2 + A p_\parallel^2} \right\}. \quad (4)$$

With $A = 1$, $\beta_\perp = \beta_\parallel = \beta$ this is the ordinary Jüttner function. Expanding the root in the limit $c \rightarrow \infty$ reproduces the ordinary nonrelativistic anisotropic Maxwell–Boltzmann distribution. Extensions to drifting or non-ideal gases are straightforward.

This in principle trivial result was anticipated first without proof by Gladd (1983) in application to the whistler instabil-

ity in weakly relativistic anisotropic plasmas¹. Normalization, the purpose of Jüttner's effort, yields

$$C = N\sqrt{A}\beta_{\perp}/4\pi(mc)^3 K_2(\beta_{\perp}) \quad (5)$$

with $K_2(\beta_{\perp})$ the Bessel function, trivially containing the anisotropy factor A .

The distribution Eq. (4) is covariant, valid in time-like slices of Minkowski space. Explicit manifestly covariant isotropic versions have been provided numerically as well (cf. Chacón-Acosta et al., 2010; Curado et al., 2016). Since $F(\mathbf{p})$ is a scalar phase space density, its manifestly covariant version is $F(x^{\nu}, p^{\nu})\sqrt{-g}$ for both isotropic and anisotropic cases. $g < 0$ is the determinant of the metric tensor $g_{\mu\nu}$ in $(+---)$ metric, a version to be applied in curvilinear coordinates. In general relativistic four-space, $\mu, \nu = 0, 1, 2, 3$, and $\mathbf{p} \rightarrow p^{\nu}$ is the four-momentum. Operator interpretation of the three-momentum $\mathbf{p} = \hbar\mathbf{k}$ relates any of these versions to quantum field theory.

Jüttner's anisotropic distribution is useful for analytical or numerical (as in Gladd, 1983) calculations. In particle-in-cell simulations the initial distribution is prescribed. In practice there is little need to choose it in the simulations to satisfy the Jüttner equilibrium requirement. Solving for all relativistic particle orbits in their self-consistent fields, the initial distribution readily adjusts itself to the physical distribution that evolves under the mutual interactions.

As a side product, this straightforward rigorous derivation indicates that in relativistic media the isotropic temperature T and its inverse $\beta = 1/T$ should be understood as vectors (confirming Nakamura, 2009, who suggested it for different reasons). In presence of anisotropy they become tensors. Including particle spins requires a slightly different treatment.

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