Incomplete Symbols — Definite Descriptions Revisited*

November 25, 2014

Abstract

We investigate incomplete symbols, i.e. definite descriptions with scope-operators. Russell famously introduced definite descriptions by contextual definitions; in this article definite descriptions are introduced by rules in a specific calculus that is very well suited for proof-theoretic investigations. That is to say, the phrase 'incomplete symbols' is formally interpreted as to the existence of an elimination procedure. The last section offers semantical tools for interpreting the phrase 'no meaning in isolation' in a formal way.

1 Introduction

According to Russell a definite description has the following form:

the so-and-so

where 'the' is in the singular and 'so-and-so' is a (possible) complex expression. According to Russell (also in earlier writings[1]) definite descriptions do not belong to the category of singular terms.[2] Russell introduces these

*This research is supported by the Alexander von Humboldt Foundation.

[1]Especially *On Denoting* (1905), *OD* for short.

[2]That so-and-so can be a complex expression seems plausible from Russells considerations in *OD*, p.479: "[...] ... a phrase such as any one of the following: a man, some man, any man, every man, all men, the present King of England, the present King of France, the centre of mass of the solar system at the first instance of the twentieth century, the revolution of the earth round the sun, the revolution of the sun round the earth. Thus a phrase is denoting solely in virtue of its form.

by use of *contextual definitions* and claims that definite descriptions are *incomplete symbols*, and having *no meaning in isolation*[3]. For him definite descriptions occur in two contexts (in a more modern notation):

(1) $B(\imath x A(x))$

(2) $E!\imath x A(x)$

These formulas containing the definite descriptions are conceived as formal counterparts of English sentences, e.g.:

(1′) The present King of France is bald.
(2′) The present King of France exists.

Russell was the first to notice that scope-operators allow to mark the occurrence of the negation-sign in a sentence with definite descriptions. Lets take one of Russell's infamous examples:

(3) The present King of France is not bald.

This sentence has two different readings:

(4) The present King of France is *not* bald.
(5) It is *not* the case that the present King of France is bald.

(4) and (5) are represented in the *PM* as:

(6) $[\imath x A(x)]\neg B(\imath x A(x))$
(7) $\neg[\imath x A(x)]B(\imath x A(x))$

The logic of sentences as (3) is governed by a *contextual definition*, i.e.

(*14.01) $[\imath x A(x)]B(\imath x A(x)) \leftrightarrow \exists x\forall y((A(y) \leftrightarrow x = y) \wedge B(x))$

According to (*14.01) both (6) and (7) are interpreted as:[4]

[3] As it will be seen later 'incomplete symbol' will be interpreted syntactically and 'no meaning in isolation' semantically in our approach.

[4] In Russell (1905) speaking of a *secondary* respectively *primary* occurrence of a definite description.

narrow scope:

(8) $[\iota x A(x)]\neg B(\iota x A(x)) \leftrightarrow \exists x \forall y((A(y) \leftrightarrow x = y) \wedge \neg B(x))$

and

wide scope:

(9) $\neg[\iota x A(x)]B(\iota x A(x)) \leftrightarrow \neg\exists x \forall y((A(y) \leftrightarrow x = y) \wedge B(x))$

Without scope-operators the distinction between would collapse:

(10) $\neg B(\iota x A(x)) \leftrightarrow \neg\exists x \forall y((A(y) \leftrightarrow x = y) \wedge B(x))$

or

(11) $\neg B(\iota x A(x)) \leftrightarrow \exists x \forall y((A(y) \leftrightarrow x = y) \wedge \neg B(x))$[5]

In addition to (*14.01) Russell licenses the transition from from (*14.01) to:

(*14.101) $B(\iota x A(x)) \leftrightarrow \exists x \forall y((A(y) \leftrightarrow x = y) \wedge B(x))$

This transition is licensed by a convention quoted below. But first of all we present the contextual definition concerning definite descriptions in existential contexts:

(*14.02) $E!\iota x A(x) \leftrightarrow \exists x \forall y(A(y) \leftrightarrow x = y)$

The convention that allows to go from (*14.01) to (*14.101) is stated as follows:

> It will be found that, in most cases in which descriptions occur, their scope is, in practice, the smallest proposition enclosed in dots or brackets in which they are contained. Thus for example
>
> $$[(\iota x)(\phi x)].\psi(\iota x)(\phi x). \rightarrow .[(\iota x)(\phi x)].\chi(\iota x)(\phi x)$$
>
> will occur much more frequently than
>
> $$[(\iota x)(\phi x)] : \psi(\iota x)(\phi x). \rightarrow .\chi(\iota x)(\phi x).$$

[5](10) and (11) are equivalent under the condition that $\iota x A(x)$ exists.

> For this reason it is convenient to decide that, when the scope of an occurrence of $(\iota x)(\phi x)$ is the smallest proposition, enclosed in dots or other brackets, in which the occurrence in question is contained, the scope need not to be indicated by "$[(\iota x)(\phi x)]$." [...] This convention enables us, in the vast majority of cases that actually occur, to dispense with the explicit indication of the scope of a descriptive symbol; and it will be found that the convention agrees very closely with the tacit conventions of ordinary language on this subject. (*PM*, p.71)

This *convention* can (or could be) formalized in a suitable setting, however, as it stands here, we think that it is something outside the formal system. Clearly, after 100 years (or more) we now have a precise understanding of object- and meta-language something that might have been not always present in the *PM*.

It is rather tempting to think that (*14.101) can be added to first order logic. In a review by Church (1963)[6] he observes that this has been done by Schock (1962). Schock wanted to treat definite descriptions as *proper* terms and not introduced by contextual definitions. However, this approach leads straightforwardly to contradiction – as Schock (1962) observes by letting A be the negation of a logical truth and B be a logical truth. Church notes (1963, p.105) that (a) this contradiction was known to Russell (among others) and (b) that the equivalence expressed by (*14.101) (and viewed as an axiom-schema) has to be replaced by a somewhat weaker principle.

The present paper approaches this topic as follows: on the one hand, this paper is inspired by Russell's work on definite descriptions and it is also set up to give formal interpretation of the famous Russell dicta that definite descriptions are *incomplete symbols* and that they have *no meaning in isolation*. Briefly, we understand the the phrase "incomplete symbol" as a syntactic notion, i.e. definite descriptions (proper or not) can be eliminated; and the phrase "no meaning in isolation" is interpreted as a semantical thesis (more on this will follow below). On the other hand, definite descriptions are seen as proper terms in the sense that they are not introduced by way of contextual definitions.

The logic presented here is somewhat half-way between Church's suggestion to weaken the principle (*14.101) (seen as an axiom-schema) and Russell's original proposal. We think that the presence of scope-operators allow for distinctions that cannot be made without them and therefore decided

[6] The review in question is in Schock (1962).

to develop a logic with scope-operators. In contexts where scope-operators seem to be superfluous they can be considered as idle.

1.1 Related research and plan of this paper

In the following sections of this paper we shall develop a logic containing $\imath$-terms that is inspired by Russell's ideas and insights on this matter.

As it is well known there is a huge amount of literature on definite descriptions; and is therefore impossible to mention every single work done in this field. It is sure worth mentioning that there are at least four big strains in this area of philosophical logic: (1) Russell (and Russell-inspired (this paper is in this strain)) theories (cf. eg. Kaplan (1972), Neale (1990), Scott (1991), Grabmayer (et.al) (2011); also Oppenheimer & Zalta (1991), (2011)–the 2011-article pointing towards *computational metaphysics*); (2) theories that are inspired by the Fregean approaches to definite descriptions (cf. e.g. Carnap (1960, p.35ff.), Kalish (et.al, 1980) (for Frege- and Russell-like approaches)), (3) a Hilbert-Bernays (initial) approach (Hilbert & Bernays (1934 & 1939)), i.e. a $\imath$-term can only be introduced if the corresponding uniqueness conditions are (formally) provable in the theory in question (cf, eg. Lambert (2003), (1999)). Last not least (4): theories that have been developed in the framework of free logic(s); which are inspired by all of the before mentioned (cf. eg. van Fraassen & Lambert (1967), Bencivenga (et.al) (1986), Bostock (1997), for an application in philosophy of science: Lewis (1970)).

Section 2 of this paper is devoted to developing a proof-theoretic approach to definite descriptions. More specifically, we develop a Tait-calculus first, this will be extended by equality, then by the $\imath$-rule and finally (and optionally) with an existence-predicate. The main result is a version of Gentzen's *Hauptsatz* that ensures e.g. consistency of the respective logic(s).

A Tait-calculus is simply a truncated Gentzen-style sequent calculus. It has the advantage (over a more standard sequent calculus) that the proof-theoretic meta-results (e.g. cut-elimination theorem) are very quickly established. Furthermore, given that we want to shed light on Russell's dictum that definite descriptions are "incomplete symbols" this version of the Tait-calculus allows us to formulate a concise formulation of the elimination procedure.

Grabmayer (et.al) (2011) tackled the *Hydra-problem* posed (most famously) by Kripke (2005). Grabmayer (et.al) chose a *term rewriting method*. Our approach in this paper, when it comes to the elimination of $\imath$-terms (section 3) is more closely related to Kleene's (2000) approach.

2 Language and Logic

2.1 A language $\mathcal{L}$: basic symbols

- Individual variables: $v_0, v_1, v_2, \dots$ (denoted by $x, y, z, x_1, \dots$)
- Individual constants: $c_0, c_1, c_2, \dots$ (denoted by $a, b, a_1, \dots$)
- $\wedge$, $\vee$, $\exists$, $\forall$, $\imath$
- Countably many predicate symbols: P_i^n with arity n (denoted by $P, Q, R \dots$.)

Definition 1 (Atomic formulas, literals)

- An atomic formula is an expression $P_i^n(t_0, \dots, t_n)$ where P_i^n is an n-ary predicate symbol and $t_1, \dots, t_n$ are individual variables or individual constants.
- An expression of the form A, $\neg A$, where A is an atomic formula is called a (positive, negative) literal.

Definition 2 (Simultaneous recursive definition of formulas and $\imath$-terms)

(i) Every variable is a term and every literal is a formula.

(ii) If A, B are formulas then $(A \wedge B)$, and $(A \vee B)$ are formulas.

(iii) If A is a formula then $\exists xA$, $\forall xA$ are formulas and $\imath xA$ is a $\imath$-term.

(iv) If $\imath xA_1$, ..., $\imath xA_n$ are $\imath$-terms and B is a formula then $[\imath xA_1, \dots \imath xA_n]$ $B(\imath xA_1, \dots \imath xA_n)$ is a formula.

(v) Nothing else is a formula, a term or a $\imath$-term.

We use s, t (with our without subscript) as syntactic variables ranging over individual variables and individual constants, u, w as syntactic variables ranging over $\imath$-terms and A, B, ... for formulas; $*$ for $\wedge$, $\vee$ and Q for $\exists, \forall, \imath$. Γ, Δ are sets of formulas.

Informally, we understand formulas of the form $[\imath xA_1, \dots \imath xA_n]B(\imath xA_1, \dots \imath xA_n)$ as follows: apply a certain procedure (which will become clearer later) to it such that $\imath xA_1$ is in $B(\imath xA_1, \dots \imath xA_n)$ is analyzed as the procedure prescribes; then proceed to the next $\imath$-term etc. Whereas the left-most $\imath$-term in the scope-operator refers to the left-most $\imath$-term in B. We say more on notation in section 2.2.

Definition 3 (Definition of the negation ($\mathsf{neg}(A)$) of a formula A)

(i) If A is atomic then $\mathsf{neg}(A) := \neg A$ and $\mathsf{neg}(\neg A) := A$.

(ii) $\mathsf{neg}(A \wedge B) := \mathsf{neg}(A) \vee \mathsf{neg}(B)$, $\mathsf{neg}(A \vee B) := \mathsf{neg}(A) \wedge \mathsf{neg}(B)$.

(iii) $\mathsf{neg}(\forall x A) := \exists x \mathsf{neg}(A)$, $\mathsf{neg}(\exists x A) := \forall x \mathsf{neg}(A)$.

(iv) $\mathsf{neg}([\imath x A, \ldots]B_t(\imath x A)) := \forall x \exists y \mathsf{neg}((A_t(y) \leftrightarrow x = y) \wedge [\ldots]B_t(x))$ (where the number of $\imath$-terms of $\forall x \exists y \mathsf{neg}((A_t(y) \leftrightarrow x = y) \wedge [\ldots]B_t(x)) < \mathsf{neg}([\imath x A, \ldots]B_t(\imath x A)))$.

Corollary 1 $\mathsf{neg}(A)$ is a formula; $\mathsf{neg}(\mathsf{neg}(A)) = A$.

The set of free variables of an expression E, FV(E), is defined as usual. Let X be a set of expressions: $\mathrm{FV}(X) := \bigcup\{\mathrm{FV}(E) : E \in X\}$. If A and A' only differ in their names of the bound variables then A and A' are identified.

Substitution A substitution is a mapping σ : Vars $\longrightarrow$ T with dom(σ) := $\{x \in \mathsf{Var} : \sigma(x) \neq x\}$. The *updates* $\sigma^t_y(x)$:= (i) t, if $x = y$, (ii), $\sigma(x)$ otherwise. $(x_1/t_1, \ldots, x_n/t_n)$ denotes the substitution σ with $\sigma(x)$ is (i) t_i if $x = x_i$ or (ii) x otherwise. If $\sigma = (x_1/t_1, \ldots, x_n/t_n)$ and E is an expression then $E\sigma$ denotes the result of simultaneously substituting the terms $t_1, \ldots, t_n$ for the variables $x_1, \ldots, x_n$ respectively.

Definition 4 A rule r is *closed under substitution (of individual constants and individual variables)* iff the following holds for every r-inference $\mathsf{I} = \dfrac{\Gamma_0 \ldots \Gamma_{n-1}}{\Gamma}$: If σ is a substitution such that (Eig(I) = {x} $\Rightarrow x\sigma \in \mathsf{Var} \setminus \mathrm{FV}(\Gamma\sigma)$), then $\mathsf{I}\sigma = \dfrac{\Gamma_0\sigma \ldots \Gamma_{n-1}\sigma}{\Gamma\sigma}$ is also an r-inference.[7]

Definition 5 (Rank of formulas and terms) The *rank* of a formula A or a term u is the maximum length of a branch in its construction tree. Formally, this is defined by simultaneous recursive definition was follows:

(r1) $|A| = 0$ if A is a literal.

(r2) $|A * B| = \max(|A|, |B|) + 1$ for binary operators $*$, i.e. $\wedge, \vee$.

[7]Definitions 3, 4, and 5 are all taken from Buchholz (2002/03), p.1ff - with minor modifications.

(r3) $|*A| = |A| + 1$ for unary operators $*$, i.e. $\forall x, \exists x, \imath x$.

(r4) $|[\imath xA]B(\imath xA)| = |B| + |A|$.

The number of $\imath$-terms occurring in a formula A (not counting its scope) is called the $\imath$-weight of A, $\imath|A|$.

This is to say, the $\imath$-weight of a formula A is encoded in this definition. E.g., $|[\imath v_1 P^1_1(v_1), \imath v_2 P^2_2(v_1, v_2)]P^2_1(\imath v_1 P^2_2(v_1),\ \imath v_2 P^2_1(v_1, v_2))| = 2$ which is also (in this case) its $\imath$-weight. Furthermore, the clause (r4) states formally that the scope does not increase the rank of the formula.

The clause (r4) reflects that the scope-operator does not add to the rank of a formula. The scope-operator serves as (in this approach) as an indicator of how to analyze formulas of the form $[\imath xA_1, \ldots \imath xA_n]B(\imath xA_1, \ldots \imath xA_n)$.

Officially there is no biconditional in the language $\mathcal{L}$. However, we think of a biconditional of the form $A \leftrightarrow B$ as defined as: $(\neg A \vee B) \wedge (\neg B \vee A)$.

Definition 6 (cut-rank, height (of a derivation))
The *cut-rank* of a derivation d is $\mathsf{crk}(d) := sup\{rk(C) + 1 : C$ cut-formula of $d\}$. A derivation d is called *cutfree* if $\mathsf{crk}(d) = 0$.
The *height* of a derivation d – $\mathsf{hgt}(d)$ – is recursively defined as follows: $\mathsf{hgt}(d) := sup_{i<n}(\mathsf{hgt}(d_i) + 1)$ where $d_0, \ldots d_{n-1}$ are the *immediate subderivations* of d $(0 \le n \le 2)$. The last inference of d is denoted by $\mathsf{last}(d)$.[8]

2.2 On notation

Our main targets are expressions of the form:

$$[\imath x_1 A^1, \imath x_2 A^2, \ldots, \imath x_n A^n]B(\imath x_1 A^1, \imath x_2 A^2, \ldots, \imath x_n A^n)$$

As we have said before, the scope of this expression, i.e. $[\imath x_1 A^1, \imath x_2 A^2, \ldots, \imath x_n A^n]$, does not add to the logical complexity (or rank) of a formula. It is merely a syntactical device to indicate several things:

(a) The scope indicates the occurrences of $\imath x_i A^i$ in B. Granted that there are $\imath$-terms, $\imath x_i A^i$ and $\imath x_j A^j$ with $i = j$ in the scope, then $\imath x_i A^i$ and $\imath x_j A^j$ refer simply to different occurrences in B. (Example below.)

(b) The (natural) number n might not be identical with number of $\imath$-terms occurring in it. This is so because $\imath$-terms may have a very complex structure. An example for this is (possibly slightly outdated

[8]cf. Buchholz (2002/03), p.2f.

and old-fashioned): 'The first born child of its father inherits the fatherly farm.' This sentence can be formalized as: $R(\imath x Q(x, \imath y P(x,y)), \imath z S(z, \imath y P(z,y)))$. In section 2.4 we extend the language with an existence-predicate: similar remarks apply to this extension.

(c) The scope is a syntactic device that allows for unique readability of the formula in question. For example and informally speaking, consider this formula: $[\imath x P(x), \imath x S(x)] Q(\imath x P(x), \imath x P(x))$. This will be interpreted as: apply some rules to the leftmost $\imath$-term (occurrence) in the formula that follows the scope.

(d) Last not least: The scope also serves as indicator for *wide* and *narrow* readings (as mentioned in the introduction).

(e) We do make use of α-conversion.

Consider the formula $[\imath x P(x), \imath x P(x)] Q(\imath x P(x), \imath x P(x))$ as an example for our logic (and later on elimination procedure (section 8)):

$$[\imath x P(x), \imath x P(x)] Q(\imath x P(x), \imath x P(x))$$

is – following our notational conventions – formally interpreted as:

$$\exists x' \forall y((P(y) \leftrightarrow x' = y) \wedge [\imath x P(x)] Q(x', \imath x P(x)))$$

and this is in turn formally interpreted as:

$$\exists x' \forall y((P(y) \leftrightarrow x' = y) \wedge \exists x'' \forall y'((P(y') \leftrightarrow x'' = y') \wedge Q(x', x'')))$$

However, $\exists x' \forall y((P(y) \leftrightarrow x' = y))$ and $\exists x'' \forall y'((P(y') \leftrightarrow x'' = y'))$ are notational variants of each other. So, finally the following equivalence should hold:

$$\begin{gathered} [\imath x P(x), \imath x P(x)] Q(\imath x P(x), \imath x P(x)) \\ \leftrightarrow \\ \exists x \forall y((P(y) \leftrightarrow x = y) \wedge Q(x, x)) \end{gathered}$$

In the course of the paper some further notational conventions will be made; mainly in order to achieve easy readability.

2.3 Logics

We develop the calculus $\mathsf{T}_\imath$ in a piecemeal fashion. First, T is introduced, which is simply a first order logic without equality; second, the equality-predicate is added with its respective rules,$\mathsf{T}^=$. And, finally, we present the calculus, $\mathsf{T}_\imath$, which includes the $\imath$-rule. The language of $\mathsf{T}_\imath$ is a restriction of $\mathcal{L}$, i.e. $\mathcal{L}$ without $\imath$-terms.

Definition 7 (A first order Tait-calculus: T)

(Ax) $\Gamma, A, \neg A$ if A is a literal

$$(\wedge)\ \frac{\Gamma, A_0 \qquad \Gamma, A_1}{\Gamma, A_0 \wedge A_1} \qquad\qquad (\vee)\ \frac{\Gamma, A_k}{\Gamma, A_0 \vee A_1}\ (k \in \{0,1\})$$

$$(\forall)\ \frac{\Gamma, A}{\Gamma, \forall x A}\ x \notin \mathrm{FV}(\Gamma) \qquad\qquad (\exists)\ \frac{\Gamma, A_x(t)}{\Gamma, \exists x A}$$

$$(\mathrm{Cut})\ \frac{\Gamma, C \qquad \Gamma, \neg C}{\Gamma}$$

Lemma 1 (Substitution) The rules of $\mathsf{T}_\imath$ are closed under substitution of simple singular terms: $\mathsf{T}_\imath \vdash^k_m \Gamma \Rightarrow \mathsf{T}_\imath \vdash^k_m \Gamma\sigma$.

Lemma 2 (Weakening) $\mathsf{T}_\imath \vdash^k_m \Gamma \,\&\, \Gamma \subseteq \Gamma' \Rightarrow \mathsf{T}_\imath \vdash \Gamma'$.[9]

Lemma 3 (Full, Inversion) The following rules are height-preserving invertible:

(Full) $\mathsf{T}_\imath \vdash \Gamma, C, \neg C$ for all C.

(I$\vee$) If then $\mathsf{T} \vdash^k_m \Gamma, A_0 \vee A_1$, then then $\mathsf{T} \vdash^n \Gamma, A_k$; $k \in \{0,1\}$.

(I$\wedge$) If $\mathsf{T} \vdash^k_m \Gamma, A_0 \wedge A_1$, then $\mathsf{T} \vdash^n \Gamma, A_0$ and then $\mathsf{T} \vdash^n \Gamma, A_1$.

(I$\forall$) If $\mathsf{T} \vdash^k_m \Gamma, \forall x A$, then $\mathsf{T} \vdash^n \Gamma, A_x(t)$.

Lemma 4 (Cut-lemma) $\mathsf{T} \vdash^k_m \Gamma, C$, $\mathsf{T} \vdash^l_m \Gamma, \neg C \Longrightarrow \mathsf{T} \vdash^{k+l}_m \Gamma$

Theorem 1 (Cut-elimination) Cut is eliminable from T.[10]

[9] The proofs of both lemmata are as in Buchholz (2002/03).
[10] More detailed proofs of the cut-lemma, and Cut-elimination are stated below.

From this follows naturally the following consistency (cf. Tait (1968), p.209) of $\mathbf{T}$:

Corollary 2 (Consistency)
Every derivable set of atoms includes an axiom.

In order to deal with equality we extend the formal language in the usual way, and say the $s = t$ is a formula of the formal language in question. Following Gentzen (1934/35) we could add further equality axioms to the sequent calculus and then obtain a version of Gentzen's *erweiterter Hauptsatz*. The equality axioms for $\mathbf{T}_\imath$ have the form:

$$\text{(Eax1) } \Gamma, t = t \quad \text{(Eax2)} \quad \Gamma, \neg(s = t), \neg P_x(s), P_x(t)$$

In (Eax2) P is an n-ary atomic predicate. From these axioms symmetry of equality is easily derivable:

$$\frac{s = s \qquad \neg(s = t), \neg(s = s), t = s}{\neg(s = t), t = s}$$

However, the cut cannot be avoided.

In analogy to the work of Negri/von Plato (2001, ch. 6, esp. 138ff.) and Negri/von Plato (1998, p.429f.) we extend our Tait-caluculus not with axioms but with rules for equality. Thereby, a cut on equality formulas can be avoided. The rules for equality have the following form:

Definition 8 ($\mathbf{T}$ with equality, $\mathbf{T}^=$)

$$\text{(E1) } \frac{\Gamma, \neg(t = t)}{\Gamma} \qquad \text{(E2) } \frac{\Gamma, \neg(s = t), \neg P_x(s), P_x(t)}{\Gamma, \neg(s = t), \neg P_x(s)}$$

Where the last rule (E2) is formulated for each (atomic) predicate P.

From (E1) and (E2) we can prove (by induction on the rank of A) the following:

$$\frac{\Gamma, \neg(s = t), \neg A_x(s), A_x(t)}{\Gamma, \neg(s = t), \neg A_(s)} \text{ (fullRepl)}$$

The rule (fullRepl) enables us to prove the Replacement schema: $\Gamma, \neg s = t, \neg A_x(s), A_x(t)$

So without further ado, we can state the following theorem:

Theorem 2
Cut is eliminable from $\mathsf{T}^{=}$.

Finally, we want to add the $\imath$-rule; we thereby allow for the equality predicate to be flanked with $\imath$-terms.

Definition 9 (A Tait-calculus with $\imath$-terms: $\mathsf{T}_\imath$)

$$(\imath)\ \frac{\Gamma, \exists x \forall y((A(y) \leftrightarrow x = y) \land B(x))}{\Gamma, [\imath x A(x)]B(\imath x A(x))}\ \dagger$$

† The number of $\imath$-terms is zero in $\exists x \forall y((A(y) \leftrightarrow x = y) \land B(x))$, and Γ.

Theorem 3 $\mathsf{T}_\imath \vdash \Gamma, C, \neg C$ for all C with $\imath|C| > 0$, $|C| > 0$.

Proof. The first part of proof where $\imath|C| = 0$, is completely analogous to the standard case. There are two main cases to consider: (1) $\imath|C| = 1$, (2) $\imath|C| > 1$. Instead of writing $\exists x \forall y((A(y) \leftrightarrow x = y) \land B(x))$ we allow ourselves the notational abbreviation: $\exists 1 x AB(x)$; and instead of writing $\forall x \exists y \neg((A(y) \leftrightarrow x = y) \land B(x))$ we use $\neg \exists 1 x AB(x)$.

Case 1, $\imath|C| = 1$:

$$\cfrac{\cfrac{\exists 1 x AB(x),\ \neg \exists 1 x AB(x)}{[\imath x A(x)]B(\imath x A(x)),\ \neg \exists 1 x AB(x)}}{[\imath x A(x)]B(\imath x A(x)),\ \neg[\imath x A(x)]B(\imath x A(x))}$$

Case 2: As in case and by the use of the IH, i.e. the theorem holds for $n-1$ $\imath$-terms; we proceed by the left-most $\imath$-term. In order to avoid to much notational mess, we allow ourselves some conventional ease: instead of writing $[\imath x_1 A^1, \imath x_2 A^2, \ldots, \imath x_n A^n]B(\imath x_1 A^1, \imath x_2 A^2, \ldots, \imath x_n A^n)$, we simply write: $[\imath_1, \imath_2, \ldots \imath_n]B$; furthermore, instead of writing: $\exists x \forall y((A^1(y) \leftrightarrow x = y) \land [\imath x_2 A^2, \ldots, \imath x_n A^n]B_t(x, \imath x_2 A^2, \ldots, \imath x_n A^n))$, we write: $\exists 1 x(A^1(x) \land [\imath_2, \ldots \imath_n]B)$. Similarly for $\neg \exists x \forall y((A^1(y) \leftrightarrow x = y) \land [\imath x_2 A^2, \ldots, \imath x_n A^n]B_t(x, \imath x_2 A^2, \ldots, \imath x_n A^n)$, we write: $\neg \exists 1 x(A^1(x) \land [\imath_2, \ldots \imath_n]B)$.

$$\dfrac{\dfrac{\exists 1x(A^1(x) \wedge [\imath_2, \ldots \imath_n]B), \neg\exists 1x(A^1(x) \wedge [\imath_2, \ldots \imath_n]B)}{[\imath_1, \imath_2, \ldots \imath_n]B, \neg\exists 1x(A^1(x) \wedge [\imath_2, \ldots \imath_n]B)}}{[\imath_1, \imath_2, \ldots \imath_n]B, \neg[\imath_1, \imath_2, \ldots \imath_n]B}$$

What this theorem shows is that the $\imath$-weight is reduced when following a derivation from the conclusion up to its premisses. This fact will be of particular importance in the cut-elimination theorem—and is stated in the following corollary:

Corollary 3 ($\imath$-weight reduction)

(a) If $\mathbf{T}_\imath \vdash^n \Gamma, [\imath_1, \imath_2, \ldots \imath_n]B$ with $\imath$-weight of $[\imath_1, \imath_2, \ldots \imath_n]B$ is r and $\imath_1$ is the principal $\imath$-term, then $\mathbf{T}_\imath \vdash^{n-1} \Gamma, \exists 1x(A^1(x) \wedge [\imath_2, \ldots \imath_n]B)$, where $\imath|\exists 1x(A^1(x) \wedge [\imath_2, \ldots \imath_n]B)| < \imath|[\imath_1, \imath_2, \ldots \imath_n]B|$.

(b) If $\mathbf{T}_\imath \vdash^n \Gamma, \neg[\imath_1, \imath_2, \ldots \imath_n]B$ with $\imath$-weight of $\neg[\imath_1, \imath_2, \ldots \imath_n]B$ is r and $\imath_1$ is the principal $\imath$-term, then $\mathbf{T}_\imath \vdash^{n-1} \Gamma, \neg\exists 1x(A^1(x) \wedge [\imath_2, \ldots \imath_n]B)$, where $\imath|\neg\exists 1x(A^1(x) \wedge [\imath_2, \ldots \imath_n]B)| < \imath|\neg[\imath_1, \imath_2, \ldots \imath_n]B|$.

Theorem 5 and its corollaries will prove their importance in the elimination of (Cut).

The characteristic theorems
The characteristic theorems of Russell's original proposal are deducible in $\mathbf{T}_\imath$ which is seen by the following derivations:

$$\dfrac{\exists 1xAB(x), \neg\exists 1xAB(x)}{[\imath xA(x)]B(\imath xA(x)), \neg\exists 1xAB(x)}$$

$$\dfrac{\neg\exists 1xAB(x), \exists 1xAB(x)}{\neg[\imath xA]B(\imath xA), \exists 1xAB(x)}$$

The inversion lemma holds also for the extended calculus.

Lemma 5 (Cut-lemma) $\mathbf{T}_\imath \vdash^k_m \Gamma, C$, $\mathbf{T}_\imath \vdash^l_m \Gamma, \neg C \Longrightarrow \mathbf{T}_\imath \vdash^{k+l}_m \Gamma$.

Proof by induction on $k + l$. Assume $d \vdash^k_m \Gamma, C$ and $e \vdash^l_m \Gamma, \neg C$.

Again, instead of writing $\exists x \forall y((A(y) \leftrightarrow x = y))$ we allow ourselves the notational abbreviation: $\exists 1xAB(x)$; and instead of writing $\forall x \exists y((A(y) \leftrightarrow$

$x = y)$) we use $\neg\exists 1xAB(x)$. We follow Buchholz (2002/03, p.5) and have to distinguish the following cases. 1. C is not a principal formula of $\mathsf{last}(d)$, respectively symmetric to $\mathsf{last}(d)$, $\neg C$ is not a principal formula of $\mathsf{last}(e)$. 2. C is a principal formula of $\mathsf{last}(d)$, $\neg C$ is a principal formula of $\mathsf{last}(e)$. 2.1 C is a literal, i.e. $\{C, \neg C\} \subseteq \Gamma \cup \{C\}$ and $\{C, \neg C\} \subseteq \Gamma \cup \{\neg C\}$, so $\{C, \neg C\} \subseteq \Gamma$ and $\vdash^{k+l}_m \Gamma$.

2.2 $C = \exists xA$, then $\neg C = \forall x \neg A$. By i.h.: $\vdash^{k-1}_m \Gamma, C, A_x(t)$ and $\vdash^{l-1}_m \Gamma, \neg C, \neg A$, the following derivation gives the required result:

$$\dfrac{\dfrac{\vdash^{k-1}_m \Gamma, C, A_x(t) \qquad \dfrac{\vdash^{l}_m \Gamma, \neg C}{\vdash^{l}_m \Gamma, \neg C, A_x(t)}}{\vdash^{k-1+l}_m \Gamma, A_x(t)} \qquad \dfrac{\dfrac{\vdash^{k}_m \Gamma, C}{\vdash^{k}_m \Gamma, C, \neg A_x(t)} \qquad \dfrac{\vdash^{l-1}_m \Gamma, \neg C, \neg A}{\vdash^{l-1}_m \Gamma, \neg C, \neg A_x(t)}}{\vdash^{l-1+k}_m \Gamma, \neg A_x(t)}}{\vdash^{k+l}_m \Gamma}$$

2.2′ $C = \forall xA, A_0 \wedge A_1, A_0 \vee A_1$ are analogous to 2.2. The next case (2.2″), where $C = [\imath xA]B(\imath xA)$ and $\neg C$ is $\neg\exists 1xAB(x)$, is also treated similar to 2.2.

2.2″: $C = [\imath xA]B(\imath xA)$ and $\neg C$ is $\neg\exists 1xAB(x)$.

$$\dfrac{\dfrac{\vdash^{k-1}_m \Gamma, C, \exists 1xAB(x) \qquad \dfrac{\vdash^{l}_m \Gamma, \neg C}{\vdash^{l}_m \Gamma, \neg C, \exists 1xAB(x)}}{\vdash^{k-1+l}_m \Gamma, \exists 1xAB(x)} \qquad \dfrac{\dfrac{\vdash^{k}_m \Gamma, C}{\vdash^{k}_m \Gamma, C, \neg\exists 1xAB(x)} \qquad \dfrac{\vdash^{l-1}_m \Gamma, \neg C, \neg\exists 1xAB(x)}{\vdash^{l-1}_m \Gamma, \neg C, \neg\exists 1xAB(x)}}{\vdash^{l-1+k}_m \Gamma, \neg\exists 1xAB(x)}}{\vdash^{k+l}_m \Gamma}$$

We assumed–tacitly–that both the $\imath$-weight of $\exists 1xAB(x)$ is strictly smaller than the $\imath$-weight of $[\imath xA]B(\imath xA)$ and $\imath$-weight of $\neg\exists 1xAB(x)$ is strictly smaller than the $\imath$-weight of $[\imath xA]B(\imath xA)$. In more general terms this means that instead of the usual tuple (with a slight abuse of notation) $\langle crk, hgt\rangle$ the induction proceeds on a triple $\langle crk, hgt, \imath\rangle$; then the following holds: $\langle crk, hgt, \imath\rangle < \langle crk', hgt', \imath'\rangle$ iff (i) $(crk < crk')$ or (ii) $(crk = crk'$ and $hgt < hgt')$ or (iii) $(crk = crk'$ and $hgt = hgt'$ and $\imath < \imath')$. This is a well ordering on $\mathbb{N}^3$.

Theorem 4 (Cut-elimination) Cut is eliminable from $\mathbf{T}_\imath$:
If $\mathbf{T}_\imath \vdash^{k}_{m+1} \Gamma$, then $\mathbf{T}_\imath \vdash^{2^k}_{m} \Gamma$.

Proof (as in Buchholz (2002/03), p.5) by induction on k: let $d \vdash^k_{m+1} \Gamma$ and asstume that $\mathsf{last}(d) = \dfrac{\Gamma, C \ldots \Gamma, \neg C}{\Gamma}$ with $|C| = m$. If $\vdash^{k-1}_{m+1} \Gamma, C$ and $\vdash^{k-1}_{m+1} \Gamma, \neg C$, then by induction hypothesis $\vdash^{2^{k-1}}_{m+1} \Gamma, C$ and $\vdash^{2^{k-1}}_{m+1} \Gamma, \neg C$, then by Lemma 6 $\vdash^{2^{k-1}+2^{k-1}}_{m} \Gamma$

Definition 10 (Sub-formula and sub-term property of $\mathbf{T}_\imath$)
B is a *subformula/subterm* of A, if B can be obtained from A by finitely many steps of the kind

- $\mathsf{Q}xA \mapsto A_x(t)$, Q is $\exists, \forall$, or
- $A_0 * A_1 \mapsto A_i$, or
- $[\imath xA, \ldots]B(\imath xA) \mapsto \exists x \forall y((A(y) \leftrightarrow x = y) \wedge [\ldots]B(x))$.

Corollary 4 (Subformula, subterm property)
Let d be a cut-free derivation of Γ in $\mathbf{T}_\imath$ then every formula is a subformula or a subterm (i.e. a $\imath$-term) of some A of Γ.

2.4 Existence as $E!$

Although the main vein of this paper is not on existence, we could nonetheless develop a logic that includes an existence-predicate. We do not want to enter here the philosophical debate whether existence is predicate or not; we adopt Russell's position (here) that the existence-predicate can only be applied to to (definite) descriptions and not to individual constants or individual variables.

For this end $E!$ is introduced as an additional logical predicate to $\mathcal{L}$ and add to the simultaneous recursive definition the following clause: If u is an $\imath$-term, then $E!u$ is a formula. We add to definition 3: $\mathsf{neg}(E!\imath xA) := \forall y \exists x \mathsf{neg}(A(y) \leftrightarrow x = y)$. Corollary 1 extends also to the augmented language. As a notational device we let $\exists 1xA$ abbreviate $\exists x \forall y(A(y) \leftrightarrow x = y)$ and $\neg \exists 1xA$ abbreviate $\forall x \exists y \neg(A(y) \leftrightarrow x = y)$.

Without further ado, we go on and define $\mathbf{T}^{E!}_\imath$ as an extension of $\mathbf{T}_\imath$ as follows:

Definition 11 (A Tait-calculus with $\imath$-terms and $E!$: $\mathbf{T}^{E!}_\imath$)

$$(E!)\ \frac{\Gamma, \exists x \forall y(A(y) \leftrightarrow x = y)}{\Gamma, E!\imath xA}\ \dagger$$

† The number of $\imath$-terms is zero in $\exists x\forall y(A(y) \leftrightarrow x = y)$, and Γ.

The next already familiar theorem holds also for $\mathsf{T}_\imath^{E!}$.

Theorem 5 $\mathsf{T}_\imath^{E!} \vdash \Gamma, C, \neg C$ for all C with $\imath|C| > 0$, $|C| > 0$.

Proof. We only state the new cases.

Case 1, $\imath|C| = 1$:

$$\cfrac{\cfrac{\exists 1xA,\ \neg\exists 1xA}{E!\imath xA,\ \neg\exists 1xA}}{E!\imath xA,\ \neg E!\imath xA}$$

Case 2: As in case and by the use of the IH, i.e. the theorem holds for $n-1$ $\imath$-terms; we proceed by the left-most $\imath$-term. Again, we use some notational relief. We write $\exists 1xA(x, \imath_2, \ldots, \imath_n)$ instead of $\exists x\forall y(A(y, \imath_2 A^2, \ldots, \imath_n A^n))$; similarly for its negation.

$$\cfrac{\cfrac{\exists 1x_1A(x_1, \imath_2, \ldots, \imath_n), \neg\exists 1xA(x, \imath_2, \ldots, \imath_n)}{E!\imath_1 A^1(x, \imath_2, \ldots, \imath_n), \neg\exists 1xA(x, \imath_2, \ldots, \imath_n)}}{E!\imath x_1 A^1(x_1, \imath_2, \ldots, \imath_n), \neg E!\imath_1 A^1(x, \imath_2, \ldots, \imath_n)}$$

The remaining parts of lemma 3, the lemmata on weakening and substitution are proved as above.

Corollary 5 ($\imath$-weight reduction)

(a) If $\mathsf{T}_\imath^{E!} \vdash^n \Gamma, E!\imath x_1 A^1(x_1, \imath_2, \ldots, \imath_n)$ with $\imath$-weight of $E!\imath x_1 A^1(x_1, \imath_2, \ldots, \imath_n)$ is r and $\imath x_1 A^1$ is the principal $\imath$-term, then $\mathsf{T}_\imath^{E!} \vdash^{n-1} \Gamma, \exists 1xA(x, \imath_2, \ldots, \imath_n)$, where $\imath|\exists 1xA(x, \imath_2, \ldots, \imath_n)| < \imath|[E!\imath x_1 A^1(x_1, \imath_2, \ldots, \imath_n)|$.

(b) If $\mathsf{T}_\imath^{E!} \vdash^n \Gamma, \neg E!\imath x_1 A^1(x_1, \imath_2, \ldots, \imath_n)$ with $\imath$-weight of $\neg E!\imath x_1 A^1(x_1, \imath_2, \ldots, \imath_n)$ is r and $\imath_1$ is the principal $\imath$-term, then $\mathsf{T}_\imath^{E!} \vdash^{n-1} \Gamma, \neg\exists 1xA(x, \imath_2, \ldots, \imath_n)$, where $\imath|\neg\exists 1xA(x, \imath_2, \ldots, \imath_n)| < \imath|\neg E!\imath x_1 A^1(x_1, \imath_2, \ldots, \imath_n)|$.

The characteristic theorems
The characteristic theorems of Russell's original proposal are deducible in $\mathsf{T}_\imath^{E!}$ which is seen by the following derivations:

$$\frac{\exists 1 x_1 A(x_1, \imath_2, \ldots, \imath_n), \neg\exists 1 x A(x, \imath_2, \ldots, \imath_n)}{E!\imath_1 A^1(x, \imath_2, \ldots, \imath_n), \neg\exists 1 x A(x, \imath_2, \ldots, \imath_n)}$$

$$\frac{\neg\exists 1 x A(x, \imath_2, \ldots, \imath_n)), \exists 1 x A(x, \imath_2, \ldots, \imath_n)}{\neg E!\imath_1 A^1(x, \imath_2, \ldots, \imath_n), \exists 1 x A(x, \imath_2, \ldots, \imath_n)}$$

The Cut-lemma also holds for $\mathsf{T}_\imath^{E!}$:

Lemma 6 (Cut-lemma) $\mathsf{T}_\imath^{E!} \vdash_m^k \Gamma, C,\ \mathsf{T}_\imath^{E!} \vdash_m^l \Gamma, \neg C \Longrightarrow \mathsf{T}_\imath^{E!} \vdash_m^{k+l} \Gamma.$

We outline only the crucial case.

$$\dfrac{\dfrac{\vdash_m^{k-1} \Gamma, C, \exists 1 x A \qquad \dfrac{\vdash_m^l \Gamma, \neg C}{\vdash_m^l \Gamma, \neg C, \exists 1 x A}}{\vdash_m^{k-1+l} \Gamma, \exists 1 x A} \qquad \dfrac{\dfrac{\vdash_m^k \Gamma, C}{\vdash_m^k \Gamma, C, \neg\exists 1 x A} \qquad \dfrac{\vdash_m^{l-1} \Gamma, \neg C, \neg\exists 1 x A}{\vdash_m^{l-1} \Gamma, \neg C, \neg\exists 1 x A}}{\vdash_m^{l-1+k} \Gamma, \neg\exists 1 x A}}{\vdash_m^{k+l} \Gamma}$$

This establishes the Cut-theorem:

Theorem 6 (Cut-elimination) Cut is eliminable from $\mathsf{T}_\imath^{E!}$:
If $\mathsf{T}_\imath^{E!} \vdash_{m+1}^k \Gamma$, then $\mathsf{T}_\imath^{E!} \vdash_m^{2^k} \Gamma$.

2.5 Conservativity

Definition 12 A theory/logic T is conservative over a theory/logic T' with respect to Γ iff if $T \vdash \Gamma$ then $T' \vdash \Gamma$.

Theorem 7 (Conservativity)

(a) If $\mathsf{T}_\imath^{E!} \vdash \Gamma$, where Γ is $E!$-free, then $\mathsf{T}_\imath \vdash \Gamma$.

(b) If $\mathsf{T}_\imath \vdash \Gamma$, where Γ is $\imath$-term-free, then $\mathsf{T}^= \vdash \Gamma$.

(c) $\mathsf{T}^= \vdash \Gamma$, where Γ is $=$-free, then $\mathsf{T} \vdash \Gamma$.

(c) is well known (cg. Troesltra/Schwichtenberg (2000, p.134f.)). *Proof* (of (a) – the proof (b) is analogous to that of (a)) by induction on the length of a derivation. Suppose that

$$\frac{\Gamma, \exists 1xA}{\Gamma, E!\imath xA}$$

is the topmost instance in a given derivation. If $\Gamma, \exists 1xA$ is an axiom and $\exists 1xA$ is not a literal, so Γ (without $\exists 1xA$) is an axiom and also its conclusion, i.e. Γ (without $\exists 1xA$) is an axiom. From this facts we obtain a trivial derivation of Γ. Now suppose that the premiss of $(E!)$-inference comes from a one- or two-premiss rule, say:

$$\cfrac{\cfrac{\Gamma'', \exists 1xA}{\Gamma', \exists 1xA}\,(\mathrm{R})}{\Gamma, E!\imath xA}\,(E!)$$

This derivation can be transformed into:

$$\cfrac{\cfrac{\Gamma'', \exists 1xA}{\Gamma', E!\imath xA}\,(E!)}{\Gamma}\,(\mathrm{R})$$

Now, since the the height of the premiss of the $(E!)$-inference is of lower degree the inductive hypothesis is applicable and the required result is obtained. If the inference is $(E!)$-inference with $E!\imath xA$ as its principal formula, then the height of the derivation of its premiss is shorter, the inductive hypothesis is applicable and result is obtained.

3 Elimination of $\imath$-terms

For this end we define inductively a function * as follows:

Definition 13 (Inductive definition of *)
The inductive definition proceeds on the $\imath$-weight, i.e. the number of $\imath$-terms, of a formula.

(i) If $\imath|C| = 0$ then C^* is C.

(ii) If $\imath|C| > 0$ then:

(ii.i) If C is of the form $[\imath_1, \imath_2, \dots \imath_n]B$, where $\imath_1$ is the leftmost $\imath$-term and B is not of the form $\neg C$, then C^* is $\exists 1x(A^{1*}(x) \wedge [\imath_2, \dots \imath_n]B)$.

($ii.ii$) If C is of the form $\neg[\imath_1, \imath_2, \dots \imath_n]B$, where $\imath_1$ is the leftmost $\imath$-term, then C^* is $\neg\exists 1x(A^{1*}(x) \wedge [\imath_2, \dots \imath_n]B)$.

($ii.iii$) If C is of the form $[\imath_1, \imath_2, \dots \imath_n]\neg B$, where $\imath_1$ is the leftmost $\imath$-term, then C^* is $\exists 1x(A^{1*}(x) \wedge [\imath_2, \dots \imath_n]\neg B)$.

(iii) The other cases, eg. C is of the form $D \vee E$, $D \wedge E$, $\exists xD$, and $\forall xD$ are not treated explicitly.

Theorem 8 (Elimination theorem)
If $\mathbf{T}_\imath \vdash \Gamma, C$ with $\imath|C| \geqslant 0$ then there is a formula C^* such that $\mathbf{T}_\imath \vdash \Gamma, C \leftrightarrow C^*$ and $\mathbf{T}_\imath \vdash \Gamma^*, C^*$, where $\imath|\Gamma| = 0$ and $\mathbf{T}$ is defined as $\mathbf{T}_\imath$ but without ($\imath$), the language of $\mathbf{T}_\imath$ is modified accordingly.

The *proof* is established by the following procedure.

A terminating elimination procedure

By hypothesis there is a derivation of Γ, C in $\mathbf{T}_\imath$.

Case 1: $\imath|\Gamma| = \imath|C| = 0$; then $\mathbf{T} \vdash \Gamma, C$.

Case 2: $\imath|C| > 0$, $\imath|\Gamma| = 0$:

Stage a If C is of the form $[\imath_1, \imath_2, \dots \imath_n]B$, where $\imath_1$ is the leftmost $\imath$-term and with the conditions as described in definition 13 ($ii.i$) then $\mathbf{T}_\imath \vdash \Gamma, [\imath_1, \imath_2, \dots \imath_n]B \leftrightarrow \exists 1x(A^1(x) \wedge [\imath_2, \dots \imath_n]B)$, where $\imath|\exists 1x(A^1(x) \wedge [\imath_2, \dots \imath_n]B)| < \imath|[\imath_1, \imath_2, \dots \imath_n]B|$. If $\imath|\exists 1x(A^1(x) \wedge [\imath_2, \dots \imath_n]B)| > 0$ then repeat stage a or go stages b or c, else go to case 1.

Stage b If C is of the form $\neg[\imath_1, \imath_2, \dots \imath_n]B$, where $\imath_1$ is the leftmost $\imath$-term and with the conditions as described in definition 13 ($ii.ii$) then $\mathbf{T}_\imath \vdash \Gamma, \neg[\imath_1, \imath_2, \dots \imath_n]B \leftrightarrow \neg\exists 1x(A^1(x) \wedge [\imath_2, \dots \imath_n]B)$, where $\imath|\neg\exists 1x(A^1(x) \wedge [\imath_2, \dots \imath_n]B)| < \neg[\imath_1, \imath_2, \dots \imath_n]B|$. If $\imath|\neg\exists 1x(A^1(x) \wedge [\imath_2, \dots \imath_n]B)| > 0$ then repeat stage b or go stages a or c, else go to case 1.

Stage c If C is of the form $[\imath_1, \imath_2, \dots \imath_n]\neg B$, where $\imath_1$ is the leftmost $\imath$-term and with the conditions as described in definition 13 ($ii.iii$) then $\mathbf{T}_\imath \vdash \Gamma, [\imath_1, \imath_2, \dots \imath_n]\neg B \leftrightarrow \exists 1x(A^1(x) \wedge [\imath_2, \dots \imath_n]\neg B)$, where $\imath|\exists 1x(A^1(x) \wedge [\imath_2, \dots \imath_n]\neg B)| < [\imath_1, \imath_2, \dots \imath_n]\neg B|$. If $\imath|\exists 1x(A^1(x) \wedge [\imath_2, \dots \imath_n]\neg B)| > 0$ then repeat stage b or go stages a or c, else go to case 1.

Case 3: $\imath|C| > 0$, $\imath|\Gamma| > 0$. The procedure starts with the formula C and follows the procedure of case 2, stages *a*—*c* but without the else-parts. The procedure is continued with the leftmost formula F of Γ with $\imath|F| > 0$ unless F is the only formula of Γ with $\imath|F| > 0$. Let G be the next formula in Γ with $\imath|G| > 0$; then again this procedure is continued as in case 2, stages *a*—*c* (without the else-parts). Continued applications of this routine eventually ends with the rightmost formula H of Γ with $\imath|H| > 0$; the routine is continued with case 2, stages *a*—*c*.

This establishes both soundness and completeness of $\mathbf{T}_\imath$ via soundness and completeness of $\mathbf{T}$.

Conjecture 1
We conjecture that the order of elimination (granted that it is uniquely specified) does not matter; i.e. given two (separable) elimination procedures E_1 and E_2 then the outcomes C^{E_1} and C^{E_2} of a formula C (of some specified formal language $\mathcal{L}$) are logically equivalent.

Cut-elimination via $\mathbf{T}$
Theorem 8 establishes an indirect cut-elimination theorem for $\mathbf{T}_\imath$ via the cut-elimination theorem of $\mathbf{T}$. Suppose—as for the elimination theorem—that $\mathbf{T}_\imath \vdash \Gamma, C$ with $\imath|\Gamma| > 0$, and $\imath|C| > 0$; then by theorem 8 there is a derivation of Γ^*, C^* in $\mathbf{T}$ with $\imath|\Gamma^*| = \imath|C^*| = 0$. By the cut-elimination theorem for $\mathbf{T}$ there is a cut-free derivation of Γ^*, C^* in $\mathbf{T}$.

Furthermore, we know that each formula G^* of Γ^* and C is provably equivalent (in $\mathbf{T}_\imath$) with the corresponding formula G of Γ and C. Trivially, there is also a cut-free derivation of Γ^*, C^* in $\mathbf{T}_\imath$. By replacing each G^* of Γ^* with its corresponding G of Γ and C^* with C in the sequent Γ^*, C^* (the end-sequent of the derivation) a new (and provably equivalent) end-sequent Γ, C is obtained.

4 "No meaning in isolation" — Semantics

We said earlier in this paper that the formal system $\mathbf{T}_\imath$ is not closed under substitution of $\imath$-terms. This holds also for $\mathbf{T}_\imath^{E!}$.

We follow Russell's intuition on definite description. For Russell $a = a$ is a theorem of, say, predicate logic with equality, but he claims that $\imath xA = \imath xA$ is not; and its truth depends contingently on the world (or rather model).

We would have an arbitrarily chosen but fixed object in domain (for each model) that would act as a denotation for empty $\imath$-terms. However, by doing so, formulas of the form $\imath xA = \imath xA$ would come out as logical truths – and this is not what Russell had in mind.

We can put this more formally, by stating a semantics for first-order predicate logic (as e.g. is done by Shoenfield (1998, p.18f.)) including $\imath$-terms. An interpretation $\Im$ consists of a tuple $\langle D, \varphi \rangle$ that satisfies the following conditions: (i) D is a non-empty set; (ii) φ is a function such that for each individual constant c of $\mathcal{L}$, $\varphi(c) \in D$ and $\varphi(P^n) \subseteq D^n$ for each n-ary predicate of $\mathcal{L}$. As usual we state a definition of truth – but in order to so, we add a new constant c_d for each element of D in $\mathcal{L}$, this ensures a proper treatment of the quantifiers:

($\Im$1) $\varphi(P^n(u_1, \ldots u_n)) = 1$ iff $\langle \varphi(u_1), \ldots, \varphi(u_n) \rangle \in \varphi(P^n)$.

($\Im$2) $\varphi(u = v)$ iff $\varphi(u) = \varphi(v)$.

($\Im$3) $\varphi(\neg A) = 1$ iff $\varphi(A) = 0$.

($\Im$4) $\varphi(A \vee B) = 1$ iff $\varphi(A) = 1$ or $\varphi(B) = 1$.

($\Im$5) $\varphi(A \wedge B) = 1$ iff $\varphi(A) = 1$ and $\varphi(B) = 1$.

($\Im$6) $\varphi(\exists xA) = 1$ iff there is a $d \in D$ such that: $\varphi(A_x(c_d)) = 1$.

($\Im$7) $\varphi(\forall xA) = 1$ iff for all $d \in D$ such that: $\varphi(A_x(c_d)) = 1$.

($\Im$8) $\varphi([\imath xA]B(\imath xA)) = 1$ iff {there is exactly one $d \in D$ such that $\varphi(A_x(c_d)) = 1$ and for all $d \in D$ holds: if $\varphi(A_x(c_d)) = 1$, then $\varphi(B_x(c_d)) = 1$}.

If the full language is taken into account, i.e. a language that includes $E!$, then we have to add the following condition:

($\Im$9) $\varphi(E!\imath A)$ iff there is exactly one $d \in D$ such that $\varphi(A(c_d)) = 1$.

The definitions of *model*, *valid*, *logical consequence*, are defined standardly. As we mentioned above it is now easily seen that formulas of the form $\imath xA = \imath xA$ are not valid.

Theorem 9 (Soundness)

(*a*) $\mathbf{T}_\imath$ is sound.

(*b*) $\mathbf{T}_\imath^{E!}$ is sound.

The *proof* is routine.

Conjecture 2
$\mathsf{T}_\imath$ and $\mathsf{T}_\imath^{E!}$ are complete with respect to the above semantics.

The proof of completeness typically requires some more technical apparatus; so, this work has to be carried out in a different paper.

5 Concluding remarks

We developed logics with $\imath$-terms in a Russellian spirit and proved the *Hauptsatz* for them. On the logical side there are still some open issues that could be addressed in future research: we mentioned already a completeness result, but furthermore it would be interesting if other metalogically celebrated results are obtainable for these logics, e.g. interpolation and Beth-definability. Especially the last one could possibly be fruitfully put to use if it comes to the analysis of theoretical terms.

Contrary to what has been developed in this paper it is rather possible that Russell's philosophical ideas on definite descriptions would be more suitably carved up as *contextually defined* expressions—as Russell originally suggested. In this case a definite description would be seen a *metalinguistic* expression that is context-definitionally equivalent to some other expression. Kaplan (1972) proposed the view Russell is especially vague if it come to incomplete symbols (cf. eg. Grabmayer (et.al) (2011, p.367ff.), however, a modern up-to-date study of contextual definitions and definite descriptions introduced by contextual definitions might still be a fruitful philosophical endeavor.

Acknowledgement The author is indebted to J. Czermak, G. Dorn, B. Fitelson, H. Leitgeb, O. Hjortland, P. Oppenheim, G. Sauermoser, and an anonymous referee.

References

Bencivenga, E., Lambert, K. & van Fraassen, B. (1986): *Logic, Bivalence and Denotation*, Ridgeview.

Bostock, D: (1997): *Intermediate Logic*, Oxford University Press.

Buchholz, H. (2002/03): *Beweistheorie*; available online: www.mathematik.uni-muenchen.de/ buchholz/articles/beweisth.ps

Carnap, R. (1960): *Meaning and Necessity*, The University of Chicago Press.

Church, A. (1963): "Review", in: The Journal of Symbolic Logic 28/1, 1963, pp.105–106.

van Fraassen, B., Lambert, K. (1967): "On Free Description Theory", in: Zeitschrift für mathematische Logik und Grundlagen der Mathematik 13, pp.225-246.

Gentzen, G. (1934/35): "Untersuchungen über das logische Schließen", in: Mathematische Zeitschrift vol.39, 1934/35, pp.176-210 and 405-431.

Grabmayer, C., Leo, J., van Oostrom, V & Visser, A: "On the termination of Russell's Description Elimination Algorithm", in: The Review of Symbolic Logic 4/3, pp.367-393.

Kalish, D., Montague, R. & D., Mar (1980): *Logic - Techniques of Formal Reasoning*, 2^{nd} edition, Oxford University Press.

Kaplan, D. (1972): "What is Russell's Theory of Definite Descriptions?", in: Pears, D.F. (1972, ed.): *Bertrand Russell: a Collection of Critical Essays*, Doubleday.

Kleene, Stephen C. (2000): *Introduction to Metamathematics*, 13^{th} edition, Wolters-Noordhoff Publishing.

Kripke, S. (2005): "Russell's Notion of Scope", in: Mind 114, pp.1005-1037.

Hilbert, D., Bernays, P. (1934): *Grundlagen der Mathematik*, vol 1., Berlin: Springer.

Hilbert, D., Bernays, P. (1939): *Grundlagen der Mathematik*, vol. 2., Berlin: Springer.

Lambert, K. (2003): “The Hilbert-Bernays Theory of Definite Descriptions”, in: Karel Lambert (ed., 2003): *Free Logic: Selected Essays*, Cambridge: The University Press, pp.44-68.

Lambert, K. (1999): “Logically Proper Definite Descriptions”, in: dialectica 53 3/4, pp. 271-282.

Lewis, D. (1970): “How to Define Theoretical Terms?”, in: The Journal of Philosophy 67/13, pp.427-446.

Neale, S. (1990): *Descriptions*, MIT Press.

Negri, S. & von Plato, J. (2001): *Structural Proof Theory*, Cambridge University Press.

Negri, S. & von Plato, J. (1998): “Cut Elimination in the Presence of Axioms”, in: The Bulletin of Symbolic Logic 4, pp.418-435.

Oppenheimer, P. E., Zalta, Edward N. (2011) “A Computationally-Discovered Simplification of the Ontological Argument”, in: Australasian Journal of Philosophy, 89, pp.333-350.

Oppenheimer, P. E., Zalta, Edward N. (1991) “On the Logic of the Ontological Argument”, in: Philosophical Perspectives 5, 1991, pp. 509–529.

Russell, B. (1993): *Introduction to Mathematical Philosophy*, Dover Publications.

Russell, B. (1905): “On Denoting”, in Mind (New Series), 14/56, pp.479-493.

Scott, D. (1991): “Existence and Description in Formal Logic”, in: Lambert, K. (ed. 1991): *Philosophical Applications of Free Logic*, Oxford University Press, pp. 28-48.

Schock, R. (1962): “Some remarks on Russell’s treatment of definite description”, in Logique et analyze, vol. 5, pp.77–80.

Shoenfield, J. (1998): *Mathematical Logic*, Addison-Wesley.

Tait, W.W. (1968): "Normal derivability in classical logic", in: Barwise, J. (1968): The Syntax and Semantics of Infinitary Languages, Lecture Notes in Mathematics Nr. 72. Berlin: Springer.

Troelstra, A. & Schwichtenberg, H. (2000): *Basic Proof Theory*, 2^{nd} edition, Cambridge University Press.

Whiethead, A.N., Russell, B. (1910): *Principia Mathematica, vol. I*, Cambridge University Press.